Proceedings of Dynamic Systems and Applications 4 (2004) 1- xxx

ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF PARAMETER-DEPENDENT MATRIX RICCATI DIFFERENTIAL EQUATIONS WITH RESPECT TO THE PARAMETER

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ABSTRACT. Starting from well known asymptotic estimates for the solutions of linear systems of differential equations $\dot{X} = B(t,\lambda)X$ with respect to $\lambda \to \infty$ we use the fundamental representation formula for the solutions of Riccati differential equations proved in [5] in order to determine the leading terms in the asymptotics of the solutions of parameter-dependent matrix Riccati differential equations.

AMS (MOS) Subject Classification. 34B20, 34E05, 34L25.

1. INTRODUCTION

Let $I := [a,b]$ with $a \le 0 \le b$ and $S \subset \mathbb{C}$ be given. We consider matrix Riccati differential equations for an unknown $m \times n$ matrix function $W \colon I \times S \to \mathbb{C}$ of the form

$$\dot{W} = B_{21}(t,\lambda) + B_{22}(t,\lambda)W - WB_{11}(t,\lambda) - WB_{12}(t,\lambda)W, \tag{1}$$

where the coefficient $B = \left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right)$ depends on $t \in I$ and on the parameter $\lambda \in S$.

According to Radon's Lemma (see [1, Section 3]) equation (1) is in a restricted sense equivalent to the associated linear system

$$\dot{X} = B(t,\lambda)X, \quad t \in I,\, \lambda \in S. \tag{2}$$

It is the main purpose of this note to show how asymptotic estimates for the solution of the linear system (2) with respect to the parameter λ can be used for the analysis of the asymptotic behavior of the solutions of the matrix Riccati differential equation (1) for $\lambda \to \infty$. This is achieved by combining the estimates derived for the solutions of (2) via Radon's Lemma with a representation formula for the solutions of (1) that has been derived in [5] in a slightly different setting and which is adapted here to the situation under consideration.

We present examples that indicate how the general results derived in this paper can be used to derive the Weyl-Titchmarsh M-function asymptotics for matrix valued Schrödinger operators, for Dirac-type operators and many others operators of similar type.

2. ASYMPTOTIC ESTIMATES FOR THE SOLUTIONS OF FIRST ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Throughout the last century various results on the asymptotic behavior of the solutions of linear differential systems of the form (2) for $\lambda \to \infty$ and also (if I is unbounded) for $x \to \pm\infty$ were derived since such estimates served as a basic tool in the study of boundary eigenvalue problems associated with such differential equations – compare for example [4], where a very general class of boundary eigenvalue problems has been studied using the asymptotic estimates for the solutions of (2) derived by Trjitzinsky [8] and Turrittin [9, 10] (see also Tretter [7], where the work of [4] has been continued).

In principle by the method presented below we can derive asymptotic estimates for any type of matrix Riccati differential equations (1), for which we can derive adequate asymptotic estimates for the solutions of the associated linear system (2). For sake of definiteness and for convenience we confine below to the situation which is based on the asymptotic estimates for the solutions of (2) derived in the recent textbook of Mennicken and Möller [6] and we use the same notation (see also [7]). Alternatively we could start from other estimates for the solutions of (2) – for example from [10] or [8] – and obtain similar results under different assumptions on the coefficients of (2).

Assumption 2.1. *For $\lambda = \rho^d$ with some $d \in \mathbb{N}$ there exists a matrix function $S(\cdot,\rho) \in M_{m+n}(W^1_\infty(a,b))$ which is a polynomial of degree $q \in \mathbb{N}_0$ in ρ and which is invertible in $M_{m+n}(W^1_\infty(a,b))$ for sufficiently large ρ with $S(\cdot,\rho)^{-1} = O(\rho^{\hat{q}})$ with respect to the norm in $M_{m+n}(L_\infty(a,b))$ for some $\hat{q} \in \mathbb{Z}$ such that*

$$S^{-1}(\cdot,\rho)B(\cdot,\rho^d)S(\cdot,\rho) = A(\cdot,\rho), \tag{3}$$

where for some $k \in \mathbb{N}_0$ and $\gamma > 0$,

$$A(\cdot,\rho) = \rho A_1 + A_0 + \sum_{j=1}^{k} \rho^{-j} A_{-j} + \rho^{-k-1} A_{-k-1}(\cdot,\rho), \qquad |\rho| > \gamma, \tag{4}$$

with the following properties:

i) $A_1 \in M_{m+n}(W^k_\infty(a,b))$ *and* $A_{-j} \in M_{m+n}(W^{k-j}_\infty(a,b))$, $j = 0, 1, \dots, k$;

ii) $A_{-k-1}(\cdot,\rho) \in M_{m+n}(L_\infty(a,b))$ *for* $|\rho| > \gamma$ *and is bounded in* $M_{m+n}(L_\infty(a,b))$ *as* $\rho \to \infty$;

iii) $$A_1 = \begin{pmatrix} r_0 I_{n_0} & & & 0 \\ & r_1 I_{n_1} & & \\ & & \ddots & \\ 0 & & & r_l I_{n_l} \end{pmatrix},$$

where I_{n_ν} denotes the n_ν-dimensional unit matrix, $\sum_{\nu=0}^{l} n_\nu = p$, and r_ν are functions such that $r_\nu = 0$ for some $\nu \in \{0, 1, \dots, l\}$, without loss of generality $r_0 = 0$ (where we allow n_0 to be 0, whereas $n_\nu > 0$ for $\nu > 0$),

$$(r_\nu - r_\mu)^{-1} \in L_\infty(a,b), \qquad \nu,\mu = 0,1,\dots,l,\ \nu \neq \mu,$$

$$r_\nu(t) - r_\mu(t) = |r_\nu(t) - r_\mu(t)| e^{i\varphi_{\nu\mu}}, \qquad \nu,\mu = 0,1,\dots,l,\ x \in [a,b],$$

with some constants $\varphi_{\nu\mu} \in \mathbb{R}$.

Theorem 2.2. *Let $k \in \mathbb{N}_0$, $\gamma > 0$ and let $A(\cdot,\rho)$, $\rho \in \mathbb{C}$, be such that (4) holds and the assumptions i), ii) and iii) are fulfilled. For $x \in [a,b]$ and $\rho \in \mathbb{C}$ we set*

$$E_\nu(t,\rho) := \exp\left(\rho \int_0^t r_\nu(\xi)\, d\xi\right) I_{n_\nu}, \quad \nu = 0, 1, \dots, l,$$

and

$$E(t,\rho) := \operatorname{diag}(E_0(t,\rho), E_1(t,\rho), \dots, E_l(t,\rho)).$$

Then there exists a fundamental matrix $Y(\cdot,\rho)$ of

$$y' = A(\cdot,\rho)y$$

such that for $|\rho| > \gamma$

$$Y(\cdot,\rho) = \left(\sum_{r=0}^{k} \frac{1}{\rho^r} P^{[r]} + \frac{1}{\rho^k} R_k(\cdot,\rho)\right) E(\cdot,\rho) \tag{5}$$

where $R_k(\cdot,\rho) \in M_{m+n}(W^1_\infty(a,b))$, $|\rho| > \gamma$, $R_k(\cdot,\rho) = o(1)$, $\frac{1}{\rho}R_k'(\cdot,\rho) = o(1)$ with respect to the norm in $M_{m+n}(L_\infty(a,b))$, and where $P^{[r]} \in M_{m+n}(W^{k+1-r}_\infty(a,b))$ are determined by

$$\begin{aligned}
&P^{[0]}A_1 - A_1P^{[0]} = 0, \quad P^{[0]}(0) = I_{m+n},\\
&P^{[r]\prime} - \sum_{j=0}^{r} A_{-j}P^{[r-j]} + P^{[r+1]}A_1 - A_1P^{[r+1]} = 0, \quad r = 0,1,\dots,k-1,\\
&P^{[k]\prime}_{\nu\nu} - A_{0,\nu\nu}P^{[k]}_{\nu\nu} = \sum_{\substack{q=0\\ q\neq\nu}}^{l} A_{0,\nu q}P^{[k]}_{q\nu} + \sum_{j=1}^{k}\sum_{q=0}^{l} A_{-j,\nu q}P^{[k-j]}_{q\nu}, \quad \nu = 0,1,\dots,l,
\end{aligned} \tag{6}$$

with

$$A_j =: (A_{j,\nu\mu})^l_{\nu,\mu=0}, \qquad P^{[r]} =: \left(P^{[r]}_{\nu\mu}\right)^l_{\nu,\mu=0}$$

defined according to the block structure of A_1. Moreover, the leading coefficient $P^{[0]}$ is invertible in $M_{m+n}(W^{k+1}_\infty(a,b))$.

Remark 2.3. (i) *Since A_1 is diagonal, the matrix $P^{[0]}$ is block-diagonal with*

$$\begin{aligned}
&P^{[0]}_{\nu\mu} = 0, \quad \nu,\mu = 0,1,\dots,l,\ \nu \neq \mu,\\
&P^{[0]\prime}_{\nu\nu} - A_{0,\nu\nu}P^{[0]}_{\nu\nu} = 0, \quad P^{[0]}_{\nu\nu}(0) = I_{n_\nu}, \quad \nu = 0,1,\dots,l.
\end{aligned}$$

In particular, $P^{[0]} = I_{m+n}$ if $A_0 = 0$.

(ii) *If $Y(\cdot,\rho)$ denotes the fundamental matrix estimated in Theorem 2.2 then*

$$X(\cdot,\rho) := S(\cdot,\rho)Y(\cdot,\rho) =: (x_1(\cdot,\rho),\dots,x_{m+n}(\cdot,\rho))\,E(t,\rho) \tag{7}$$

is a fundamental matrix of (2) with

$$x_j(t,\rho) =: \sum_{r=0}^{k} \frac{1}{\rho^r}\xi_{jr}(t) + o\left(\frac{1}{\rho^k}\right) \tag{8}$$

and

$$E(t,\rho) = \operatorname{diag}(E_0(t,\rho),E_1(t,\rho),\dots,E_l(t,\rho)) = \operatorname{diag}\left(e^{\rho\mu_1(t)},\dots,e^{\rho\mu_{m+n}(t)}\right). \tag{9}$$

3. REPRESENTATION OF THE SOLUTIONS OF (1)

Below we use the following abbreviations for vectors associated with

$$x_\nu = (x_{\nu,1},\dots,x_{\nu,n+m})^T, \quad 1 \le \nu \le n+m,$$

where we suppress the arguments (t,ρ):

$$\tilde{x}_\nu := (x_{\nu,1},\dots,x_{\nu,n})^T$$

and for $1 \le i \le m$, $1 \le j \le n$

$$x_\nu(i,j) := (x_{\nu,1}, \ldots, x_{\nu,j-1}, x_{\nu,n+i}, x_{\nu,j+1}, \ldots, x_{\nu,n})^T.$$

Let $W_0(\rho) \in \mathbb{C}^{m\times n}$ for $|\rho| > \gamma$ be given and denote by $X(\cdot,\rho)$ the fundamental matrix considered in Remark 2.3. Then $X(\cdot,\rho)$ is invertible for $|\rho| > \gamma$; hence the solution $C(\rho) = (c_1(\rho), \ldots, c_{m+n}(\rho))$ of the linear equation

$$X(0,\rho)C(\rho) = \begin{pmatrix} I_n \\ W_0(\rho) \end{pmatrix} \tag{10}$$

is well defined. If we set

$$\begin{pmatrix} P(t,\rho) \\ Q(t,\rho) \end{pmatrix} := X(t,\rho)C(\rho), \tag{11}$$

then it follows from Radon's Lemma (see [1, Theorem 3.1.1]) and can be easily verified by differentiation that

$$W(t,\rho) := Q(t,\rho)P(t,\rho)^{-1}, \quad t \ge 0,\ |\rho| > \gamma, \tag{12}$$

defines the solution of the matrix Riccati differential equation (1) with $W(0,\rho) = W_0(\rho)$ while $P(t,\rho)$ is invertible. Using the preceding results and notations we obtain analogously to the proof of [5, Theorem 1] (see also the remark following this theorem and [1, Chapter 3]):

Theorem 3.1. *Let $W(\cdot,\rho) = (w_{ij}(\cdot,\rho))_{\substack{1\le i\le m\\ 1\le j\le n}}$ be the solution (defined by* (10) *–* (12)*) of* (1) *with $W(0,\rho) = W_0(\rho)$ and denote by $J(\rho)$ its maximal interval of existence around $t = 0$. Then for $|\rho| > \gamma$ and $t \in J(\rho)$ we have for $W(\cdot,\rho)$ the representation*

$$w_{ij}(t,\rho) = \tag{13}$$

$$\frac{\displaystyle\sum_{1\le k_1<\ldots<k_n\le m+n} e^{(\mu_{k_1}(t)+\ldots+\mu_{k_n}(t))\rho} |x_{k_1}(i,j), \ldots, x_{k_n}(i,j)|(t,\rho) \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}{\displaystyle\sum_{1\le k_1<\ldots<k_n\le m+n} e^{(\mu_{k_1}(t)+\ldots+\mu_{k_n}(t))\rho} |\widetilde{x}_{k_1}, \ldots, \widetilde{x}_{k_n}|(t,\rho) \begin{vmatrix} c_{k_1} \\ \vdots \\ c_{k_n} \end{vmatrix}}.$$

Formula (13) nicely displays the asymptotic behavior of the solutions of (1) on t, ρ and also on the initial data (which determine C via (11)). In [5] and [1, Chapter 3] the authors have analyzed the asymptotic dependence of the solutions of (1) (for constant ρ) with respect to the (real or complex) argument t – below we shall discuss their dependence on ρ for $t \in [a,b]$.

4. ASYMPTOTIC BEHAVIOR OF $W(t,\rho)$ for $\rho \to \infty$

¿From (13) it is obvious that the limits

$$\lim_{r\to\infty} w_{ij}(t, re^{i\varphi})$$

exist, provided one of the exponential terms $e^{(\mu_{k_1}(t)+\ldots+\mu_{k_n}(t))\rho}$ dominates all others for fixed φ and $\rho = re^{i\varphi} \to \infty$. More precisely we have the following result:

Theorem 4.1. *Assume that there exist $a_1, b_1 \in I$, $\delta > 0$, $\varphi \in [0, 2\pi[$ and integers κ_j with $1 \le \kappa_1 < \ldots < \kappa_n \le m+n$ such that (with $\xi_{k,j}$ and μ_j as in* (8) *and* (9)*)*

(i) (Disconjugacy Condition)

$$D(t) := \left|\tilde{\xi}_{\kappa_1,0}(t), \ldots, \tilde{\xi}_{\kappa_n,0}(t)\right| \neq 0 \quad \textit{for} \quad a \leq 0 < a_1 \leq t \leq b_1 \leq b, \tag{14}$$

(ii) (Dichotomy Condition)

$$\operatorname{Re}\left[(\mu_{\kappa_1}(t) + \ldots \mu_{\kappa_n}(t))e^{i\varphi}\right] \geq \operatorname{Re}\left[(\mu_{k_1}(t) + \ldots \mu_{k_n}(t))e^{i\varphi}\right] + \delta \tag{15}$$

for $\alpha \leq \varphi \leq \beta$, $a_1 \leq t \leq b_1$ *and all* $(k_1, \ldots, k_n) \neq (\kappa_1, \ldots, \kappa_n)$ *with* $1 \leq k_1 < \ldots < k_n \leq n+m$,

(iii)

$$\begin{vmatrix} c_{\kappa_1}(\rho) \\ \vdots \\ c_{\kappa_n}(\rho) \end{vmatrix} \neq 0 \quad \textit{for} \quad \rho = re^{i\varphi} \quad \textit{with} \quad r \geq \gamma_0 \geq \gamma.$$

Then (using the notation of Section 3) $\det P(t,\rho) \neq 0$ *for* $t \in [a_1, b_1]$ *and* $\rho \in \mathcal{S}^R_{\alpha,\beta} = \{re^{i\varphi} : r \geq R,\ \alpha \leq \varphi \leq \beta\}$ *where* $R = R(a_1, b_1, \alpha, \beta, C)$*, and* $W(t, re^{i\varphi})$ *has (uniformly) in the sector* $\mathcal{S}^R_{\alpha,\beta}$ *an asymptotic expansion of the form*

$$W(t, re^{i\varphi}) = \sum_{\nu=0}^{k} W_\nu(t) r^{-\nu} e^{-i\nu\varphi} + o(r^{-k}) \quad \textit{for} \quad r \to \infty, \tag{16}$$

where, in particular,

$$W_0(t) = \frac{1}{D(t)}\left(w^0_{ij}(t)\right) \quad \textit{with} \quad w^0_{ij}(t) = \left|\xi_{\kappa_1,0}(i,j), \ldots, \xi_{\kappa_n,0}(i,j)\right|(t). \tag{17}$$

Here $W_1, \ldots, W_k$ *can be determined from* (8) *and* (13)*; notice that* $W_0, \ldots, W_k$ *depend on the sector* $\mathcal{S}^R_{\alpha,\beta}$.

Proof. From the assumptions (i), (ii) it follows that $\det P(t, re^{i\varphi}) \neq 0$ for $a_1 \leq t \leq b_1$ provided r is sufficiently large ($r \geq R$). Therefore we infer from (13) – (15) that $W(t, re^{i\varphi})$ differs from $\hat{W}(t, re^{i\varphi}) := \left(\hat{w}_{ij}(t, re^{i\varphi})\right)$ with

$$\hat{w}_{ij}(t, re^{i\varphi}) = \frac{|x_{k_1,0}(i,j), \ldots, x_{k_n,0}(i,j)|(t, re^{i\varphi})}{|\tilde{x}_{k_1,0}, \ldots, \tilde{x}_{k_n,0}|(t, re^{i\varphi})}$$

only by a term that tends exponentially fast to 0 for $a_1 \leq t \leq b_1$, $\alpha \leq \varphi \leq \beta$ as $\mathcal{S}^R_{\alpha,\beta} \ni \lambda \to \infty$. Consequently it follows easily from (8) that (16) and (17) are valid. □

In many applications (1) and (2) are such that

$$B(t,\lambda) = \lambda B_1 + B_0(t) + \lambda^{-1} B_{-1}(t) \tag{18}$$

with integrable or sufficiently smooth B_0 and B_{-1} and a constant, diagonalizable matrix B_1. Then there exists a Jordan basis matrix $V = (v_1, \ldots, v_{n+m})$ of B_1 such that

$$A_1 := V^{-1} B_1 V = \operatorname{diag}(\lambda_1, \ldots, \lambda_{m+n}). \tag{19}$$

In this case (using the notation of Assumption 2.1) we have $\lambda = \rho$ (i.e. $d = 1$) and we can choose $S(\cdot, \rho) \equiv V$. Consequently,

$$A_0(t) + \lambda^{-1} A_{-1}(t) := V^{-1} B(t,\rho) V - \rho A_1 = V^{-1}\left[B_0(t) + \lambda^{-1} B_{-1}(t)\right] V;$$

moreover (using again the notation of Section 2) $r_\nu(t) \equiv \mu_\nu$,

$$E(t,\rho) = \operatorname{diag}(e^{\rho\lambda_1 t}, \ldots, e^{\rho\lambda_{m+n} t})$$

and the functions $\xi_{j,0}$, defined in (8), satisfy

$$(\xi_{j,0}(t), \ldots, \xi_{m+n,0}(t)) = VP^{[0]}(t).$$

If $\kappa_1 < \kappa_2 < \ldots < \kappa_n$ are fixed, then define $\tilde{P} \in \mathbb{C}^{n\times n}$ and $\tilde{Q} \in \mathbb{C}^{m\times n}$ by

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} := (v_{\kappa_1}, \ldots, v_{\kappa_n}) \tag{20}$$

and assume that $\tilde{P}^{-1}$ exists and that there exists $\hat{P}$ with

$$(\xi_{\kappa_1,0}(t), \ldots, \xi_{\kappa_n,0}(t)) = (v_{\kappa_1}, \ldots, v_{\kappa_n})\, \hat{P}(t). \tag{21}$$

Since $P^{[0]}$ is block-diagonal (21) is fulfilled if $\lambda_j \notin \{\lambda_{\kappa_1}, \ldots, \lambda_{\kappa_n}\}$ for $j \notin \{\kappa_1, \ldots, \kappa_n\}$. Therefore it follows from the proof of [5, Theorem 1] and the subsequent remarks that $V_0(t) \equiv \tilde{Q}\tilde{P}^{-1}$ is – according to Theorem 4.1 – the leading term of the solution $W(t, \rho)$, defined by (10) – (12):

Corollary 4.2. *Let $B(t, \lambda)$ have the special form (18) with $B_0 \in W^0_\infty(I)$ and let there exist $\kappa_1, \ldots, \kappa_n$ such that (21) holds and, moreover,*

(i) $\det \tilde{P} \neq 0$,
(ii) $\mathrm{Re}\left[(\lambda_{\kappa_1} + \ldots \lambda_{\kappa_n})e^{i\varphi}\right] \geq \mathrm{Re}\left[(\lambda_{k_1} + \ldots \lambda_{k_n})e^{i\varphi}\right] + \delta$ *for* $\alpha \leq \varphi \leq \beta$ *and all* $(k_1, \ldots, k_n) \neq (\kappa_1, \ldots, \kappa_n)$ *with* $1 \leq k_1 < \ldots < k_n \leq n + m$,
(iii)

$$\begin{vmatrix} c_{\kappa_1}(\rho) \\ \vdots \\ c_{\kappa_n}(\rho) \end{vmatrix} \neq 0 \quad \textit{for} \quad \rho = re^{i\varphi} \in \mathcal{S}^R_{\alpha,\beta} \quad \textit{with} \quad R \geq R_0.$$

Then $\lim_{r\to\infty} W(t, re^{i\varphi}) = \tilde{Q}\tilde{P}^{-1}$ *for* $\alpha \leq \varphi \leq \beta$.

If B_0 is sufficiently smooth, it is possible to determine further coefficients W_ν in the representation (16) of W.

5. EXAMPLES

Recently Clark and Gesztesy [2, 3] published some interesting papers where they determined explicitly the so-called high-energy asymptotics for Weyl-Tichmarsh matrices associated with matrix-valued Schrödinger operators and general Dirac-type operators by appealing to the theory of matrix Riccati equations (see [2, 3] for the basic notions from Weyl-Tichmarsh theory and for further references). Both problems are related to Hamiltonian systems of the form

$$J\dot{\psi}(t, \lambda) = [\lambda M(t) + N(t)]\psi(t, \lambda) \quad \text{with} \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{22}$$

where

(i) in the Dirac-type case

$$M(t) = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad N(t) = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = N^*(t),$$

and

(ii) in the Schrödinger-type case

$$M(t) = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \quad N(t) = \begin{pmatrix} N_{11} & 0 \\ 0 & I_n \end{pmatrix} = N^*(t),$$

and where all the coefficients are at least locally integrable in the sense of Lebesgue. We further assume below that N is such that (22) verifies Assumption 2.1 for some $k \in \mathbb{N}_0$.

In both cases the Weyl-Titchmarsh function is determined by the solution of an initial value problem for the matrix Riccati equation associated with (22) which is given

(i) in the Dirac-type case by

$$\begin{aligned}\dot W(t,\lambda) = &-W(t,\lambda)[\lambda I_n + N_{22}(t)]W(t,\lambda) - N_{12}(t)W(t,\lambda)\\ &- W(t,\lambda)N_{21}(t) - [N_{11}(t) + \lambda I_n]\,,\end{aligned} \tag{23}$$

with $\det\big(iI_n \pm W(0,\lambda)\big) \neq 0$ for $\mp \operatorname{Im}\lambda > 0$,

(ii) in the Schrödinger-type case by

$$\dot W(t,\lambda) = -W(t,\lambda)^2 - [N_{11}(t) + \lambda I_n] \tag{24}$$

with $\det\big(i\sqrt{\lambda}\, I_n - W(0,\lambda)\big) \neq 0$ for $\operatorname{Im}\sqrt{\lambda} > 0$.

Case (i): The coefficient matrix of the corresponding linear system (2) is here

$$B(t,\lambda) = \lambda B_1 + B_0(t)$$

with

$$B_1 = J^{-1} = -J \quad \text{and} \quad B_0(t) = J^{-1}N(t);$$

in the notation used in Assumption 2.1 we have $\lambda = \rho$ and $m = n$. If we define

$$V := \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22}\end{pmatrix} := \begin{pmatrix} I_n & I_n \\ iI_n & -iI_n\end{pmatrix}, \quad \text{hence} \quad V^{-1} = \frac{1}{2i}\begin{pmatrix} iI_n & I_n \\ iI_n & -I_n\end{pmatrix},$$

we obtain

$$A_1 := V^{-1}B_1V = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n\end{pmatrix},$$

i.e. here B_1 has the eigenvalues $\lambda_1 = \ldots = \lambda_n = i$ and $\lambda_{n+1} = \ldots = \lambda_{2n} = -i$. Since we choose here $V(\cdot,\rho) \equiv S$, (5) and (7) yield that

$$C(\rho) = V^{-1}\begin{pmatrix} I \\ W(0,\rho)\end{pmatrix} + o(1) = \frac{1}{2i}\begin{pmatrix} iI_n + W(0,\rho) \\ iI_n - W(0,\rho)\end{pmatrix} + o(1).$$

Therefore the assumptions of Corollary 4.2 are fulfilled for $(\kappa_1,\ldots,\kappa_n) = (1,\ldots,n)$ and $\varphi \in\,]-\pi,0[$ (resp. for $(\kappa_1,\ldots,\kappa_n) = (n+1,\ldots,2n)$ and $\varphi \in\,]0,\pi[$). Hence we obtain

$$\begin{aligned}\lim_{r\to\infty} W(t,re^{i\varphi}) &= V_{21}V_{11}^{-1} = iI_n \qquad &&\text{for} \quad -\pi+\varepsilon \le \varphi \le -\varepsilon < 0,\\ \lim_{r\to\infty} W(t,re^{i\varphi}) &= V_{22}V_{12}^{-1} = -iI_n \qquad &&\text{for} \quad 0 < \varepsilon \le \varphi \le \pi-\varepsilon.\end{aligned}$$

Notice that here the first (resp. second) limit can be read off directly from the first (resp. last) n columns of V.

Case (ii): Since

$$J^{-1}\begin{pmatrix} I_n & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -I_n & 0\end{pmatrix}$$

has the $2n$-fold eigenvalue 0 and a Jordan canonical form with n Jordan blocks $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, we cannot apply Corollary 4.2 directly. But an application of the shearing transformation

$$Z := \begin{pmatrix} \sqrt{\lambda}\, I_n & 0 \\ 0 & I_n \end{pmatrix} X$$

reduces

$$\dot{X} = J^{-1} \left[\lambda \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{11}(t) & 0 \\ 0 & I_n \end{pmatrix} \right] X$$

to the equation

$$\dot{Z} = \left[\rho \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} + \frac{1}{\rho} \begin{pmatrix} 0 & 0 \\ -N_{11}(t) & 0 \end{pmatrix} \right] Z \quad \text{with} \quad \rho = \sqrt{\lambda} = \sqrt{r}\, e^{i\varphi/2}$$

which has the same leading term as we had in (22) in Case (i) with λ replaced by ρ. Consequently an application of Theorem 4.1 yields similarly as before (for $\det\big(i\sqrt{\lambda}\, I_n - W(0,\lambda)\big) \neq 0$)

$$\lim_{r\to\infty} W(t, re^{i\varphi}) + i\sqrt{r}\, e^{i\varphi/2} I = 0 \quad \text{for} \quad 0 < \varepsilon \leq \varphi \leq \pi - \varepsilon.$$

Similarly we can determine in Case (i) and Case (ii) also the limits of fractional linear transformed functions

$$T(t,\lambda) = \big(T_{11}W(t,\lambda) + T_{12}\big)\big(T_{21}W(t,\lambda) + T_{22}\big)^{-1}$$

since it is well known that $T(\cdot,\lambda)$ solves also a matrix Riccati equation.

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