On a class of rational matrix differential equations arising in stochastic control

G. Freiling *, A. Hochhaus

Department of Mathematics, University of Duisburg, D-47048 Duisburg, Germany

Abstract

We prove a monotonicity and a comparison theorem for the solutions of the rational matrix differential equation (1.4) appearing in stochastic control and derive existence and convergence results for the solutions of (1.4). Moreover we present conditions ensuring that the corresponding algebraic matrix equation (1.5) has a stabilizing solution.

Key words: Rational matrix differential equations, generalized Riccati differential equations, generalized stabilizability and detectability, comparison theorem, existence and convergence results.
1991 MSC: 34A34, 34C10, 93E03

1 Introduction

In many applications (see [34]) there appear linear-quadratic stochastic optimal control problems of the form

$$\left.\begin{aligned} dx(t) &= \Big[A(t)x(t) + B(t)u(t)\Big]\,dt + \Big[C(t)x(t) + D(t)u(t)\Big]\,dw(t), \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.1}$$

$$J(u) := E\left\{ x(t_f)^T Q_f x(t_f) + \int_{t_0}^{t_f} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right\}, \tag{1.2}$$

where the state $x(t)$ and the control $u(t)$ are stochastic processes of dimensions n and m, respectively, and where $w(t)$ is a one-dimensional standard Wiener process (Brownian motion) on some probability space $(\Omega, \mathcal{F}, P)$. Moreover

* Corresponding author.
Email address: freiling@math.uni-duisburg.de (G. Freiling).

it is assumed here that $u(t)$ is non-anticipating with respect to $w(t)$, x_0 is independent of $w(t)$ and that A, B, C and D are sufficiently smooth matrix-functions.

Analogously to the proof of Theorem 6.1 in [34] (see also Proposition 2.1 in [3]) it can be shown that the optimal control for (1.1), (1.2) is determined by the solution of the following terminal value problem for a generalized Riccati matrix differential equation (where we suppress the argument t):

$$
\begin{aligned}
-\dot{P} = A^T P + PA + Q + C^T PC - \left(S + PB + C^T PD\right) \\
\times \left(R + D^T PD\right)^+ \left(S + PB + C^T PD\right)^T, \quad P(t_f) = Q_f.
\end{aligned}
\tag{1.3}
$$

More precisely we have

Theorem 1.1 *Let P be the solution of the terminal value problem* (1.3)*. If P exists on the interval $[t_0, t_f]$ such that in addition*

$$R(t) + D(t)^T P(t)D(t) \geq 0$$

and

$$\operatorname{Ker}\left[R(t) + D(t)^T P(t)D(t)\right] \subseteq \operatorname{Ker}\left[S(t) + P(t)B(t) + C(t)^T P(t)D(t)\right]$$

for all $t \in [t_0, t_f]$, then

$$J(u) \geq E\left\{x_0^T P(t_0)x_0\right\},$$

where equality holds if $u(t) = F(t)x(t)$ with the feedback matrix

$$F(t) = -\left[R(t) + D(t)^T P(t)D(t)\right]^+ \left[S(t) + P(t)B(t) + C(t)^T P(t)D(t)\right]^T.$$

The main purpose of this note is to show that several of the nice properties of standard matrix Riccati equations remain valid for the solutions of a general class of rational matrix equations which contains (1.3) as a particular case; for convenience we confine to the autonomous case.

For this purpose we consider below matrix differential equations of the form

$$
\begin{aligned}
-\dot{X} = A^* X + XA + Q + \Pi_1(X) - [S + XB + \Pi_{12}(X)] \\
\times [R + \Pi_2(X)]^+ [S + XB + \Pi_{12}(X)]^*
\end{aligned}
\tag{1.4}
$$

and the corresponding algebraic equations

$$
\begin{aligned}
&A^*X + XA + Q + \Pi_1(X) - [S + XB + \Pi_{12}(X)] \\
&\quad \times [R + \Pi_2(X)]^+[S + XB + \Pi_{12}(X)]^* = 0
\end{aligned} \tag{1.5}
$$

where Z^+ is the Moore-Penrose inverse of a matrix Z and A, B, Q, R and S are given matrices of sizes $n \times n$, $n \times m$, $n \times n$, $m \times m$ and $n \times m$, respectively, such that

$$
T := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}
$$

is hermitian. We assume further that the operator $\Pi\colon \mathcal{H}^n \to \mathcal{H}^{n+m}$ with

$$
\Pi(X) := \begin{bmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{bmatrix}
$$

is linear and positive, i.e. $X \geq 0$ implies $\Pi(X) \geq 0$. Here, $\mathcal{H}^n$ stands for the real vector space of hermitian matrices of size n, and by $X \geq 0$ (or $X > 0$) it is denoted that $X = X^*$ is positive semidefinite (or positive definite).

Notice that in (1.3) Π takes the form $\Pi(X) = \begin{bmatrix} C & D \end{bmatrix}^T X \begin{bmatrix} C & D \end{bmatrix}$.

Equations of the form (1.4) and certain special cases have been studied recently (see [8], [12], [23] and [34]) and can be considered as generalized Riccati-type equations.

In the case where $R > 0$ and $\Pi \equiv 0$ (1.4) reduces to the continous-time Riccati differential equation

$$
-\dot{X} = A^*X + XA + Q - (S + XB)R^{-1}(S + XB)^*,
$$

and for $R > 0$, $\Pi_2 \equiv 0$ and $\Pi_{12} \equiv 0$ (1.4) coincides with

$$
-\dot{X} = A^*X + XA + Q + \Pi_1(X) - (S + XB)R^{-1}(S + XB)^*.
$$

The latter class of linearly perturbed Riccati differential equations appears among others in control problems with stochastically jumping parameters (see [13], [14], [17], [20], [21], [30], [33] and the numerous literature cited therein); the corresponding algebraic equations and inequalities play also an important role in the application of the Lyapunov-Krasovskii method to linear time-delay systems (see [25], [31]).

First steps concerning the theory of the rational matrix differential equation (1.4) have been performed by Hinrichsen and Pritchard [23], by Ait Rami et al. [3,4] and by Chen et al. [8–10], who obtained unter additional assumptions sufficient conditions for the existence of the solutions of (1.4) on a given interval for certain initial values (see also [18]). The algebraic equation (1.5) has been studied recently in detail by Damm and Hinrichsen [12].

Dragan and Morozan [15] considered in the case of time-varying coefficients coupled systems of differential equations which can be transformed to the form (1.4); they investigate properties of stabilizing and bounded solutions of these differential equations and provide a theorem on the existence of the maximal solution - the case of periodic coefficients is studied as well.

The aim of this paper is to provide a unified treatment of the class of equations (1.4). In Sections 2 and 3 of this note we recall several notations and preliminary results concerning Schur complements and linearly perturbed Lyapunov equations. In Section 4 we show that the solutions of (1.4) depend in particular monotonically on T and on a given initial or terminal value. As a consequence of Theorem 4.5 we derive two corollaries concerning the existence of the solutions of (1.4) on $(-\infty, t_f]$ which extend the existence result that has been derived recently in [8]. The main contribution of Section 5 is Theorem 5.9 where we present sufficient conditions for the existence and the uniqueness of the stabilizing solution X_+ of (1.5). Furthermore we show in Section 6 that under adequate definiteness, stabilizability and detectability assumptions the solution of (1.4) converges for any positive semidefinite terminal value to the stabilizing solution of (1.5).

There exist also discrete-time versions of our results – details can be found in [19].

2 The Schur complement

In this section we present some notations and preliminary results from matrix analysis.

Remark 2.1 *The* Moore-Penrose inverse *of an $m \times n$ matrix Z is the unique $n \times m$ matrix Z^+ satisfying the conditions*

$$Z^+ Z Z^+ = Z^+, \quad Z Z^+ Z = Z, \quad (Z^+ Z)^* = Z^+ Z, \quad (Z Z^+)^* = Z Z^+.$$

It has the following properties (see [28]):

(i) $(Z^+)^+ = Z$.
(ii) $(Z^*)^+ = (Z^+)^*$.
(iii) $(\lambda Z)^+ = \lambda^{-1} Z^+$ *for all* $\lambda \neq 0$.
(iv) $\operatorname{Ker} Z^+ = \operatorname{Ker} Z^*$, $\operatorname{Im} Z^+ = \operatorname{Im} Z^*$.
(v) If Z is hermitian or positive semidefinite, then so is Z^+.

Lemma 2.2 ([2], Theorem 9.17) *Assume that Z is an $m \times n$ matrix and W is a $p \times n$ matrix. Then the following statements are equivalent:*

(i) $\operatorname{Ker} Z \subseteq \operatorname{Ker} W$.
(ii) $W = WZ^+Z$.
(iii) $W^+ = Z^+ZW^+$.

Lemma 2.3 ([5], Theorem 1) *Let H be a hermitian matrix of size $n+m$ with*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix}$$

where L is $n \times n$ and M is $m \times m$. Then:

(i) H is positive semidefinite if and only if

$$M \geq 0, \quad L - NM^+N^* \geq 0 \quad \text{and} \quad \operatorname{Ker} M \subseteq \operatorname{Ker} N.$$

(ii) If $M > 0$ then H is positive semidefinite if and only if $L - NM^{-1}N^* \geq 0$.
(iii) H is positive definite if and only if $M > 0$ and $L - NM^{-1}N^* > 0$.

The matrix $H/M := L - NM^+N^*$ is called the *Schur complement* of M in H.

Corollary 2.4 *If*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix}$$

is a hermitian matrix then H admits a Schur decomposition

$$H = \begin{bmatrix} I & NM^+ \\ 0 & I \end{bmatrix} \begin{bmatrix} L - NM^+N^* & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} I & 0 \\ M^+N^* & I \end{bmatrix}$$

if and only if $\operatorname{Ker} M \subseteq \operatorname{Ker} N$.

The following lemma provides the basis for the proof of a comparison theorem for rational matrix differential equations of the form (1.4); under the additional assumption $\operatorname{Ker} M = \operatorname{Ker} \tilde{M}$ it was already proved in [11].

Lemma 2.5 *Let*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix} \quad \text{and} \quad \tilde{H} = \begin{bmatrix} \tilde{L} & \tilde{N} \\ \tilde{N}^* & \tilde{M} \end{bmatrix}$$

be hermitian $(n+m) \times (n+m)$ matrices, where L and $\tilde{L}$ are $n \times n$ matrices. Define $H_d := H - \tilde{H}$, $M_d := M - \tilde{M}$ and $N_d := N - \tilde{N}$. If

$$\operatorname{Ker} M \subseteq \operatorname{Ker} N, \quad \operatorname{Ker} \tilde{M} \subseteq \operatorname{Ker} \tilde{N} \quad \text{and} \quad \operatorname{Ker} M_d \subseteq \operatorname{Ker} N_d,$$

then

$$H/M - \tilde{H}/\tilde{M} - H_d/M_d = (NM^+\tilde{M} - \tilde{N})(M_d^+ + \tilde{M}^+)(NM^+\tilde{M} - \tilde{N})^*.$$

PROOF. From the definition of H/M we infer

$$\begin{aligned}
&H/M - \tilde{H}/\tilde{M} - H_d/M_d \\
&= -NM^+N^* + \tilde{N}\tilde{M}^+\tilde{N}^* + (N - \tilde{N})M_d^+(N - \tilde{N})^* \\
&= \begin{bmatrix} N & -\tilde{N} \end{bmatrix} \begin{bmatrix} M_d^+ - M^+ & M_d^+ \\ M_d^+ & M_d^+ + \tilde{M}^+ \end{bmatrix} \begin{bmatrix} N & -\tilde{N} \end{bmatrix}^* . \qquad (2.1)
\end{aligned}$$

With the assumptions of the lemma it follows from Lemma 2.2 that $N = NM^+M$ and $\tilde{N} = \tilde{N}\tilde{M}^+\tilde{M}$. Using the properties of the Moore-Penrose inverse we get

$$NM^+\tilde{M}(M_d^+ + \tilde{M}^+)\tilde{N}^* = N(M_d^+ + M^+)\tilde{N}^* - NM^+M_dM_d^+\tilde{N}^*$$

and

$$\begin{aligned}
&NM^+\tilde{M}(M_d^+ + \tilde{M}^+)\tilde{M}M^+N^* \\
&= N(M_d^+ + M^+)N^* - NM_d^+M_dM^+N^* - NM^+M_dM_d^+N^*.
\end{aligned}$$

From the last two identities it follows with $N_d = N_dM_d^+M_d$ that

$$\begin{aligned}
&(NM^+\tilde{M} - \tilde{N})(M_d^+ + \tilde{M}^+)(NM^+\tilde{M} - \tilde{N})^* \\
&= N(M_d^+ + M^+)N^* - N(M_d^+ + M^+)\tilde{N}^* - \tilde{N}(M_d^+ + M^+)N^* \\
&\qquad - NM^+M_dM_d^+(N - \tilde{N})^* - (N - \tilde{N})M_d^+M_dM^+N^* \\
&\qquad + \tilde{N}(M_d^+ + \tilde{M}^+)\tilde{N}^* \\
&= N(M_d^+ - M^+)N^* - NM_d^+\tilde{N}^* - \tilde{N}M_d^+N^* + \tilde{N}(M_d^+ + \tilde{M}^+)\tilde{N}^* \\
&= \begin{bmatrix} N & -\tilde{N} \end{bmatrix} \begin{bmatrix} M_d^+ - M^+ & M_d^+ \\ M_d^+ & M_d^+ + \tilde{M}^+ \end{bmatrix} \begin{bmatrix} N & -\tilde{N} \end{bmatrix}^* ,
\end{aligned}$$

and this yields together with (2.1) the assertion of the lemma. □

Corollary 2.6 *Let*

$$H = \begin{bmatrix} L & N \\ N^* & M \end{bmatrix} \quad \text{and} \quad \tilde{H} = \begin{bmatrix} \tilde{L} & \tilde{N} \\ \tilde{N}^* & \tilde{M} \end{bmatrix}$$

be hermitian $(n+m) \times (n+m)$ *matrices, where* L *and* $\tilde{L}$ *are* $n \times n$ *matrices. Assume that*

$$\operatorname{Ker} M \subseteq \operatorname{Ker} N \quad \text{and} \quad \operatorname{Ker} \tilde{M} \subseteq \operatorname{Ker} \tilde{N}.$$

If $H \geq \tilde{H}$ *and* $\tilde{M} \geq 0$ *then the difference* $H/M - \tilde{H}/\tilde{M}$ *is positive semidefinite.*

PROOF. Let $H_d := H - \tilde{H}$, $M_d := M - \tilde{M}$ and $N_d := N - \tilde{N}$. From $H_d \geq 0$ we infer with Lemma 2.3 (i) that $M_d \geq 0$, $H_d/M_d \geq 0$ und $\operatorname{Ker} M_d \subseteq \operatorname{Ker} N_d$. Therefore the assumptions of Lemma 2.5 are satisfied. Moreover M_d^+ and $\tilde{M}^+$ are positive semidefinite. Consequently Lemma 2.5 yields the assertion of the corollary. □

3 Lyapunov equations and stability

In this section we consider the linearly perturbed algebraic Lyapunov equation

$$A^*X + XA + \Pi_1(X) + Q = 0, \tag{3.1}$$

where A and Q are given $n \times n$ matrices, Q is hermitian and $\Pi_1 \colon \mathcal{H}^n \to \mathcal{H}^n$ is a positive linear operator. For the investigation of (3.1) we need some preliminary facts.

First we extend some results of Damm and Hinrichsen [12] which are based on the concept of resolvent positive operators in ordered Banach spaces. More information about ordered Banach spaces can be found in Section 7.1 of [35].

Throughout this article we endow the real vector space $\mathcal{H}^n$ of all hermitian $n \times n$ matrices with the scalar product $\langle A, B \rangle = \operatorname{tr} A^*B$ and the induced Frobenius norm $\|A\|_F := \langle A, A \rangle^{1/2}$. Notice that $\mathcal{H}^n$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle$; moreover $\mathcal{H}^n$ is ordered since the cone $\mathcal{H}^n_+$ of all positive semidefinite matrices defines an order relation on $\mathcal{H}^n$ by

$$A \geq B \iff A - B \in \mathcal{H}_n^+;$$

this order is used subsequently.

In the next lemma we recall some properties of the trace of a product of matrices.

Lemma 3.1 ([22]) *(i) For all matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, $\operatorname{tr} AB = \operatorname{tr} BA$.*

(ii) Let $A, B \in \mathcal{H}^n_+$ with $B > 0$. Then, $\operatorname{tr} AB \geq 0$, with equality holding if and only if $B = 0$.

(iii) Let $A, B \in \mathcal{H}^n_+$. Then, $\operatorname{tr} AB \geq 0$, with equality holding if and only if $AB = 0$.

Now we introduce the concept of resolvent positive operators. Details on this topic can be found in Section 3.11 of [7].

Definition 3.2 *An operator $\mathcal{T} \colon \mathcal{H}^n \to \mathcal{H}^n$ is called* inverse positive *if the* *inverse $\mathcal{T}^{-1}$ exists and is positive. The operator $\mathcal{T}$ is called* resolvent positive *if $\lambda\mathcal{I} - \mathcal{T}$ is inverse positive for all sufficiently large $\lambda \in \mathbb{R}$.*

For any linear continuous operator $\mathcal{T}$ we denote by

$$r(\mathcal{T}) := \sup_{\lambda\in\sigma(\mathcal{T})} |\lambda| \quad \text{and} \quad s(\mathcal{T}) := \sup_{\lambda\in\sigma(\mathcal{T})} \operatorname{Re}\lambda$$

the spectral radius and the spectral bound of $\mathcal{T}$, respectively.

Define the continuous-time Lyapunov operator $\mathcal{L}_A$ by

$$\mathcal{L}_A \colon \mathcal{H}^n \to \mathcal{H}^n, \;\; X \mapsto A^*X + XA.$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (counted with multiplicities), then the eigenvalues of $\mathcal{L}_A$, considered as an operator from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n}$, are the n^2 numbers $\lambda_j + \lambda_k$, $1 \leq j,k \leq n$. If all eigenvalues of A lie in the open left half-plane then $-\mathcal{L}_A$ is inverse positive, and the inverse of $\mathcal{L}_A$ is given by (see Section 5.3 in [27])

$$\mathcal{L}_A^{-1}(X) = -\int_0^\infty e^{A^*t}Xe^{At}\,dt.$$

Since $\lambda\mathcal{I} - \mathcal{L}_A = -\mathcal{L}_{A-\lambda/2I}$ is inverse positive for $\lambda > 2s(\mathcal{L}_A)$ it follows from Definition 3.2 that $\mathcal{L}_A$ is resolvent positive.

The next theorem generalizes Lyapunovs stability theorem. A slightly modified version of this result can already be found in [17] (see also [16], Section III); in the case of time-varying coefficients a similar result has been proved in [14], Proposition 4.6.

Theorem 3.3 ([12]) *The following statements are equivalent:*

(i) All eigenvalues of A lie in the open left half-plane and $r\left(\mathcal{L}_A^{-1}\Pi_1\right) < 1$.
(ii) $-(\mathcal{L}_A - \Pi_1)$ is inverse positive.
(iii) There is some $X > 0$ such that $(\mathcal{L}_A + \Pi_1)(X) < 0$.
(iv) For any $Q > 0$ equation (3.1) has a unique solution $X > 0$.
(v) $\mathcal{L}_A + \Pi_1$ is c-stable.

If any one of these conditions is fulfilled then A is called c-stable relative to Π_1.

It turns out that the classical definitions of stabilizability and detectability have to be replaced in our situation by the following generalizations.

Definition 3.4 *A pair (A, B) of matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ is said to be* c-stabilizable relative to Π *if there is a matrix F such that $A + BF$ is c-stable relative to $\left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]$.*

According to Theorem 3.3 (A, B) is c-stabilizable relative to Π if and only if the inequality

$$(A + BF)^*X + X(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F \end{bmatrix} < 0$$

is fulfilled by a pair (F, X) with $X > 0$.

Notice that the concept of mean square stabilizability used in [17] is in the special case considered there equivalent to c-stabilizability relative to Π.

Definition 3.5 *A pair (C, A) of matrices $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$ is said to be* c-detectable relative to Π_1 *if there is a matrix $L \in \mathbb{C}^{n\times m}$ such that $A + LC$ is c-stable relative to Π_1.*

Next we formulate a necessary condition for c-detectability which corresponds to the well-known Hautus critierion.

Lemma 3.6 *If there exist a positive semidefinite matrix $V \neq 0$ with $CV = 0$ and some $\lambda \geq 0$ such that*

$$(\mathcal{L}_A + \Pi_1)^{adj}(V) = \lambda V, \tag{3.2}$$

then (C, A) is not c-detectable relative to Π_1.

PROOF. We assume that (C, A) is c-detectable relative to Π_1. Then according to Theorem 3.3 there exist matrices $L \in \mathbb{C}^{n\times m}$ and $X > 0$ such that

$$(\mathcal{L}_{A+LC} + \Pi_1)(X) < 0. \tag{3.3}$$

From $CV = 0$ and Lemma 3.1, (i), we infer that $\langle V, \mathcal{L}_{A+LC}(X)\rangle = \langle V, \mathcal{L}_A(X)\rangle$. Using hypothesis (3.2) and Lemma 3.1, (iii), it follows now that

$$\begin{aligned} 0 \leq \lambda\langle V, X\rangle &= \langle(\mathcal{L}_A + \Pi_1)^{adj}(V), X\rangle \\ &= \langle V, \mathcal{L}_A(X) + \Pi_1(X)\rangle \\ &= \langle V, \mathcal{L}_{A+LC}(X) + \Pi_1(X)\rangle \leq 0. \end{aligned}$$

Hence $\langle V, \mathcal{L}_{A+LC}(X) + \Pi_1(X)\rangle = 0$, and Lemma 3.1, (ii), yields now in view of (3.3) that $V = 0$, which contradicts the hypotheses. □

The following lemma generalizes results known from stability theory in the special case $\Pi_1 \equiv 0$.

Lemma 3.7 *Suppose $Q \geq 0$ and (3.1) has a solution $X \geq 0$.*

(i) If $Q > 0$ then A is c-stable relative to Π_1 and we have $X > 0$.
(ii) If (Q, A) is c-detectable relative to Π_1 then A is c-stable relative to Π_1.

PROOF. (i) Let us assume that A is not c-stable relative to Π_1. Then from Theorem 3.3 it follows that $s(\mathcal{L}_A + \Pi_1) \geq 0$ and now the Krein-Rutman-Theorem for resolvent positive operators (see [12], Theorem 3.7) shows that there is some $\lambda \geq 0$ and a matrix $V \in \mathcal{H}_n^+ \setminus \{0\}$ such that $(\mathcal{L}_A + \Pi_1)^{adj}(V) = \lambda V$. So we have

$$0 \leq \langle V, Q\rangle = -\langle V, (\mathcal{L}_A + \Pi_1)(X)\rangle = -\lambda\langle V, X\rangle \leq 0.$$

Hence $\langle V, Q\rangle = 0$, and since Q is positive definite, it follows from Lemma 3.1, (ii), that $V = 0$. This contradiction proves that A is c-stable relative to Π_1 and from Theorem 3.3 we obtain that the unique solution of (3.1) is positive definite.

(ii) If A is not c-stable relative to Π_1, then it follows from the proof above that there is a nonzero matrix $V \geq 0$ such that $\langle V, Q\rangle = 0$. Since V and Q are both positive semidefinite we obtain $QV = 0$ which contradicts by Lemma 3.6 the c-detectability of (Q, A) relative to Π_1. □

4 Existence and comparison theorems

In this section we present a general comparison theorem which allows the comparison of solutions of two rational matrix differential equations. As corollaries we derive two existence results. To formulate the comparison theorem we define $D(\mathcal{R})$ as the set of all $X \in \mathcal{H}^n$ such that

$$R + \Pi_2(X) \geq 0 \quad \text{and} \quad \operatorname{Ker}[R + \Pi_2(X)] \subseteq \operatorname{Ker}[S + XB + \Pi_{12}(X)]$$

and the rational matrix operator $\mathcal{R}\colon D(\mathcal{R}) \to \mathcal{H}^n$ by

$$\begin{aligned} \mathcal{R}(X) = {} & A^*X + XA + Q + \Pi_1(X) - [S + XB + \Pi_{12}(X)] \\ & \times [R + \Pi_2(X)]^+[S + XB + \Pi_{12}(X)]^*. \end{aligned} \tag{4.1}$$

If $\Pi_{12} \equiv 0$ and $\Pi_2 \equiv 0$ then $D(\mathcal{R}) = \mathcal{H}^n$ if $R \geq 0$ and $\operatorname{Ker} R \subseteq \operatorname{Ker}\left[\begin{smallmatrix} S \\ B \end{smallmatrix}\right]$. In the general situation we have

Lemma 4.1 *If $\hat{X} \in D(\mathcal{R})$ and $\operatorname{Ker}\left[R + \Pi_2(\hat{X})\right] \subseteq \operatorname{Ker} B$, then*

$$X \in D(\mathcal{R}) \quad \textit{for all} \quad X \geq \hat{X}.$$

In particular, $\mathcal{H}_+^n$ is contained in $D(\mathcal{R})$ if $R \geq 0$ and $\operatorname{Ker} R \subseteq \operatorname{Ker}\left[\begin{smallmatrix} S \\ B \end{smallmatrix}\right]$.

PROOF. From $R+\Pi_2(\hat{X}) \geq 0$ and $X \geq \hat{X}$ we infer that

$$R+\Pi_2(X) \geq R+\Pi_2(\hat{X}) \geq 0,$$

consequently

$$R+\Pi_2(X) \geq \Pi_2(X-\hat{X}) \geq 0.$$

These inequalities imply that

$$\operatorname{Ker}[R+\Pi_2(X)] \subseteq \operatorname{Ker}\left[R+\Pi_2(\hat{X})\right] \tag{4.2}$$

and

$$\operatorname{Ker}[R+\Pi_2(X)] \subseteq \operatorname{Ker}\Pi_2(X-\hat{X}) \subseteq \operatorname{Ker}\Pi_{12}(X-\hat{X}),$$

where the last inclusion is obtained by applying Lemma 2.3, (i), to the matrix $H := \Pi(X-\hat{X})$. Using (4.2) and the assumptions fulfilled by $\operatorname{Ker}\left[R+\Pi_2(\hat{X})\right]$ we get

$$\operatorname{Ker}[R+\Pi_2(X)] \subseteq \operatorname{Ker}\left[S+\hat{X}B+\Pi_{12}(\hat{X})\right]$$

and

$$\operatorname{Ker}[R+\Pi_2(X)] \subseteq \operatorname{Ker} B \subseteq \operatorname{Ker}\left[(X-\hat{X})B\right].$$

Combining the preceding relations we obtain finally

$$\operatorname{Ker}[R+\Pi_2(X)] \subseteq \operatorname{Ker}[S+XB+\Pi_{12}(X)],$$

and toghether with $R+\Pi_2(X) \geq 0$ it follows that $X \in D(\mathcal{R})$. If in particular $R \geq 0$ and $\operatorname{Ker} R \subseteq \operatorname{Ker}\left[\begin{smallmatrix} S \\ B \end{smallmatrix}\right]$, then $\hat{X}=0$ fulfills the assumptions of the lemma. In this case $\mathcal{H}_+^n$ is contained in $D(\mathcal{R})$. □

It is obvious that $\mathcal{R}(X)$ is the Schur complement of the so-called *dissipation matrix*

$$\Lambda(X) := \begin{bmatrix} A^*X+XA+Q+\Pi_1(X) & S+XB+\Pi_{12}(X) \\ [S+XB+\Pi_{12}(X)]^* & R+\Pi_2(X) \end{bmatrix}. \tag{4.3}$$

Consequently, by Lemma 2.3, (i), the quadratic matrix inequality $\mathcal{R}(X) \geq 0$ and the linear matrix inequality $\Lambda(X) \geq 0$ are equivalent on $D(\mathcal{R})$.

We agree that in this paper all statements concerning solutions X of (1.4) and (1.5) are made under the additional hypothesis $X(t) \in D(\mathcal{R})$ on its domain of definition and $X \in D(\mathcal{R})$, respectively.

Lemma 4.2 *If X is a hermitian $(n \times n)$ matrix such that*

$$\mathrm{Ker}[R + \Pi_2(X)] \subseteq \mathrm{Ker}[S + XB + \Pi_{12}(X)] \tag{4.4}$$

then

$$\mathcal{R}(X) = (A + BF)^* X + X(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* [T + \Pi(X)] \begin{bmatrix} I \\ F \end{bmatrix}, \tag{4.5}$$

where

$$F = F(X) := -[R + \Pi_2(X)]^+ [S + XB + \Pi_{12}(X)]^*. \tag{4.6}$$

PROOF. From Lemma 2.2 it follows that condition (4.4) is equivalent to

$$-F^*[R + \Pi_2(X)] = S + XB + \Pi_{12}(X).$$

So, if we rewrite $\mathcal{R}(X)$ as

$$\mathcal{R}(X) = A^* X + XA + Q + \Pi_1(X) - F^*[R + \Pi_2(X)]F,$$

we obtain

$$\begin{aligned} &\mathcal{R}(X) - \begin{bmatrix} I \\ F \end{bmatrix}^* [T + \Pi(X)] \begin{bmatrix} I \\ F \end{bmatrix} \\ &= A^* X + XA \\ &\quad - \Big\{ S + \Pi_{12}(X) + F^*[R + \Pi_2(X)] \Big\} F \\ &\quad - F^* \Big\{ [S + \Pi_{12}(X)]^* + [R + \Pi_2(X)]F \Big\} \\ &= (A + BF)^* X + X(A + BF). \qquad \square \end{aligned}$$

Lemma 4.3 *Let X_1 and X_2 be hermitian $(n \times n)$ matrices such that*

$$\mathrm{Ker}[R + \Pi_2(X_i)] \subseteq \mathrm{Ker}[S + X_i B + \Pi_{12}(X_i)], \quad i = 1, 2.$$

For $i = 1, 2$ define

$$F_i := F(X_i) = -[R + \Pi_2(X_i)]^+ [S + X_i B + \Pi_{12}(X_i)]^*.$$

Then the following identities hold:

$$\begin{aligned} \mathcal{R}(X_1) = &(A + BF_2)^* X_1 + X_1(A + BF_2) \\ &- (F_2 - F_1)^*[R + \Pi_2(X_1)](F_2 - F_1) \\ &+ \begin{bmatrix} I \\ F_2 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_2 \end{bmatrix} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\mathcal{R}(X_2)-\mathcal{R}(X_1) = &(A+BF_2)^*(X_2-X_1)+(X_2-X_1)(A+BF_2)\\ &+(F_2-F_1)^*[R+\Pi_2(X_1)](F_2-F_1)\\ &+\begin{bmatrix} I\\ F_2\end{bmatrix}^* \Pi(X_2-X_1)\begin{bmatrix} I\\ F_2\end{bmatrix}. \qquad (4.8)\end{aligned}$$

PROOF. Using Lemma 4.2 we get

$$\begin{aligned}\mathcal{R}(X_1) &= (A+BF_1)^*X_1+X_1(A+BF_1)\\ &\quad+\begin{bmatrix} I\\ F_1\end{bmatrix}^*[T+\Pi(X_1)]\begin{bmatrix} I\\ F_1\end{bmatrix}\\ &=(A+BF_2)^*X_1+X_1(A+BF_2)\\ &\quad+Q+\Pi_1(X_1)+F_1^*[R+\Pi_2(X_1)]F_1\\ &\quad+[S+X_1B+\Pi_{12}(X_1)]F_1-X_1BF_2\\ &\quad+F_1^*[S+X_1B+\Pi_{12}(X_1)]^*-F_2^*B^*X_1\\ &=(A+BF_2)^*X_1+X_1(A+BF_2)\\ &\quad+Q+\Pi_1(X_1)-F_1^*[R+\Pi_2(X_1)]F_1\\ &\quad+\Big\{S+\Pi_{12}(X_1)+F_1^*[R+\Pi_2(X_1)]\Big\}F_2\\ &\quad+F_2^*\Big\{[S+\Pi_{12}(X_1)]^*+[R+\Pi_2(X_1)]F_1\Big\}\\ &=(A+BF_2)^*X_1+X_1(A+BF_2)\\ &\quad-(F_2-F_1)^*[R+\Pi_2(X_1)](F_2-F_1)\\ &\quad+\begin{bmatrix} I\\ F_2\end{bmatrix}^*[T+\Pi(X_1)]\begin{bmatrix} I\\ F_2\end{bmatrix},\end{aligned}$$

which proves (4.7). Subtracting this from (4.5) with $X := X_2$, we obtain (4.8). □

To formulate the announced comparison theorem we introduce another rational matrix operator $\tilde{\mathcal{R}}\colon D(\tilde{\mathcal{R}}) \to \mathcal{H}^n$ with

$$\begin{aligned}\tilde{\mathcal{R}}(X) = &A^*X+XA+\tilde{Q}+\Pi_1(X)-[\tilde{S}+XB+\Pi_{12}(X)]\\ &\times[\tilde{R}+\Pi_2(X)]^+[\tilde{S}+XB+\Pi_{12}(X)]^* \qquad (4.9)\end{aligned}$$

where we assume that $\tilde{Q}$ and $\tilde{R}$ are hermitian and where $D(\tilde{\mathcal{R}})$ denotes the set of all $X \in \mathcal{H}^n$ such that

$$\tilde{R}+\Pi_2(X) \geq 0 \quad \text{and} \quad \operatorname{Ker}[\tilde{R}+\Pi_2(X)] \subseteq \operatorname{Ker}[\tilde{S}+XB+\Pi_{12}(X)].$$

With these notations we have

Lemma 4.4 *Let $X \in D(\tilde{\mathcal{R}})$ be given. If*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix}, \tag{4.10}$$

then

$$X \in D(\mathcal{R}) \quad \textit{and} \quad \mathcal{R}(X) \geq \tilde{\mathcal{R}}(X).$$

PROOF. Inequality (4.10) implies, in particular, that $R \geq \tilde{R}$, and consequently

$$R + \Pi_2(X) \geq \tilde{R} + \Pi_2(X) \geq 0. \tag{4.11}$$

Furthermore, we have

$$R + \Pi_2(X) \geq R - \tilde{R} \geq 0.$$

From these two inequalities it follows that

$$\operatorname{Ker}[R + \Pi_2(X)] \subseteq \operatorname{Ker}\left[\tilde{R} + \Pi_2(X)\right] \subseteq \operatorname{Ker}\left[\tilde{S} + XB + \Pi_{12}(X)\right]$$

and

$$\operatorname{Ker}[R + \Pi_2(X)] \subseteq \operatorname{Ker}\left[R - \tilde{R}\right] \subseteq \operatorname{Ker}\left[S - \tilde{S}\right],$$

where the last inclusion follows from the fact that (4.10) also implies $\operatorname{Ker}\left[R - \tilde{R}\right] \subseteq \operatorname{Ker}\left[S - \tilde{S}\right]$ (see Lemma 2.3, (i)). Combining the two relations above we get

$$\operatorname{Ker}[R + \Pi_2(X)] \subseteq \operatorname{Ker}\left[S + XB + \Pi_{12}(X)\right],$$

and together with (4.11), we obtain $X \in D(\mathcal{R})$.

If we associate the matrix

$$\tilde{\Lambda}(X) := \begin{bmatrix} A^*X + XA + \tilde{Q} + \Pi_1(X) & \tilde{S} + XB + \Pi_{12}(X) \\ [\tilde{S} + XB + \Pi_{12}(X)]^* & \tilde{R} + \Pi_2(X) \end{bmatrix}$$

with (4.9), just as $\Lambda(X)$ is associated with (4.1), it follows from (4.10) that $\Lambda(X) \geq \tilde{\Lambda}(X)$, and now an application of Corallary 2.6 yields the statement of the lemma. $\square$

Theorem 4.5 (Comparison theorem) *Let $\mathcal{I} \subset \mathbb{R}$ be some interval and $t_f \in \mathcal{I}$. Assume that X_2 and X_1 are on $\mathcal{I}$ solutions of $-\dot{X}_2 = \mathcal{R}(X_2)$ and $-\dot{X}_1 = \tilde{\mathcal{R}}(X_1)$, respectively. If*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix},$$

then $X_2(t_f) \geq X_1(t_f)$ implies that $X_2(t) \geq X_1(t)$ for $t \in \mathcal{I} \cap (-\infty, t_f]$.

PROOF. Define $X := X_2 - X_1$ and $F_i := F(X_i)$, $i = 1, 2$. According to Lemma 4.4 we have $X_1(t) \in D(\mathcal{R})$ for $t \in \mathcal{I}$ and using (4.8) we infer, that X is a solution of the differential equation

$$\begin{aligned} -\dot{X} &= \mathcal{R}(X_2) - \tilde{\mathcal{R}}(X_1) \\ &= \mathcal{R}(X_2) - \mathcal{R}(X_1) + \mathcal{R}(X_1) - \tilde{\mathcal{R}}(X_1) \\ &= \hat{A}^*(t)X + X\hat{A}(t) + \hat{Q}(t) + \hat{\Pi}(t, X), \end{aligned}$$

where

$$\hat{A}(t) := A + BF_2(t), \quad \hat{\Pi}(t, X) = \begin{bmatrix} I \\ F_2(t) \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F_2(t) \end{bmatrix}$$

and

$$\begin{aligned} \hat{Q}(t) :=& \mathcal{R}\big(X_1(t)\big) - \tilde{\mathcal{R}}\big(X_1(t)\big) \\ &+ \big[F_2(t) - F_1(t)\big]^* \big[R + \Pi_2\big(X_1(t)\big)\big] \big[F_2(t) - F_1(t)\big]. \end{aligned}$$

Now Lemma 4.4 implies that $\hat{Q}(t) \geq 0$ for all $t \in \mathcal{I}$. Therefore Theorem 2.1, (ii), in [33] yields the statement of the theorem. □

Theorem 4.5 generalizes the well-known comparison theorem for standard Riccati equations. It shows that the solutions of (1.4) depend monotonically on $\left(\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right)$ and on the terminal value X_f.

Subsequently we present two corollaries showing how the comparison theorem can be used to derive existence results and upper and lower bounds for the solutions of (1.4).

Corollary 4.6 *Let $\mathcal{I} \subset \mathbb{R}$ be some interval and $t_f \in \mathcal{I}$. If X_ℓ and X_u are on $\mathcal{I}$ hermitian solutions of $-\dot{X}_\ell \leq \mathcal{R}(X_\ell)$ and $-\dot{X}_u \geq \mathcal{R}(X_u)$, respectively, with*

$$\operatorname{Ker}\big[R + \Pi_2\big(X_\ell(t)\big)\big] \subseteq \operatorname{Ker} B \quad \textit{for} \quad t \in \mathcal{I} \cap (-\infty, t_f], \tag{4.12}$$

then $X_\ell(t_f) \le X_f \le X_u(t_f)$ *implies that the solution* X *of*

$$-\dot{X} = \mathcal{R}(X), \quad X(t_f) = X_f \tag{4.13}$$

exists on $\mathcal{I} \cap (-\infty, t_f]$ *and fulfills there the inequality*

$$X_\ell(t) \le X(t) \le X_u(t). \tag{4.14}$$

PROOF. By the hypotheses, there exists a hermitian matrix function $Q_\ell \ge 0$ such that $-\dot{X}_\ell = \tilde{R}(X_\ell)$ with $\tilde{Q} = Q - Q_\ell$, $\tilde{R} = R$ and $\tilde{S} = S$. Since $X_f \ge X_\ell(t_f)$ we obtain from Theorem 4.5 (which holds also in the time-varying case – see Remark 4.8) that $X(t) \ge X_\ell(t)$ for $t \in \mathcal{I} \cap (-\infty, t_f]$. Substituting Q_ℓ by $Q_u \le 0$ the right inequality in (4.14) follows analogously. Notice that X cannot blow up as long as (4.14) holds. Hence $X(t)$ exists for $t \in \mathcal{I} \cap (-\infty, t_f]$.

It remains to show that $X(t) \in D(\mathcal{R})$ for all $t \in \mathcal{I}$ with $t \le t_f$. Using Lemma 4.1 this results immediately from (4.12) and the fact that $X(t) \ge X_\ell(t)$ for all $t \in \mathcal{I} \cap (-\infty, t_f]$. □

Corollary 4.7 *Assume that* $\operatorname{Ker} R \subseteq \operatorname{Ker} B$ *and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \ge 0. \tag{4.15}$$

If $X_f \ge 0$ *then the solution* X *of* (4.13) *exists on* $(-\infty, t_f]$ *and fulfills there the inequality*

$$0 \le X(t) \le X_u(t)$$

where X_u *is the solution of the linear equation*

$$-\dot{X} = A^*X + XA + Q + \Pi_1(X), \quad X(t_f) = X_f. \tag{4.16}$$

PROOF. Since (4.16) is a linear differential equation its solution X_u exists on $(-\infty, t_f]$ whereas X exists a priori only on a certain interval $(t^-, t_f]$.

Define $D := X_u - X$. Then D satisfies the differential equation

$$-\dot{D} = A^*D + DA + Q_u + \Pi_1(D), \quad D(t_f) = 0,$$

where

$$Q_u = [S + XB + \Pi_{12}(X)][R + \Pi_2(X)]^+[S + XB + \Pi_{12}(X)]^* \ge 0.$$

From Theorem 2.1, (ii), in [33] we obtain now that $D(t) \ge 0$ for $t \in (t^-, t_f]$, i.e. $X_u(t) \ge X(t)$, for $t \in (t^-, t_f]$.

With $\tilde{Q} = SR^+S^*$, $\tilde{R} = R$ and $\tilde{S} = S$ the corresponding differential equation $-\dot{X} = \tilde{\mathcal{R}}(X)$ has the trivial solution. Since $\tilde{Q} \geq Q$ and $X(t_f) \geq 0$ it follows from Theorem 4.5 that $X(t) \geq 0$ for $t \in (t^-, t_f]$. Hence, X is bounded from below and above as long as it exists for $t \leq t_f$ and therefore it follows that $t^- = -\infty$.

Finally it follows from (4.15) that in particular $R \geq 0$ and $\operatorname{Ker} R \subseteq \operatorname{Ker} S$. Since $\operatorname{Ker} R \subseteq \operatorname{Ker} B$, Lemma 4.1 implies that $\mathcal{H}^n_+ \subseteq D(\mathcal{R})$. Consequently $X(t) \in D(\mathcal{R})$ for all $t \leq t_f$. □

Remark 4.8 *We mention that all the results obtained in Section 4 remain valid if the coefficients of* (1.4) *depend on t and the assumptions used are valid for all t.*

5 Existence of constant solutions

One way to find a solution of a nonlinear operator equation $\mathcal{T}(x) = 0$, where $\mathcal{T} \colon \mathcal{X} \to \mathcal{Y}$ is a Fréchet-differentiable operator between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is the so-called Newton-Kantorovich procedure. Its formulas are of the form

$$\mathcal{T}'_{x_i}(x_{i+1} - x_i) = -\mathcal{T}(x_i), \quad i = 0, 1, 2, \ldots,$$

where $\mathcal{T}'_{x_i}$ is the Fréchet derivative of $\mathcal{T}$ at x_i. To apply this method to the algebraic matrix equation $\mathcal{R}(X) = 0$ we first need the Fréchet derivative $\mathcal{R}'_X$. Since

$$\mathcal{R}'_X(H) = \lim_{t \to 0} \frac{\mathcal{R}(X + tH) - \mathcal{R}(X)}{t} \equiv \frac{d}{dt} \mathcal{R}(X + tH) \bigg|_{t=0},$$

provided $\mathcal{R}'_X$ exists, it is easy to obtain the following lemma (see [12]):

Lemma 5.1 *For $X \in \mathcal{H}^n$ with $\det[R + \Pi_2(X)] \neq 0$ and the corresponding feedback matrix $F = F(X)$ the Fréchet derivative $\mathcal{R}'_X \colon \mathcal{H}^n \to \mathcal{H}^n$ is given by*

$$\mathcal{R}'_X(H) = (A + BF)^* H + H(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(H) \begin{bmatrix} I \\ F \end{bmatrix}.$$

In particular

$$\mathcal{R}'_X(X) = \mathcal{R}(X) - \begin{bmatrix} I \\ F \end{bmatrix}^* T \begin{bmatrix} I \\ F \end{bmatrix}.$$

The following theorem generalizes the main result derived by Damm and Hinrichsen in [12] who applied the classical Newton-Kantorovich procedure and used stronger assumptions; for standard Riccati equations the result has been proved in [29]. Our proof follows the idea presented in [6] (see also [26]). We construct a sequence $\{X_i\}_{i=1}^{\infty}$ defined by

$$(A+BF_i)^*X_{i+1}+X_{i+1}(A+BF_i) + \begin{bmatrix} I \\ F_i \end{bmatrix}^* [T+\Pi(X_{i+1})] \begin{bmatrix} I \\ F_i \end{bmatrix} + \frac{1}{i+1} I = 0, \quad i=0,1,2,\ldots, \tag{5.1}$$

with

$$F_i := -[R+\Pi_2(X_i)]^+[S+X_iB+\Pi_{12}(X_i)]^*, \quad i=1,2,\ldots, \tag{5.2}$$

where X_1 is some adequate initial matrix. This algorithm differs from the Newton-Kantorovich procedure by the additional term $\frac{1}{i+1} I$ which is used for technical reasons; here the sequence $\left(\frac{1}{i+1}\right)$ could be replaced by any strictly decreasing sequence (a_i) with $a_i > 0$ and $a_i \to 0$ for $i \to \infty$.

Theorem 5.2 *Assume that* (A,B) *is c-stabilizable relative to* Π *and that there exists a matrix* $\hat{X} \in D(\mathcal{R})$ *with* $\operatorname{Ker}[R+\Pi_2(\hat{X})] \subseteq \operatorname{Ker} B$ *for which* $\mathcal{R}(X) \geq 0$. *Then there exists a solution* $X_+ \in D(\mathcal{R})$ *of* $\mathcal{R}(X)=0$ *such that* $X_+ \geq X$ *for every solution of* $\mathcal{R}(X) \geq 0$ *with* $\operatorname{Ker}[R+\Pi_2(X)] \subseteq \operatorname{Ker} B$. *Moreover, all the eigenvalues of*

$$A_+ := A - B[R+\Pi_2(X_+)]^+[S+X_+B+\Pi_{12}(X_+)]^* \tag{5.3}$$

lie in the closed left half-plane.

PROOF. By the hypotheses, there exists a matrix $\hat{X} \in D(\mathcal{R})$ with

$$\mathcal{R}(\hat{X}) = Q - \hat{Q} \tag{5.4}$$

where $\hat{Q}$ is a hermitian matrix such that $\hat{Q} \leq Q$.

Since (A,B) is c-stabilizable relative to Π, there is an F_0 such that $A_0 := A+BF_0$ is c-stable relative to $\left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]$. We are going to use the unique solution X_1 of the linearly perturbed Lyapunov equation

$$A_0^*X_1 + X_1A_0 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T+\Pi(X_1)] \begin{bmatrix} I \\ F_0 \end{bmatrix} + I = 0$$

as initial value for the sequence defined by (5.1). If we define $\hat{F} := F(\hat{X})$ then as in the proof of (4.7) we get

$$\mathcal{R}(\hat{X}) = A_0^*\hat{X} + \hat{X}A_0 - (F_0 - \hat{F})^*[R + \Pi_2(\hat{X})](F_0 - \hat{F}) + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T + \Pi(\hat{X})] \begin{bmatrix} I \\ F_0 \end{bmatrix}.$$

From this equation it follows together with (5.4) that

$$\begin{aligned} 0 &= A_0^* X_1 + X_1 A_0 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* [T + \Pi(X_1)] \begin{bmatrix} I \\ F_0 \end{bmatrix} + I - \mathcal{R}(\hat{X}) + Q - \hat{Q} \\ &= A_0^*(X_1 - \hat{X}) + (X_1 - \hat{X})A_0 + \begin{bmatrix} I \\ F_0 \end{bmatrix}^* \Pi(X_1 - \hat{X}) \begin{bmatrix} I \\ F_0 \end{bmatrix} + V, \end{aligned} \tag{5.5}$$

where

$$V = Q - \hat{Q} + (F_0 - \hat{F})^*[R + \Pi_2(\hat{X})](F_0 - \hat{F}) + I > 0.$$

Since A_0 is c-stable relative to $\left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_0 \end{smallmatrix}\right]$, Theorem 3.3 shows that $X_1 > \hat{X}$.

Starting with A_0, F_0, X_1, we construct recursively the three sequences of matrices $\{A_i\}_{i=0}^{\infty}$, $\{F_i\}_{i=0}^{\infty}$, $\{X_i\}_{i=1}^{\infty}$, where X_i and F_i are defined by (5.1) and (5.2), respectively, and where

$$A_i = A + BF_i, \quad i = 0, 1, \ldots.$$

We show by induction that for all $i = 1, 2, \ldots$

(a)$_i$ A_i is c-stable relative to $\left[\begin{smallmatrix} I \\ F_i \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_i \end{smallmatrix}\right]$,
(b)$_i$ $X_i > X_{i+1}$,
(c)$_i$ $X_{i+1} > \hat{X}$.

Notice that (b)$_i$ and (c)$_i$ imply (in view of Lemma 4.1) in particular that $X_i, X_{i+1} \in D(\mathcal{R})$.

The proof follows the scheme (c)$_{i-1}$ $\Rightarrow$ (a)$_i$ $\Rightarrow$ (b)$_i$ and (c)$_i$. Assume that (c)$_{m-1}$ holds. We show that (a)$_m$ is valid. Letting $X_1 := X_m$, $X_2 := X_{m-1}$ in (4.7) and applying (5.1), we get

$$\mathcal{R}(X_m) = -(F_m - F_{m-1})^*[R + \Pi_2(X_m)](F_m - F_{m-1}) - \frac{1}{m} I.$$

Together with (4.5) it follows that

$$\begin{aligned} &A_m^* X_m + X_m A_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* [T + \Pi(X_m)] \begin{bmatrix} I \\ F_m \end{bmatrix} \\ &\quad + (F_m - F_{m-1})^*[R + \Pi_2(X_m)](F_m - F_{m-1}) + \frac{1}{m} I = 0. \end{aligned} \tag{5.6}$$

Next, use (4.7) again with $X_1 := \hat{X}$, $X_2 := X_m$ and apply (5.4) to get

$$\begin{aligned} &A_m^* \hat{X} + \hat{X} A_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* [T + \Pi(\hat{X})] \begin{bmatrix} I \\ F_m \end{bmatrix} - Q + \hat{Q} \\ &- (F_m - \hat{F})^* [R + \Pi_2(\hat{X})](F_m - \hat{F}) = 0. \end{aligned}$$

Subtracting this from (5.6), we obtain

$$A_m^*(X_m - \hat{X}) + (X_m - \hat{X})A_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* \Pi(X_m - \hat{X}) \begin{bmatrix} I \\ F_m \end{bmatrix} + W = 0,$$

where

$$\begin{aligned} W = &Q - \hat{Q} + (F_m - F_{m-1})^* [R + \Pi_2(X_m)](F_m - F_{m-1}) \\ &+ (F_m - \hat{F})^* [R + \Pi_2(\hat{X})](F_m - \hat{F}) + \frac{1}{m} I \end{aligned}$$

is positive definite. Since $X_m > \hat{X}$ it follows from Theorem 3.3 that A_m is c-stable relative to $\left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]$; this proves (a)$_m$.

We now define X_{m+1} as the unique solution (necessarily hermitian) of the linearly perturbed Lyapunov equation

$$A_m^* X_{m+1} + X_{m+1} A_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* [T + \Pi(X_{m+1})] \begin{bmatrix} I \\ F_m \end{bmatrix} + \frac{1}{m+1} I = 0. \quad (5.7)$$

As in (5.5) it is found that

$$\begin{aligned} &A_m^*(X_{m+1} - \hat{X}) + (X_{m+1} - \hat{X})A_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* \Pi(X_{m+1} - \hat{X}) \begin{bmatrix} I \\ F_m \end{bmatrix} \\ &+ Q - \hat{Q} + (F_m - \hat{F})^* [R + \Pi_2(\hat{X})](F_m - \hat{F}) + \frac{1}{m+1} I = 0. \end{aligned} \quad (5.8)$$

Next it will be shown that $X_m > X_{m+1}$. Subtracting (5.7) from (5.6) we get

$$\begin{aligned} &A_m^*(X_m - X_{m+1}) + (X_m - X_{m+1})A_m + \begin{bmatrix} I \\ F_m \end{bmatrix}^* \Pi(X_m - X_{m+1}) \begin{bmatrix} I \\ F_m \end{bmatrix} \\ &+ (F_m - F_{m-1})^* [R + \Pi_2(X_m)](F_m - F_{m-1}) \\ &+ \left(\frac{1}{m} - \frac{1}{m+1} \right) I = 0. \end{aligned} \quad (5.9)$$

The last two equations, together with the fact, that A_m is c-stable relative to $\left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_m \end{smallmatrix}\right]$ imply (in view of Theorem 3.3) that $X_m > X_{m+1} > \hat{X}$; this proves (b)$_m$ and (c)$_m$.

We have obtained a nonincreasing sequence $\{X_i\}_{i=1}^{\infty}$ of hermitian matrices bounded below by $\hat{X}$. Hence

$$X_+ := \lim_{i\to\infty} X_i$$

exists and is a hermitian matrix with $X_+ > \hat{X}$. From Lemma 4.1 we obtain that $X_+ \in D(\mathcal{R})$. Passing to the limit in (5.1) when $i \to \infty$, it is found that

$$(A + BF_+)^* X_+ + X_+(A + BF_+) + \begin{bmatrix} I \\ F_+ \end{bmatrix}^* [T + \Pi(X_+)] \begin{bmatrix} I \\ F_+ \end{bmatrix} = 0,$$

which, in view of Lemma 4.2, can be rewritten as $\mathcal{R}(X_+) = 0$.

Finally, since A_i is c-stable relative to $\left[\begin{smallmatrix} I \\ F_i \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_i \end{smallmatrix}\right]$ for all $i \geq 0$, the eigenvalues of the matrix (5.3) lie in the closed left half-plane. $\square$

Corollary 5.3 *Assume that* $\operatorname{Ker} R \subseteq \operatorname{Ker} B$, (A, B) *is c-stabilizable relative to* Π *and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{5.10}$$

Then $\mathcal{R}(X) = 0$ *has a solution* $X_+ \geq 0$, *and all eigenvalues of the matrix* (5.3) *lie in the closed left half-plane.*

PROOF. According to Lemma 2.3, (i), condition (5.10) implies that $\hat{X} = 0$ is a solution of $\mathcal{R}(X) \geq 0$. Therefore an application of Theorem 5.2 yields the statement of the corollary. $\square$

Corollary 5.4 *Assume that* (A, B) *is c-stabilizable relative to* Π *and that there exists a matrix* $\hat{X} \in D(\mathcal{R})$ *with* $\operatorname{Ker}[R + \Pi_2(\hat{X})] \subseteq \operatorname{Ker} B$ *for which* $\mathcal{R}(X) > 0$. *Then there exists a solution* $X_+ \in D(\mathcal{R})$ *of* $\mathcal{R}(X) = 0$ *such that* $X_+ > X$ *for every solution of* $\mathcal{R}(X) > 0$ *with* $\operatorname{Ker}[R + \Pi_2(X)] \subseteq \operatorname{Ker} B$. *Moreover, all the eigenvalues of the matrix* (5.3) *lie in the open left half-plane.*

PROOF. Passing to the limit in (5.8) when $m \to \infty$ we obtain

$$\begin{aligned} &A_+^*(X_+ - \hat{X}) + (X_+ - \hat{X})A_+ + \begin{bmatrix} I \\ F_+ \end{bmatrix}^* \Pi(X_+ - \hat{X}) \begin{bmatrix} I \\ F_+ \end{bmatrix} \\ &= -\mathcal{R}(\hat{X}) - (F_+ - \hat{F})^*[R + \Pi_2(\hat{X})](F_+ - \hat{F}) < 0. \end{aligned}$$

Since $X_+ \geq \hat{X}$ it follows now from Lemma 3.7, (i), that A_+ is c-stable relative to $\left[\begin{smallmatrix} I \\ F_+ \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_+ \end{smallmatrix}\right]$ and that $X_+ > \hat{X}$ holds. $\square$

If X is a solution of $\mathcal{R}(X) = 0$ and if $F = F(X)$ denotes the corresponding feedback matrix then X is called *stabilizing*, if $A + BF$ is c-stable relative to $\left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]$.

Lemma 5.5 *If $\mathcal{R}(X) = 0$ has a stabilizing solution X_s, then $X_s \geq X$ for every solution X of $\mathcal{R}(X) \geq 0$. In particular, X_s is the (unique) maximal solution of $\mathcal{R}(X) = 0$.*

PROOF. Let X_s be a stabilizing solution of $\mathcal{R}(X) = 0$ and denote the corresponding feedback matrix by $F_s = F(X_s)$. For every $\hat{X}$ with $\mathcal{R}(\hat{X}) \geq 0$ there is a matrix $\hat{Q} \leq Q$ such that $\mathcal{R}(\hat{X}) = Q - \hat{Q}$. If $\hat{F} = F(\hat{X})$, then an application of (4.8) with $X_1 := \hat{X}$ and $X_2 := X_s$ yields the equation

$$
\begin{aligned}
&(A + BF_s)^*(X_s - \hat{X}) + (X_s - \hat{X})(A + BF_s) + Q - \hat{Q} \\
&\;+ (F_s - \hat{F})^*[R + \Pi_2(\hat{X})](F_s - \hat{F}) \\
&\;+ \begin{bmatrix} I \\ F_s \end{bmatrix}^* \Pi(X_s - \hat{X}) \begin{bmatrix} I \\ F_s \end{bmatrix} = 0.
\end{aligned}
$$

Hence, the difference $X_s - \hat{X}$ fulfills a linearly perturbed Lyapunov equation where $Q \geq \hat{Q}$, $R + \Pi_2(\hat{X}) \geq 0$ and $A + BF_s$ is c-stable relative to $\left[\begin{smallmatrix} I \\ F_s \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ F_s \end{smallmatrix}\right]$. Applying Theorem 3.3 we obtain $X_s \geq \hat{X}$. □

Lemma 5.6 *Assume that the following hypotheses hold:*

(i) $R > 0$, $\left(\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right) \geq 0$,

(ii) $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ *is c-detectable relative to* $\left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]$.

Then every positive semidefinite solution of $\mathcal{R}(X) = 0$ is stabilizing.

PROOF. Let $X \geq 0$ be a solution of $\mathcal{R}(X) = 0$ and denote by $F = F(X)$ the corresponding feedback matrix. From Lemma 4.2 we know that X is also a solution of the linearly perturbed Lyapunov equation

$$
(A + BF)^*X + X(A + BF) + \hat{Q} + \hat{\Pi}(X) = 0 \tag{5.11}
$$

with

$$
\begin{aligned}
\hat{Q} &= \begin{bmatrix} I \\ F \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \\
&= \begin{bmatrix} I \\ F \end{bmatrix}^* \begin{bmatrix} I & SR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - SR^{-1}S^* & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}S^* & I \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \\
&= Q - SR^{-1}S^* + \left(F + R^{-1}S^*\right)^* R \left(F + R^{-1}S^*\right)
\end{aligned} \tag{5.12}
$$

and

$$
\hat{\Pi}(X) = \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(X) \begin{bmatrix} I \\ F \end{bmatrix}.
$$

Notice that $\hat{Q} \geq 0$. We assume now that $A + BF$ is not c-stable relative to $\hat{\Pi}$. Then it follows from Theorem 3.3 and the Krein-Rutman-Theorem for resolvent positive operators that there is a matrix $V \in \mathcal{H}^n_+ \setminus \{0\}$ and some $\lambda \geq 0$ such that

$$
\left(\mathcal{L}_{A+BF} + \hat{\Pi}\right)^{adj} (V) = \lambda V. \tag{5.13}
$$

Lemma 3.1, (iii), shows now that

$$
0 \leq \langle V, \hat{Q} \rangle = -\langle V, (\mathcal{L}_{A+BF} - \hat{\Pi})(X) \rangle = -\lambda \langle V, X \rangle \leq 0
$$

and we conclude that $\langle V, \hat{Q} \rangle = 0$. From $Q \geq SR^{-1}S^*$, $R > 0$ and (5.12) it follows that

$$
\left\langle V, Q - SR^{-1}S^* \right\rangle = \left\langle V, \left(F + R^{-1}S^*\right)^* R \left(F + R^{-1}S^*\right) \right\rangle = 0, \tag{5.14}
$$

and this equations implies

$$
\left(Q - SR^{-1}S^*\right) V = 0 \tag{5.15}
$$

and

$$
\left(F + R^{-1}S^*\right) V \left(F + R^{-1}S^*\right)^* = 0. \tag{5.16}
$$

Since V has a positive semidefinite square root, we obtain $FV = -R^{-1}S^*V$. Using Lemma 3.1, (i), it is easy to see that for all $Y \in \mathcal{H}^n$ we have

$$
\left\langle V, \mathcal{L}_{A+BF}(Y) + \hat{\Pi}(Y) \right\rangle = \left\langle V, \mathcal{L}_{A-BR^{-1}S^*}(Y) + \check{\Pi}(Y) \right\rangle
$$

with

$$
\check{\Pi}(Y) = \begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}^* \Pi(Y) \begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}.
$$

From (5.13) it follows finally that

$$
\left(\mathcal{L}_{A-BR^{-1}S^*} + \check{\Pi}\right)^{adj} (V) = \lambda V,
$$

and together with (5.15) this contradicts the presupposed detectability. Hence $A + BF$ is c-stable relative to $\hat{\Pi}$. $\square$

Corollary 5.7 *Assume that*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$$

and that $(Q - SR^+S^*, A - BR^+S^*)$ *is c-detectable relative to* Π_1*. Then every positive semidefinite solution of*

$$A^*X + XA + Q + \Pi_1(X) - (S + XB)R^+(S + XB)^* = 0$$

is stabilizing.

PROOF. In this special case we have

$$F = -R^+(S + XB)^*.$$

Hence it is trivial that $\operatorname{Ker} R^+ \subseteq \operatorname{Ker} F^*$ which according to Lemma 2.2 is equivalent to $R^+RF = F$.

We precede as in the proof of Lemma 5.6. Since $R \geq 0$ we obtain from (5.14) and Lemma 3.1, (iii) that

$$V(F + R^+S^*)^*R(F + R^+S^*)V = 0.$$

Using $R \geq 0$ we get

$$R(F + R^+S^*)V = 0,$$

and multiplication from the left with R^+ yields now that $FV = -R^+S^*V$. Continuing as in the proof of Lemma 5.6 the statement of the corollary follows. $\square$

The assertion of Lemma 5.6 can be sharpened if we replace the detectability assumption by $Q > SR^+S^*$. Instead of $R > 0$ it is then sufficient to assume that $R \geq 0$ with $\operatorname{Ker} R \subseteq \operatorname{Ker} S$.

Lemma 5.8 *Assume that*

$$R \geq 0, \quad \operatorname{Ker} R \subseteq \operatorname{Ker} S \quad \textit{and} \quad Q > SR^+S^*.$$

If $X \geq 0$ *is a solution of* $\mathcal{R}(X) = 0$*, then* X *is stabilizing and positive definite.*

PROOF. Using the notation from the proof of Lemma 5.6 we obtain $\hat{Q} > 0$ (this can be easily derived from formula (5.12) where R^{-1} has to be replaced by R^{+}). Applying Lemma 3.7, (i), to equation (5.11) we then obtain the statement of the lemma. □

From Corollary 5.3, Lemma 5.5 and Lemma 5.6 we infer:

Theorem 5.9 *Assume that* $R > 0$*,*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0,$$

(A, B) *is c-stabilizable relative to* Π *and* $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ *is c-detectable relative to* $\begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}^* \Pi \begin{bmatrix} I \\ -R^{-1}S^* \end{bmatrix}$*. Then* $\mathcal{R}(X) = 0$ *has a unique positive semidefinite solution* X_+*. Moreover,* X_+ *is stabilizing and maximal among all solutions of* $\mathcal{R}(X) = 0$*.*

6 Convergence theorems

In this section we show that the same idea that is known from the theory of standard Riccati differential equations and that has been used by Abou-Kandil, Freiling and Jank in order to prove convergence for linearly perturbed Riccati differential equations ([1], Theorem 1; [20], Section 3; [21], Section 2, see also [17], Theorem 5.1) can be used in order to derive monotonicity and convergence results for the solutions of (1.4) with $X(t_f) \geq 0$.

Lemma 6.1 *Let* $\mathcal{I} \subset \mathbb{R}$ *be some interval and* $t_f \in \mathcal{I}$*. If* X *is on* $\mathcal{I}$ *a solution of* $-\dot{X} = \mathcal{R}(X)$ *such that* $R + \Pi_2\big(X(t)\big)$ *is positive definite for* $t \in \mathcal{I}$ *then* $\dot{X}(t_f) \geq 0$ *(*$\dot{X}(t_f) \leq 0$*) implies* $\dot{X}(t) \geq 0$ *(*$\dot{X}(t) \leq 0$*) for* $t \in \mathcal{I} \cap (-\infty, t_f]$*.*

PROOF. Differentiating $-\dot{X} = \mathcal{R}(X)$, we obtain

$$\begin{aligned} -\ddot{X} &= A^*\dot{X} + \dot{X}A + \Pi_1(\dot{X}) + [\dot{X}B + \Pi_{12}(\dot{X})]F \\ &\qquad + F^*\Pi_2(\dot{X})F + F^*[\dot{X}B + \Pi_{12}(\dot{X})]^* \\ &= (A + BF)^*\dot{X} + \dot{X}(A + BF) + \begin{bmatrix} I \\ F \end{bmatrix}^* \Pi(\dot{X}) \begin{bmatrix} I \\ F \end{bmatrix}. \end{aligned}$$

The statement of the lemma follows now from Theorem 2.1, (ii), in [33]. □

Lemma 6.2 *Assume that $R > 0$,*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$$

and that X_ℓ is a solution of the differential equation $-\dot{X} = \mathcal{R}(X)$ with $X_\ell(0) = 0$. If the algebraic equation $\mathcal{R}(X) = 0$ has no positive semidefinite solution then $\lim_{t\to-\infty} \|X_\ell(t)\| = \infty$. Otherwise $X_{\min} = \lim_{t\to-\infty} X_\ell(t)$ exists and is the minimal positive semidefinite solution of $\mathcal{R}(X) = 0$.

PROOF. From Corollary 4.7 we know that $X_\ell(t) \geq 0$ exists for all $t \leq 0$. In particular, it follows that $R + \Pi_2\big(X_\ell(t)\big) > 0$ for all $t \leq 0$, and since $\dot{X}_\ell(0) = SR^{-1}S^* - Q \leq 0$, it follows from Lemma 6.1 that $\dot{X}_\ell(t) \leq 0$ for all $t \leq 0$. So if $\lim_{t\to-\infty} \|X_\ell(t)\| < \infty$ then the monotonicity of $X_\ell(t)$ implies that $X_{\min} = \lim_{t\to-\infty} X_\ell(t)$ exists. Obviously $X_{\min}$ is positive semidefinite and a solution of $\mathcal{R}(X) = 0$. If $X \geq 0 = X_\ell(0)$ is another solution of $\mathcal{R}(X) = 0$, it follows from Theorem 4.5 that $X_\ell(t) \leq X$ for all $t \leq 0$. Passing to $t \to -\infty$ we obtain $X_{\min} \leq X$. □

Theorem 6.3 *Assume that the following hypotheses hold:*

(i) $R > 0$, $\left(\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right) \geq 0$,
(ii) (A, B) *is c-stabilizable relative to* Π,
(iii) $(Q - SR^{-1}S^*, A - BR^{-1}S^*)$ *is c-detectable relative to* $\left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]^* \Pi \left[\begin{smallmatrix} I \\ -R^{-1}S^* \end{smallmatrix}\right]$.

If X is a solution of $-\dot{X} = \mathcal{R}(X)$ with $X(0) \geq 0$, then $\lim_{t\to-\infty} X(t) = X_+$.

PROOF. We choose a matrix $\tilde{Q} \in \mathcal{H}^n$ such that

$$\tilde{Q} \geq Q \quad \text{and} \quad \tilde{Q} > SR^{-1}S^*.$$

Then it follows from Corollary 5.3 that the algebraic equation

$$\begin{aligned} &A^*X + XA + \tilde{Q} + \Pi_1(X) - [S + XB + \Pi_{12}(X)] \\ &\quad \times [R + \Pi_2(X)]^+ [S + XB + \Pi_{12}(X)]^* \end{aligned} \tag{6.1}$$

has a solution $\tilde{X}_+ \geq 0$ which in view of Lemma 5.8 is stabilizing and positive definite. Now there exists some $\lambda > 1$ such that $X(0) \leq \lambda \tilde{X}_+$. We consider the solution X_u of $-\dot{X} = \mathcal{R}(X)$ with $X_u(0) = \lambda \tilde{X}_+$. If X_ℓ denotes the solution of the same differential equation with $X_\ell(0) = 0$, then from Theorem 4.5 we obtain the inequality

$$X_\ell(t) \le X(t) \le X_u(t) \quad \text{for all} \quad t \le 0. \tag{6.2}$$

Under the given hypotheses the equation $\mathcal{R}(X) = 0$ has a unique positive semidefinite solution X_+. Because of Lemma 6.2 it follows that $X_\ell(t) \to X_+$ (monotonically) for $t \to -\infty$. We show now that X_u is monotonically decreasing as t is decreasing. If we multiply (6.1) with λ and substitute X by $\tilde{X}_+$, then (having in mind that $\tilde{Q} \ge Q$) we obtain an inequality of the form $\tilde{\mathcal{R}}(\lambda\tilde{X}_+) \le 0$, where $\tilde{\mathcal{R}}$ is a rational matrix operator with $\tilde{T} = \lambda T$. On the other hand we have $\dot{X}_u(0) = -\mathcal{R}\Big(X_u(0)\Big) = -\mathcal{R}(\lambda\tilde{X}_+)$. Since $\tilde{T} > T$ it follows with Lemma 4.4 that

$$\dot{X}_u(0) \ge \tilde{\mathcal{R}}(\lambda\tilde{X}_+) - \mathcal{R}(\lambda\tilde{X}_+) \ge 0.$$

Using again Lemma 6.1 this proves that X_u is monotonically increasing with $X_u(t) \ge X_\ell(t) \ge 0$ for all $t \le 0$. Hence $\lim\limits_{t\to-\infty} X_u(t) = X_+$ since X_+ is the unique positive semidefinite solution. Together with (6.2) this proves the assertion of the theorem. □

References

[1] Abou-Kandil, H.; Freiling, G.; Jank, G.: Solution and asymptotic behavior of coupled Riccati equations in jump linear systems, *IEEE Trans. Automat. Control* **39** (1994), 1631–1636.

[2] Ahlbrandt, C. D.; Peterson, A. C.: Discrete Hamiltonian systems, *Kluwer Academic Publishers Group, Dordrecht*, 1996.

[3] Ait Rami, M.; Chen, X.; Moore, J. B.; Zhou, X. Y.: Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls, *IEEE Trans. Automat. Control* **46** (2001), 428–440.

[4] Ait Rami, M.; Zhou, X. Y.: Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls, *IEEE Trans. Automat. Control* **45** (2000), 1131–1143.

[5] Albert, A.: Conditions for positive and nonnegative definiteness in terms of pseudoinverses, *SIAM J. Appl. Math.* **17** (1969), 434–440.

[6] Ando, T.: Matrix quadratic equations, *Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo*, 1988.

[7] Arendt, W.; Batty, C. J. K.; Hieber, M.; Neubrander, F.: Vector-valued Laplace Transforms and Cauchy Problems, *Birkhäuser Verlag, Basel*, 2001.

[8] Chen, S.; Li, X.; Zhou, X. Y.: Stochastic linear quadratic regulators with indefinite control weight costs, *SIAM J. Control Optim.* **36** (1998), 1685–1702.

[9] Chen, S.; Yong, J.: Stochastic linear quadratic optimal control problems, *Appl. Math. Optim.* **43** (2001), 21–45.

[10] Chen, S.; Zhou, X. Y.: Stochastic linear quadratic regulators with indefinite control weight costs II, *SIAM J. Control Optim.* **39** (2000), 1065–1081.

[11] Clements, D. J.; Wimmer, H. K.: Monotonicity of the optimal cost in the discrete-time regulator problem and Schur complements, *Automatica J. IFAC* **37** (2001), 1779–1786.

[12] Damm, T.; Hinrichsen, D.: Newton's method for a rational matrix equation occuring in stochastic control, *Linear Algebra Appl.* **332/334** (2001), 81–109.

[13] Dragan, V.; Morozan, T.: Global solutions to a game-theoretic Riccati equation of stochastic control, *J. Differential Equations* **138** (1997), 328–350.

[14] Dragan, V.; Morozan, T.: Stability and robust stabilization to linear stochastic systems described by differential equations with markovian jumping and multiplicative white noise, *Preprint No. 17/1999, Institutul de Matematica al academiei Romane* (1999).

[15] Dragan, V.; Morozan, T.: Systems of matrix rational differential equations arising in connection with linear stochastic systems with markovian jumping, *Preprint No. 9/2000, Institutul de Matematica al academiei Romane* (2000).

[16] Feng, X.; Loparo, K. A.; Ji, Y.; Chizeck, H. J.: Stochastic stability properties of jump linear systems, *IEEE Trans. Automat. Control* **37** (1992), 38–53.

[17] Fragoso, M. D.; Costa, O. L. V.; de Souza, C. E.: A new approach to linearly perturbed Riccati equations arising in stochastic control, *Appl. Math. Optim.* **37** (1998), 99–126.

[18] G. Freiling, A. Hochhaus, Basic properties of a class of rational matrix differential equations, *Proceedings of the European Control Conference (ECC 2001), Porto, 4–7 September, 2001.*

[19] Freiling, G.; Hochhaus, A.: Properties of the solutions of rational matrix difference equations. Advances in difference equations, IV, *Comput. Math. Appl.*, to appear.

[20] Freiling, G.; Jank, G.: Existence and comparison theorems for algebraic Riccati equations and Riccati differential and difference equations, *J. Dynam. Control Systems* **2** (1996), 529–547.

[21] Freiling, G.; Jank, G.; Abou-Kandil, H.: Generalized Riccati difference and differential equations, *Linear Algebra Appl.* **241/243** (1996), 291–303.

[22] Harville, D. A.: Matrix algebra from a statistician's perspective, *Springer-Verlag, New York*, 1997.

[23] Hinrichsen, D.; Pritchard, A. J.: Stochastic H^∞, *SIAM J. Control Optim.* **36** (1998), 1504–1538.

[24] Knobloch, H.-W.; Kwakernaak, H.: Lineare Kontrolltheorie, *Springer-Verlag, Berlin*, 1985.

[25] Kolmanovskii, V. B.; Richard, J.-P.: Some Riccati equations in the stability study of dynamical systems with delays, *Neural Parallel Sci. Comput.* **7** (1999), 235–252.

[26] Kučera, V.: Algebraic Riccati equation: Hermitian and definite solutions. The Riccati equation, 53–88, Comm. Control Engrg. Ser., Springer, Berlin, 1991

[27] Lancaster, P.; Rodman, L.: Algebraic Riccati equations, *Clarendon Press, Oxford*, 1995.

[28] Lancaster, P.; Tismenetsky, M.: The theory of matrices, *Academic Press Inc., Orlando*, 1985.

[29] Ran, A. C. M.; Vreugdenhil, R.: Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems, *Linear Algebra Appl.* **99** (1988), 63–83.

[30] Reid, W. T.: Riccati differential equations, *Academic Press, New York-London*, 1972.

[31] Verriest, E. I.: Robust stability and stabilization: from linear to nonlinear, *Proc. IFAC Workshop, Ancona, 2000*, 184–195.

[32] Voigt, J.: On resolvent positive operators and positive C_0-semigroups on AL-spaces, *Semigroup Forum* **38** (1989), 263–266.

[33] Wonham, W. M.: On a matrix Riccati equation of stochastic control, *SIAM J. Control* **6** (1968), 681–697.

[34] Yong, J.; Zhou, X. Y.: Stochastic controls, *Springer-Verlag, New York*, 1999.

[35] Zeidler, E.: Nonlinear functional analysis and its applications I. Fixed-point theorems, *Springer-Verlag, New York-Berlin*, 1986.