A Sampling Theorem for Transforms with Discontinuous Kernels

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We present a sampling representation for the transform $F(\lambda) = \int_{-1}^{1} f(x)\Phi(x,\lambda)\,dx,\ f \in L^2(-1,1)$, where the kernel is a solution of an n-th order differential equation $\ell(y) = \lambda y$. Here the differential expression ℓ is defined to be $\ell = \ell_1$ on $[-1,0)$ and $\ell = \ell_2$ on (0,1], ℓ_1, ℓ_2 are in general two different n-th order differential expressions. This includes the case when ℓ_1 and ℓ_2 are identical with discontinuous coefficients at $x = 0$. So, in addition to the boundary conditions, the eigenvalue problems in question contain n compatibility conditions. The problem is not necessarily self adjoint, but conditions are imposed on it in which the eigenfunctions are a Riesz basis.

Keywords: Sampling theory, expansion in eigenfunctions, Riesz bases, spectral problems with discontinuous coefficients

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1 INTRODUCTION

The connection between sampling theory and eigenvalue problems had been initiated in the middle of the last century, [30, 22]. Few papers in the subject might be found since these two sources till the last two decades, see e.g. [10, 19]. More recently, this problem attracted many authors who extended the subject to several directions and derived some applications. Among those authors are e.g., Zayed et. al. [35, 36]; Zayed [33, 34]; Butzer and Schöttler [9]; Everitt et. al. [12]; Everitt and Nasri-Roudsari [13, 14, 15]; Annaby and Zayed [2]; Annaby [3]; Annaby and Hassan [1] and Annaby and Freiling [4, 5]. In most of this work a transformation is defined, an integral transformation or integrodifferential transformation, and is reconstructed from its values at the eigenvalues of the eigenvalue problem. The kernels of the transformations are continuous functions which are solutions or Green's function of the problem. The example of [12, p. 117], is the first one to

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reconstruct transformations with a discontinuous kernel. The problem that leads to that example is

$$-(p(x)y')' = \lambda y, \quad -1 \le x \le 1, \tag{1.1}$$

$$y(-1) = y(1) = 0, \tag{1.2}$$

where

$$p(x) = \begin{cases} -1, & -1 \le x < 0, \\ 0, & x = 0, \\ 1, & 0 < x \le 1. \end{cases} \tag{1.3}$$

Let $\rho(\lambda)$ be the entire function,

$$\rho(\lambda) := \frac{1}{\sqrt{\lambda}}\left[\sinh\sqrt{\lambda}\cos\sqrt{\lambda} - \cosh\sqrt{\lambda}\sin\sqrt{\lambda}\right]. \tag{1.4}$$

Then the sampling series of the transformation

$$F(\lambda) = \int_{-1}^{1} f(x)K_-(x,\lambda)\,dx, \quad f \in L^2(-1,1) \tag{1.5}$$

has the form

$$F(\lambda) = \frac{-3\rho(\lambda)}{2\lambda}F(0) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} F(\lambda_k)\frac{-\rho(\lambda)\lambda_k}{\sinh\sqrt{\lambda_k}\cdot\sin\sqrt{\lambda_k}(\lambda-\lambda_k)}, \tag{1.6}$$

where $\{\lambda_k\}_{k=-\infty}^{\infty}$ are the zeros of $\rho(\lambda)$. Here the kernel $K_-(\cdot,\lambda)$ is defined by

$$K_-(x,\lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(\sinh\sqrt{\lambda}\cosh\sqrt{\lambda}x + \cosh\sqrt{\lambda}\sinh\sqrt{\lambda}x), & -1 \le x \le 0 \\ \frac{1}{\sqrt{\lambda}}(\sinh\sqrt{\lambda}\cos\sqrt{\lambda}x - \cosh\sqrt{\lambda}\sin\sqrt{\lambda}x), & 0 \le x \le 1. \end{cases} \tag{1.7}$$

See [12, p. 117, ff]. In the article of Zayed and García, [37] the authors discussed the situation of deriving a sampling theorem of Kramer type when the kernels are discontinuous. They have considered the following discontinuous Sturm-Liouville problem studied by Kobayashi [23]. Consider the interval $[0,\pi]$. Let $0 < d < \frac{\pi}{2}$ be a fixed point and $d_1 := d$; $d_2 := \pi - d$. The discontinuous Sturm-Liouville problem of [23] consists of the differential equation

$$-u'' + q(x)u = \lambda u, \quad 0 \le x \le \pi; \tag{1.8}$$

the boundary conditions

$$hu(0) - u'(0) = 0 = hu(\pi) + u'(\pi) \tag{1.9}$$

and the jump conditions

$$\left.\begin{aligned} u(d_1^+) = au(d_1^-), \quad u'(d_1^+) = a^{-1}u'(d_1^-) + bu(d_1^-), \\ u(d_2^-) = au(d_2^+), \quad u'(d_2^-) = a^{-1}u'(d_2^+) - bu(d_2^+), \end{aligned}\right\} \tag{1.10}$$

where $h, b \in I\!R$ and a is a positive number. Below, conditions similar to (1.10) will be called compatibility conditions. Their sampling result states that the transformation

$$F(\lambda) = \int_0^\pi f(x)u(x,\lambda)\,dx \quad f \in L^2(0,\pi) \tag{1.11}$$

can be reconstructed via the sampling form

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n)\omega'(\lambda_n)}, \tag{1.12}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ are the zeros of the function $\omega(\lambda) = -hu(\pi,\lambda) - u'(\pi,\lambda)$ and they are exactly the eigenvalues of the discontinuous Sturm-Liouville problem. The kernel $u(\cdot,\lambda)$ is a solution of (1.8) which satisfies the jump conditions together with the initial condition $hu(0) - u'(0) = 0$. We mention here that the sampling result of [37] contains the canonical product

$$P(\lambda) := \begin{cases} \displaystyle\prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right), & \text{if zero is not an eigenvalue,} \\ \displaystyle\lambda \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right), & \text{if } \lambda_0 = 0 \text{ is an eigenvalue,} \end{cases} \tag{1.13}$$

instead of the function $\omega(\lambda)$ of (1.12). This makes no confusion since, according to the Hadamard factorization theorem, $\omega(\lambda)/P(\lambda)$ is a constant, see also [21].

The goal of this paper is the study the situation in a very general case when:

i) The differential expression replacing $-\frac{d^2}{dx^2} + q(\cdot)$ in (1.8) is of arbitrary order $n \in I\!N$.

ii) The differential expression is not necessarily self adjoint. Notice that the differential expression $-\frac{d^2}{dx^2} + q(\cdot)$ is self adjoint since the function $q(\cdot)$ is an integrable real-valued function, see [23], p. 910.

iii) The differential expression is not necessarily defined by the same way throughout the interval where the problem is defined, like $[0,\pi]$ in the above case.

iv) The boundary conditions are not necessarily self adjoint or of the separated type. Conditions (1.9) are self adjoint and of the separated type.

So, we claim that the sampling theorem obtained in Section 4 below is the best one can do in this setting. But we do not claim that we generalize all the way. Actually we have two restrictions:

1. The boundary conditions as well as the compatibility conditions must be strongly regular. The definition of strong regularity is given below. For example, conditions (1.9) are strongly regular.

2. Instead of having two points of discontinuity, we have a single point.

The first restriction is important to guarantee that the system of eigenfunctions of the problem is a Riesz basis of the Hilbert space where the problem is defined.

Definition 1.1. A set $\{u_k(\cdot)\}_{k=1}^{\infty}$ is called a Riesz basis of $L^2[-1,1]$, if it is a basis, that is for any $f \in L^2[-1,1]$, there is a unique L^2-convergent expansion of f,

$$f(x) = \sum_{k=1}^{\infty} c_k u_k(x), \tag{1.14}$$

and in addition there is a constant $\gamma > 1$ such that for any $f \in L^2[-1,1]$,

$$\gamma^{-1} \sum_{k=1}^{\infty} |c_k|^2 \leq ||f||^2 \leq \gamma \sum_{k=1}^{\infty} |c_k|^2. \tag{1.15}$$

See [32] for equivalent definitions. As for the second restriction, to the best of our knowledge, the analysis of the Riesz-basis-problem with the above mentioned extensions has not been established when there are several points of discontinuity of the coefficients of the differential equation. For a detailed representation of the contour integral method and for the proof of expansion theorems for this type of problems see [27]. Fortunately, with extensions (i)–(iv) and restrictions (1)–(2) above, the analysis has been developed by Muravei in [25]. In principle it would be possible to generalize the results of Muravei and those of the present paper to the case of problems having a finite number of discontinuities. We will not study this general situation since the presentation of the results becomes very technical but the ideas used are essentially the same. In this setting, the eigenfunctions do not necessarily have the property of orthogonality, but a biorthogonal relationship holds between the eigenfunctions of the problem and those of its adjoint. This means that we cannot use the orthogonal version of Kramer's Lemma, [22], but the biorthogonal one of Higgins [20]. The next section contains this biorthogonal version and an application to eigenvalue problems taken from [3]. Section 3 defines our eigenvalue problem and introduces its properties, moreover it indicates how to compute the eigenvalues and the eigenfunctions as well as the adjoint problem. The main result will be introduced in Section 4. Section 5 is fully devoted to detailed examples. Example 5.5 at the end of this article introduces a class of new problems that has not been treated before in a general setting. Its treatment will be considered separately because the properties of the eigenvalues and the eigenfunctions differ from those considered in Section 3 below.

2 SAMPLING AND STRONGLY REGULAR PROBLEMS

This section is devoted to give a brief account about the results of [3]. Here we use the following Lemma, first derived by Higgins [20].

Lemma 2.1. *Let $[a,b] \subset I\!R$ be a bounded closed interval and let $K_i(\cdot,\lambda)$ be functions defined on $[a,b] \times \mathbb{C}$, $i = 1,2$. Let $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\lambda_k^*\}_{k=0}^{\infty}$ be sequences of numbers such that $\{K_1(\cdot,\lambda_k), K_2(\cdot,\lambda_m^*)\}_{k,m=0}^{\infty}$ satisfies*

$$\int_a^b K_1(x,\lambda_k)\overline{K}_2(x,\lambda_m^*)dx \neq 0 \ \ \text{only when} \ \ k \neq m \tag{2.1}$$

and $\{K_1(\cdot,\lambda_k)\}_{k=0}^{\infty}$ is a basis of $L^2[a,b]$. Then the transformations

$$F_i(\lambda) = \int_a^b f(x)K_i(x,\lambda)\,dx, \ \ f \in L^2(a,b), \quad \lambda \in \mathbb{C}, \ \ i = 1,2 \tag{2.2}$$

can be reconstructed via the absolutely convergent sampling series

$$F_1(\lambda) = \sum_{k=0}^{\infty} F_1(\lambda_k)\frac{\displaystyle\int_a^b K_1(x,\lambda)\overline{K}_2(x,\lambda_k^*)\,dx}{\displaystyle\int_a^b K_1(x,\lambda_k)\overline{K}_2(x,\lambda_k^*)\,dx}, \tag{2.3}$$

$$F_2(\lambda) = \sum_{k=0}^{\infty} F_2(\lambda_k^*)\frac{\displaystyle\int_a^b K_2(x,\lambda)\overline{K}_1(x,\lambda_k)\,dx}{\displaystyle\int_a^b K_2(x,\lambda_k^*)\overline{K}_1(x,\lambda_k)\,dx}. \tag{2.4}$$

The convergence of the sampling series (2.3) and (2.4) is uniform if $K_1(\cdot,\lambda)$, $K_2(\cdot,\lambda)$ satisfies further conditions, see [20] for details. The point now is, where can one find the kernels and the sequences of numbers such that the conditions of the Lemma are fulfilled. Higgins in the above mentioned reference indicated that a prototype situation, where the above theorem can be applied, is the problem

$$-y'' = \lambda y, \quad 0 \leq x \leq \pi, \quad \lambda \in \mathbb{C}, \tag{2.5}$$

$$y'(0) = y(0) + y'(\pi) = 0. \tag{2.6}$$

In [3] the situation is extended to include the general eigenvalue problem

$$p_0(x)y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \ldots + p_n(x)y(x) = \lambda y(x), \ \ a \leq x \leq b; \tag{2.7}$$

$$M_\nu(y) := \sum_{j=1}^{a} \alpha_{i\nu}y^{(j-1)}(a) + \beta_{i\nu}y^{(i-1)}(b) = 0, \quad \nu = 1,\ldots,n, \tag{2.8}$$

where a and b are finite numbers. See [3, 26] for definitions and details. In that paper, [3], it is indicated that, under the condition that the boundary conditions are strongly regular, one can find a kernel which generates all eigenfunctions of problem (2.7)–(2.8), i.e. a kernel which plays the role of $K_1(\cdot,\lambda)$ in the Lemma. In this case the eigenvalues of (2.7)–(2.8), $\{\mu_k\}_{k=0}^{\infty}$ play the role of the sequence $\{\lambda_k\}_{k=0}^{\infty}$ of the Lemma. The other kernel and sequence are extracted similarly from the problem adjoint to (2.7)–(2.8). The author in [3] used the fact that under the prescribed conditions the eigenfunctions of problem (2.7)–(2.8) as well as those of its adjoint form biorthogonal Riesz bases of $L^2[a,b]$, [24, 6].

3 THE EIGENVALUE PROBLEM

We consider subsequently the eigenvalue problem

$$Ly = \lambda y \tag{3.1}$$

for the differential operator

$$L : L^2[-1,1] \supset D(L) \to L^2[-1,1], \quad y \mapsto \ell(y). \tag{3.2}$$

The differential expression associated with L, $\ell(\cdot)$, is given by

$$\ell(y) := \begin{cases} \ell_1(y_1), & \text{for} \quad -1 \le x < 0, \\ \ell_2(y_2), & \text{for} \quad \ \ 0 < x \le 1, \end{cases} \tag{3.3}$$

where

$$\ell_i(y_i) = y_i^{(n)} + \sum_{j=1}^{n} p_{ij}(x) y_i^{(n-j)}(x), \quad i = 1, 2. \tag{3.4}$$

Here y_1 and y_2 are the restrictions of y to $[-1,0)$ and $(0,1]$ respectively. The coefficients p_{ij} satisfy for $1 \le j \le n$, $1 \le i \le 2$,

$$p_{1j} \in C^{(n-j)}([-1,0]); \;\; p_{2j} \in C^{(n-j)}([0,1]). \tag{3.5}$$

The domain of definition of L is

$$D(L) = \{y \in L^2[-1,1] \cap D_{n,0} | U_\nu(y) = 0, \;\; V_\nu(y) = 0, \;\; 1 \le \nu \le n\}, \tag{3.6}$$

where the functionals U_ν, V_ν are defined below and $D_{n,0}$ is the set of all functions that have n continuous derivatives in $[-1,0) \cup (0,1]$ such that the limits $y^{(j)}(0^-)$ and $y^{(j)}(0^+)$, $0 \le j \le n-1$, exist (as finite numbers). The boundary and compatibility conditions $U_\nu(\cdot)$ and $V_\nu(\cdot)$, $1 \le \nu \le n$ are defined respectively by:

$$\begin{aligned} U_\nu(y) &= U_{\nu,-1}(y_1) + U_{\nu,1}(y_2), \\ U_{\nu,-1}(y_1) &= \alpha_\nu y_1^{(k_\nu)}(-1) + \alpha_{\nu,k_\nu-1} y_1^{(k_\nu-1)}(-1) + \dots, \\ U_{\nu,1}(y_2) &= \beta_\nu y_2^{(k_\nu)}(1) + \beta_{\nu,k_\nu-1} y_2^{(k_\nu-1)}(1) + \dots; \end{aligned} \tag{3.7}$$

$$\begin{aligned}
&V_\nu(y) = V_{\nu,0^-}(y_1) + V_{\nu,0^+}(y_2),\\
&V_{\nu,0^-}(y_1) = \gamma_\nu y_1^{(m_\nu)}(0^-) + \gamma_{\nu,m_\nu-1} y_1^{(m_\nu-1)}(0^-) + \ldots, \qquad (3.8)\\
&V_{\nu,0^+}(y_2) = \delta_\nu y_2^{(m_\nu)}(0^+) + \delta_{\nu,m_\nu-1} y_2^{(m_\nu-1)}(0^+) + \ldots.
\end{aligned}$$

The numbers k_ν and m_ν are called the orders of U_ν and V_ν respectively. They satisfy

$$n-1 \geq k_1 \geq k_2 \geq \ldots \geq k_n, \quad k_\nu > k_{\nu+2}, \tag{3.9}$$

$$n-1 \geq m_1 \geq m_2 \geq \ldots \geq m_n, \quad m_\nu > m_{\nu+2}. \tag{3.10}$$

Moreover the numbers $\alpha_\nu, \beta_\nu, \gamma_\nu$ and δ_ν satisfy

$$|\alpha_\nu| + |\beta_\nu| > 0, \quad |\gamma_\nu| + |\delta_\nu| > 0, \quad 1 \leq \nu \leq n. \tag{3.11}$$

We assume without any loss of generality that the forms U_ν and V_ν, $1 \leq \nu \leq n$ are both normalized. This means, e.g. for U_ν that if $\widehat{U}_\nu$, $1 \leq \nu \leq n$ is any equivalent system with orders $\widehat{k}_1 \geq \widehat{k}_2 \geq \ldots \geq \widehat{k}_n$, then $\widehat{k}_\nu \geq k_\nu$, $1 \leq \nu \leq n$.

By an eigenfunction of the operator L, we mean a nontrivial function $y \in D(L)$ with

$$y(x) = \begin{cases} y_1(x), & -1 \leq x < 0, \\ y_2(x), & 0 < x \leq 1, \end{cases} \tag{3.12}$$

such that $\ell(y) = \lambda y$ for a complex number λ. In this case λ is said to be an eigenvalue with an eigenfunction $y(\cdot)$, or $y(\cdot)$ is an eigenfunction of L corresponding (belonging) to the eigenvalue λ.

The problem adjoint to (3.1), $L^* y = \lambda y$, consists of the differential expression

$$\ell^*(y) = \begin{cases} \ell_1^*(y_1), & \text{for} \quad -1 \leq x < 0, \\ \ell_2^*(y_2), & \text{for} \quad 0 < x \leq 1 \end{cases} \tag{3.13}$$

and the boundary and compatibility conditions

$$U_\nu^*(y) = 0, \quad V_\nu^*(y) = 0, \quad \nu = 1, 2, \ldots, n. \tag{3.14}$$

Similar to [11, 26], the adjoint problem (3.13)–(3.14) is computed in the following way. First, we complement the linearly independent forms U_ν and V_ν with other linearly independent forms $U_{\nu+1}, \ldots, U_{2n}$ and $V_{\nu+1}, \ldots, V_{2n}$, respectively, to have the linearly independent systems $\{U_\nu\}_{\nu=1}^{2n}$ and $\{V_\nu\}_{\nu=1}^{2n}$. The adjoint differential expression ℓ^* and the adjoint boundary and compatibility conditions are extracted from the following Lagrange's identity

$$\int_{-1}^{1} \ell(y)(x)\overline{z}(x)\,dx = \sum_{i=1}^{2n} V_i(y) V_{2n-i+1}^*(\overline{z}) + \sum_{i=1}^{2n} U_i(y) U_{2n-i+1}^*(\overline{z}) + \int_{-1}^{1} y(x)\overline{\ell^*(z)}(x)\,dx. \tag{3.15}$$

An illustrative example are given in Section 5 below, indicating how to apply (3.15) to compute the adjoint problem.

Before giving the definitions of strongly regular problems, we indicate how to obtain the eigenvalues and the eigenfunctions of problem (3.1). Let $\{y_{1\nu}(\cdot,\lambda)\}_{\nu=1}^{n}$ and $\{y_{2\nu}(\cdot,\lambda)\}_{\nu=1}^{n}$ be n linearly independent solutions (i.e. fundamental sets) of $\ell_1(y_1) = \lambda y_1$, and $\ell_2(y_2) = \lambda y_2$, respectively. Thus any solution of $\ell(y) = \lambda y$ will have the form

$$y(x,\lambda) = \begin{cases} c_{11}y_{11}(x,\lambda) + c_{12}y_{12}(x,\lambda) + \ldots + c_{1n}y_{1n}(x,\lambda), & -1 \le x < 0, \\ c_{21}y_{21}(x,\lambda) + c_{22}y_{22}(x,\lambda) + \ldots + c_{1n}y_{2n}(x,\lambda), & 0 < x \le 1. \end{cases} \tag{3.16}$$

To be an eigenfunction $y(\cdot,\lambda)$ has to satisfy the system

$$\left.\begin{aligned} \sum_{j=1}^{n} c_{1j}U_{\nu,-1}(y_{1j}) + \sum_{j=1}^{n} c_{2j}U_{\nu,1}(y_{2j}) = 0, \\ \sum_{j=1}^{n} c_{1j}V_{\nu,0^-}(y_{1j}) + \sum_{j=1}^{n} c_{2j}U_{\nu,0^+}(y_{2j}) = 0. \end{aligned}\right\} \tag{3.17}$$

System (3.17) is a linear homogeneous system of $2n$ equations in the $2n$ unknowns $c_{1j}, c_{2j}, 1 \le j \le n$. It has a nontrivial solution only for the values of λ that satisfy

$$\triangle(\lambda) = \begin{vmatrix} U_{1,-1}(y_{11}) & \ldots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \ldots & U_{1,1}(y_{2n}) \\ U_{2,-1}(y_{11}) & \ldots & U_{2,-1}(y_{2n}) & U_{2,1}(y_{21}) & \ldots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \ldots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \ldots & V_{n,0^+}(y_{2n}) \end{vmatrix} = 0. \tag{3.18}$$

Therefore the determinant $\triangle(\lambda)$ is called the characteristic determinant of problem (3.1) and its zeros are exactly the eigenvalues of the operator L.

Now we give the definition of strongly regular problems, which have properties needed for the derivation of the sampling theorem. Let us set $\lambda = -\rho^n$ and let $S_k, 0 \le k \le 2n-1$ denote the sectors of the complex ρ-plane defined by

$$\frac{k\pi}{n} \le \arg\rho \le \frac{(k+1)\pi}{n}. \tag{3.19}$$

By $\omega_1, \ldots, \omega_n$ we denote the nth roots of -1 which are enumerated such that

$$Re(\rho\omega_1) \le Re(\rho\omega_2) \le \ldots \le Re(\rho\omega_n), \quad \text{for } \rho \in S_0.$$

Now let the fundamental sets $y_{1j}(\cdot,\lambda)$ and $y_{2j}(\cdot,\lambda)$ satisfy the initial conditions

$$y_{1j}^{(k-1)}(-1,\lambda) = \delta_{kj}; \quad y_{2j}^{(k-1)}(0^+,\lambda) = \delta_{kj}, 1 \le k, j \le n. \tag{3.20}$$

Using the asymptotics of y_{1j}, y_{2j} [25], p. 119, it follows that there are numbers $\theta_{-1}, \theta_0, \theta_1$ and a function $\triangle_0$ with $\triangle_0(\lambda) \ne 0$ for $\lambda \ne 0$ such that if $|\rho| \to \infty$, $\rho \in S_0$, then

(i) for $n = 2\mu - 1$,
$$\triangle(\lambda) = \triangle_0(\lambda)([\theta_{-1}]e^{-\rho\omega_\mu} + [\theta_0] + [\theta_1]e^{\rho\omega_\mu}); \tag{3.21}$$

(ii) for $n = 2\mu$
$$\triangle(\lambda) = \triangle_0(\lambda)([\theta_{-1}]e^{-2\rho\omega_\mu} + [\theta_0] + [\theta_1]e^{2\rho\omega_\mu}); \tag{3.22}$$
here $[a] := a + O(\frac{1}{\rho})$.

The problem $Ly = \lambda y$ is called *regular* if $\theta_{-1} \neq 0 \neq \theta_1$, and it is called *strongly regular* if $\theta_0^2 \neq 4\theta_1\theta_{-1} \neq 0$. This definition extends that of Birkhoff [7, 26]. Throughout this paper, we assume that our problem is strongly regular. Thus, [25] the adjoint problem is also strongly regular. Using and generalizing the well known methods developed by Birkhoff [7] and Tamarkin [29], Muravei [25] has proved the following results.

Lemma 3.1. *L has two sequences of eigenfunctions $\{\lambda'_k\}_{k\geq 1}$ and $\{\lambda''_k\}_{k\geq 1}$ where $\lambda'_k = -(\rho'_k)^n$ and $\lambda''_k = -(\rho''_k)^n$ with [25], p. 123;*

(i) for $n = 2\mu - 1$,

$$\begin{aligned} \rho'_k &= \frac{1}{\omega_\mu}\left(2k\pi i + \ell n\ \xi' + \frac{\eta'}{k} + 0\left(\frac{1}{k^2}\right)\right), \\ \rho''_k &= \frac{1}{\omega_\mu}\left(2k\pi i + \ell n\ \xi'' + \frac{\eta''}{k} + 0\left(\frac{1}{k^2}\right)\right), \end{aligned} \tag{3.23}$$

(ii) for $n = 2\mu$

$$\begin{aligned} \rho'_k &= \frac{1}{\omega_\mu}\left(k\pi i + \frac{1}{2}\ell n\ \xi' + \frac{\eta'}{k} + 0\left(\frac{1}{k^2}\right)\right), \\ \rho''_k &= \frac{1}{\omega_\mu}\left(k\pi i + \frac{1}{2}\ell n\ \xi'' + \frac{\eta''}{k} + 0\left(\frac{1}{k^2}\right)\right), \end{aligned} \tag{3.24}$$

valid for $|k| \to \infty$, where η', η'' are constants independent of k and ξ', ξ'' are the zeros of the quadratic equation $0 = \theta_1\xi^2 + \theta_0\xi + \theta_{-1} = \theta_1(\xi - \xi')(\xi - \xi'')$.

Also another important property of strongly regular problems is that all, but a finite number, of the eigenvalues are simple from geometric and algebraic points of view. Therefore we assume in this paper for convenience that all eigenvalues are simple zeros of $\triangle(\lambda)$. We have the following theorem taken from [25].

Theorem 3.2. *The systems $\{\phi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ of eigenfunctions of a strongly regular operator L of the form (3.1) above and of its adjoint L^*, respectively, are both Riesz-bases of $L^2[-1,1]$. Moreover without any loss of generality*

$$\int_{-1}^{1} \phi_i(x)\overline{\psi}_j(x)dx = \delta_{ij}, \qquad 1 \leq i, j \leq n. \tag{3.25}$$

From now on we assume that the eigenvalues are given by the sequence $\{\lambda_k\}_{k=0}^{\infty}$. Consequently the eigenvalues of the adjoint problem will have the form $\{\overline{\lambda}_k\}_{k=0}^{\infty}$.

4 THE MAIN RESULT

In this section we consider the problem $Ly = \lambda y$ generated by the differential expression (3.3) together with the boundary conditions (3.7) and the compatibility conditions (3.8). We also consider the adjoint problem $L^*y = \lambda y$ generated by the differential expression (3.13) and conditions (3.14). The problem $Ly = \lambda y$ is assumed to be strongly regular and we assume that all eigenvalues are simple zeros of $\triangle(\lambda)$. To derive a sampling theorem, we need to find functions which generate the eigenfunctions of $Ly = \lambda y$, $\{\phi_k(\cdot)\}_{k=0}^{\infty}$, and those of the adjoint problem, $\{\psi_k(\cdot)\}_{k=0}^{\infty}$. From the simplicity of the eigenvalues, it follows that the rank of the matrix

$$\begin{pmatrix} U_{1,-1}(y_{11}) & \dots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \dots & U_{1,n}(y_{2n}) \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,n}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{pmatrix} \tag{4.1}$$

is $2n-1$ for each eigenvalue λ_k. This means that at least one of first minors of the matrix does not vanish; we assume for convenience that it is a minor of the first row for all eigenvalues. Otherwise the modifications lead to more complicated forms, cf. [3]. Thus the eigenfunction corresponding to an eigenvalue λ^* say will have the form

$$\phi^*(x) = \begin{cases} y_1(x), & -1 \le x < 0 \\ y_2(x), & 0 < x \le 1, \end{cases} \tag{4.2}$$

where

$$y_1(x) = \begin{vmatrix} y_{11}(x,\lambda^*) & \dots & y_{1n}(x,\lambda^*) & 0 & \dots & 0 \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix}, \tag{4.3}$$

$$y_2(x) = \begin{vmatrix} 0 & \dots & 0 & y_{21}(x,\lambda^*) & \dots & y_{2n}(x,\lambda^*) \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix}. \tag{4.4}$$

Similarly for the adjoint problem. This indicates that if we define

$$\mathcal{K}_1(x,\lambda) := \begin{vmatrix} y_{11}(x,\lambda) & \dots & y_{1n}(x,\lambda) & 0 & \dots & 0 \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix}, -1 \leq x < 0; \tag{4.5}$$

$$\mathcal{K}_2(x,\lambda) := \begin{vmatrix} 0 & \dots & 0 & y_{21}(x,\lambda) & \dots & y_{2n}(x,\lambda) \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix}, 0 < x \leq 1; \tag{4.6}$$

$$\mathcal{K}_1^*(x,\lambda) := \begin{vmatrix} z_{11}(x,y) & \dots & z_{1n}(x,\lambda) & 0 & \dots & 0 \\ U^*_{2,-1}(z_{11}) & \dots & U^*_{2,-1}(z_{1n}) & U^*_{2,1}(z_{21}) & \dots & U^*_{2,1}(z_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V^*_{n,0^-}(z_{11}) & \dots & V^*_{n,0^-}(z_{1n}) & V^*_{n,0^+}(z_{21}) & \dots & V^*_{n,0^+}(z_{2n}) \end{vmatrix}, -1 \leq x < 0, \tag{4.7}$$

$$\mathcal{K}_2^*(x,\lambda) := \begin{vmatrix} 0 & \dots & 0 & z_{21}(x,\lambda) & \dots & z_{2n}(x,\lambda) \\ U^*_{2,-1}(z_{11}) & \dots & U^*_{2,-1}(z_{1n}) & U^*_{2,1}(z_{21}) & \dots & U^*_{2,1}(z_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V^*_{n,0^-}(z_{11}) & \dots & V^*_{n,0^-}(z_{1n}) & V^*_{n,0^+}(z_{21}) & \dots & V^*_{n,0^+}(z_{2n}) \end{vmatrix}, 0 < x \leq 1, \tag{4.8}$$

then the function

$$\Phi(x,\lambda) = \begin{cases} \mathcal{K}_1(x,\lambda), & -1 \leq x < 0, \\ \mathcal{K}_2(x,\lambda), & 0 < x \leq 1 \end{cases} \tag{4.9}$$

generates all eigenfunctions of $Ly - \lambda y = 0$. That is $\Phi(\cdot, \lambda_k)$ is an eigenfunction corresponding to the eigenvalue λ_k. Moreover, the function

$$\Phi^*(x,\lambda) = \begin{cases} \mathcal{K}_1^*(x,\lambda), & -1 \leq x < 0, \\ \mathcal{K}_2^*(x,\lambda), & 0 < x \leq 1 \end{cases} \tag{4.10}$$

generates all eigenfunctions of the problem adjoint to (3.1). That is $\Phi^*(\cdot,\overline{\lambda}_k)$ is an eigenfunction of (3.13)–(3.14). In the previous forms the functions $y_{i1}(\cdot,\lambda),\dots,y_{in}(\cdot,\lambda)$ and $z_{i1}(\cdot,\lambda),\dots,z_{in}(\cdot,\lambda)$ are fundamental solutions of $\ell_i(y)-\lambda y=0$ and $\ell_i^*(y)-\lambda y=0$ respectively, $i=1,2$, satisfying the initial conditions

$$y_{1j}^{(m-1)}(-1,\lambda)=\delta_{jm},\quad y_{2j}^{(m-1)}(0,\lambda)=\delta_{jm}, \tag{4.11}$$

$$z_{1j}^{(m-1)}(-1,\lambda)=\delta_{jm},\quad z_{2j}^{(m-1)}(0,\lambda)=\delta_{jm}, \tag{4.12}$$

where $1\le j,m\le n$. The main result of this paper is the following theorem.

Theorem 4.1. *Let $f(x)\in L^2(-1,1)$ and let $F(\lambda)$ be the transformation*

$$F(\lambda)=\int_{-1}^{1} f(x)\Phi(x,\lambda)\,dx. \tag{4.13}$$

Then $F(\lambda)$ is an entire function that admits the sampling representation.

$$F(\lambda)=\sum_{n=0}^{\infty} F(\lambda_n)\frac{\triangle(\lambda)}{(\lambda-\lambda_n)\triangle'(\lambda_n)}. \tag{4.14}$$

The above series converges absolutely and uniformly on compact sets of the complex plane.

Proof. Let $\lambda,\lambda'\in\mathbb{C}$ be such that none of them is an eigenvalue. Replacing $y(x)$ and $z(x)$ in the Lagrange-type identity (3.15) by $\Phi(x,\lambda)$ and $\Phi^*(x,\lambda')$, we obtain

$$\begin{aligned}\int_{-1}^{1}\ell(\Phi(x,\lambda))\overline{\Phi}^*(x,\lambda')\,dx &= \sum_{i=1}^{2n} V_i(\Phi(x,\lambda))V^*_{2n-i+1}(\overline{\Phi}^*(\cdot,\lambda'))\\ &\quad+\sum_{i=1}^{n} U_i(\Phi(\cdot,\lambda))U^*_{2n-i+1}(\overline{\Phi}^*(\cdot,\lambda'))\\ &\quad+\int_{-1}^{1}\Phi(x,\lambda)\overline{\ell^*(\Phi^*(x,\lambda))}\,dx.\end{aligned} \tag{4.15}$$

Using the fact that $\Phi(\cdot,\lambda)$ and $\Phi^*(\cdot,\lambda)$ solve $\ell(y)=\lambda y$ and $\ell^*(y)=\lambda y$, respectively, we obtain

$$\begin{aligned}(\lambda-\overline{\lambda}')\int_{-1}^{1}\Phi(x,\lambda)\overline{\Phi}^*(x,\lambda')\,dx &= \sum_{i=1}^{2n} V_i(\Phi(\cdot,\lambda))V_{2n-i+1}(\overline{\Phi}^*(\cdot,\lambda'))\\ &\quad+\sum_{i=1}^{n} U_i(\Phi(\cdot,\lambda))U^*_{2n-i+1}(\overline{\Phi}^*(\cdot,\lambda')).\end{aligned} \tag{4.16}$$

From the definition of $\Phi(\cdot,\lambda)$ and the linearity of $V_i(\cdot)$ and $U_i(\cdot)$, $i=1,2,\dots,n$, we obtain for $i\in\{2,\dots,n\}$,

$$U_i(\Phi(\cdot,\lambda)) = U_{i,-1}(\mathcal{K}_1(\cdot,\lambda)) + U_{i,1}(\mathcal{K}_2(\cdot,\lambda)). \tag{4.17}$$

But from the rules for addition of determinants, we have

$$\begin{aligned}
U_i(\Phi(\cdot,\lambda)) \;&=\; \begin{vmatrix} U_{i,-1}(y_{11}) & \dots & U_{i,-1}(y_{1n}) & 0 & \dots & 0 \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix} \\
&+\; \begin{vmatrix} 0 & \dots & 0 & U_{i,1}(y_{21}) & \dots & U_{i,1}(y_{2n}) \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix} \qquad (4.18) \\
&=\; \begin{vmatrix} U_{i,-1}(y_{11}) & \dots & U_{i,-1}(y_{1n}) & U_{i,1}(y_{21}) & \dots & U_{i,1}(y_{2n}) \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix}.
\end{aligned}$$

Thus

$$U_i(\Phi(\cdot,\lambda)) = \begin{cases} 0 & , \quad i=2,3,\dots,n, \\ \triangle(\lambda) & , \quad i=1. \end{cases} \tag{4.19}$$

Hence from the second sum of (4.16), it remains only $\triangle(\lambda)U^*_{2n}(\overline{\Phi}(\cdot,\lambda'))$. Similarly we can prove that

$$V_i(\Phi(\cdot,\lambda)) = V^*_i(\overline{\Phi}^*(\cdot,\lambda')) = 0, \quad i=1,2,\dots,n. \tag{4.20}$$

Combining (4.16), (4.19) and (4.20) one obtains.

$$(\lambda-\overline{\lambda}')\int_{-1}^{1}\Phi(x,\lambda)\overline{\Phi}^*(x,\lambda')\,dx = \triangle(\lambda)U^*_{2n}(\overline{\Phi}^*(\cdot,\lambda')). \tag{4.21}$$

Let $\lambda'\to\overline{\lambda}_k$ for some fixed k, then

$$(\lambda-\lambda_k)\int_{-1}^{1}\Phi(x,\lambda)\overline{\Phi}^*(x,\overline{\lambda}_k)\,dx = \triangle(\lambda)U^*_{2n}(\overline{\Phi}^*(\cdot,\overline{\lambda}_k)). \tag{4.22}$$

Obviously $U_{2n}^*(\overline{\Phi}^*(\cdot,\overline{\lambda}_k)) \neq 0$ since otherwise we have

$$\int_{-1}^{1} \Phi(x,\lambda_k)\overline{\Phi}^*(x,\overline{\lambda}_k)\,dx = 0, \tag{4.23}$$

contradicting the biorthogonality of the eigenfunctions $\Phi(\cdot,\lambda_k)$ and $\Phi^*(\cdot,\overline{\lambda}_k)$, see [25] p. 118. From (4.22) one obtains

$$\int_{-1}^{1} \Phi(x,\lambda_k)\overline{\Phi}^*(x,\lambda_k)_k)\,dx = \triangle'(\lambda_k)U_{2n}^*(\overline{\Phi}^*(\cdot,\overline{\lambda}_k)). \tag{4.24}$$

Since both $\{\Phi(\cdot,\lambda_k)\}_{k=0}^{\infty}$ and $\{\Phi^*(\cdot,\overline{\lambda}_k)\}_{k=0}^{\infty}$ are Riesz bases of $L^2[-1,1]$ and since they are biorthogonal, we have

$$\Phi(x,\lambda) = \sum_{k=0}^{\infty} \frac{\int_{-1}^{1}\Phi(x,\lambda)\overline{\Phi}^*(x,\overline{\lambda}_k^*)\,dx}{\int_{-1}^{1}\Phi(x,\lambda_k)\overline{\Phi}^*(x,\overline{\lambda}_k)\,dx}\,\Phi(x,\lambda_k), \tag{4.25}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{\int_{-1}^{1} f(x)\Phi(x,\lambda_k)\,dx}{\int_{-1}^{1}\Phi(x,\lambda_k)\overline{\Phi}^*(x,\overline{\lambda_k})\,dx}\,\overline{\Phi}^*(x,\overline{\lambda}_k). \tag{4.26}$$

Thus, from (4.13), (4.25) and (4.26) we get, using biorthogonality,

$$F(\lambda) = \sum_{k=0}^{\infty} F(\lambda_k)\,\frac{\int_{-1}^{1}\Phi(x,\lambda)\overline{\Phi}^*(x,\overline{\lambda}_k)\,dx}{\int_{-1}^{1}\Phi(x,\lambda_k)\overline{\Phi}^*(x,\overline{\lambda}_k)\,dx}. \tag{4.27}$$

The sampling representation (4.14) results by combining (4.27) together with (4.22) and (4.24), with a pointwise convergence on $\mathbb{C}$. We omit details for the proof of the statements concerning absolute and uniform convergence as well as the analyticity of $F(\lambda)$; the reader may consult [4] or [3], where detailed proofs are given for analogous statements. $\square$
A similar result holds for the transformation

$$G(\lambda) = \int_{-1}^{1} f(x)\Phi^*(x,\lambda)\,dx, \quad f \in L^2(-1,1). \tag{4.28}$$

Here the sampling points will be $\{\overline{\lambda}_k\}_{k=0}^{\infty}$ instead of $\{\lambda_k\}_{k=0}^{\infty}$ and the function $\triangle(\lambda)$ of (4.14) should be replaced by

$$\triangle^*(\lambda) := \begin{vmatrix} U_{1,-1}(z_{11}) & \dots & U_{1,-1}(z_{1n}) & U_{1,1}(z_{21}) & \dots & U_{1,1}(z_{2n}) \\ U_{2,-1}(z_{11}) & \dots & U_{2,-1}(z_{1n}) & U_{2,1}(z_{21}) & \dots & U_{2,1}(z_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(z_{11}) & \dots & V_{n,0^-}(z_{1n}) & V_{n,0^+}(z_{11}) & \dots & V_{n,0^+}(z_{2n}) \end{vmatrix}. \tag{4.29}$$

Also it makes no difference if we replace the interval $[-1,1]$ by $[a,b]$, $-\infty < a < b < \infty$, and 0 by any $c \in (a,b)$. The differential expressions ℓ_1, ℓ_2 may also be taken to be identical.

5 WORKED EXAMPLES

The present section contains some examples illustrating the previous sections. Example 5.1, 5.2 and 5.4 are direct applications of the sampling theorem introduced in the previous section. While the first two examples include first order problems, Example 5.4 is a second order one. Example 5.3 indicates how to obtain the adjoint problem of a second order problem using the extended Lagrange's identity of Section 3 above. The last example, Example 5.5 introduces a challenging problem which cannot be treated in the same manner of Sections 3 and 4 above. In this example the strong regularity condition is omitted, so the eigenfunctions of the problem, and hence those of the adjoint one, are no longer bases.

Example 5.1. Let

$$\ell_1(y) = \ell_2(y) = \frac{dy}{dx}. \tag{5.1}$$

Consider also the boundary and compatibility conditions

$$U_1(y) = y(-1) - y(1) = 0, \tag{5.2}$$

$$V_1(y) = y(0^-) + ay(0^+) = 0, \quad a \neq 0. \tag{5.3}$$

For $\lambda \neq 0$ a fundamental set of solutions of (5.1) is (in the notation of Section 3)

$$y_{11}(x,\lambda) = e^{\lambda x}, \quad -1 \leq x < 0, \quad y_{21}(x,\lambda) = e^{\lambda x}, \quad 0 < x \leq 1. \tag{5.4}$$

Then

$$\triangle(\lambda) = \begin{vmatrix} U_{1,-1}(y_{11}) & U_{1,1}(y_{21}) \\ V_{1,0^-}(y_{11}) & V_{1,0^+}(y_{21}) \end{vmatrix} = \begin{vmatrix} e^{-\lambda} & -e^{\lambda} \\ 1 & a \end{vmatrix} = e^{\lambda} + ae^{-\lambda}. \tag{5.5}$$

Thus the boundary conditions are strongly regular. The eigenvalues of problem (5.1)–(5.3) are the zeros of $\triangle(\lambda)$, i.e. eigenvalues occur when $e^{2\lambda} = -a$. Hence the eigenvalues of the problem are for $a \neq -1$

$$\lambda_k = \frac{1}{2}[\ln(-a) + 2k\pi i], \quad k \in \mathbb{Z}. \tag{5.6}$$

Obviously only when $a = -1$, $\lambda_0 = 0$ is an eigenvalue with a constant eigenfunction. Every eigenvalue $\lambda_k \neq 0$ is simple and has an eigenfunction

$$\phi_k(x) = \begin{cases} \phi_1(x) & , \quad -1 \leq x < 0, \\ \phi_2(x) & , \quad 0 < x \leq 1, \end{cases} \tag{5.7}$$

where

$$\phi_1(x) = \begin{vmatrix} y_{11}(x,\lambda_k) & 0 \\ V_{1,0^-}(y_{11}) & V_{1,0^+}(y_{21}) \end{vmatrix} = \begin{vmatrix} e^{\lambda_k x} & 0 \\ 1 & a \end{vmatrix} = ae^{\lambda_k x}; \tag{5.8}$$

$$\phi_2(x) = \begin{vmatrix} 0 & e^{\lambda_k x} \\ 1 & a \end{vmatrix} = -e^{\lambda_k x}. \tag{5.9}$$

The set $\{\phi_k(\cdot)\}_{k\in\mathbb{Z}}$ is a Riesz basis of $L^2[-1,1]$. The kernel $\Phi(\cdot,\lambda)$ of (4.9) will have the form

$$\Phi(x,\lambda) = \begin{cases} ae^{\lambda x}, & -1 \le x < 0, \\ e^{\lambda x}, & 0 < x \le 1, \end{cases} \tag{5.10}$$

and the sampling series associated with this problem states that the integral transformation

$$F(\lambda) = a\int_{-1}^{0} f_1(x)e^{\lambda x}\,dx + \int_0^1 f_2(x)e^{\lambda x}\,dx, \quad f \in L^2[-1,1], \tag{5.11}$$

where f_1 and f_2 are the restrictions of f to $[-1,0)$ and $(0,1]$ respectively, has for $a \neq -1$ the sampling representation

$$F(\lambda) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} F(\lambda_k)\ \frac{e^{\lambda} + ae^{-\lambda}}{(\lambda-\lambda_k)(e^{\lambda_k} - ae^{-\lambda_k})}, \quad \lambda \in \mathbb{C}. \tag{5.12}$$

In the special case $a = -1$, the sampling series of (5.1) is

$$F(\lambda) = \sum_{k=-\infty}^{\infty} (-1)^k F(ik\pi)\ \frac{\sinh\lambda}{(\lambda - ik\pi)}. \tag{5.13}$$

The above example simply represents the way how the main result of this paper can be applied. But in this example, the sampling points are complex numbers. This is due to the non-selfadjointness of the problem. We could easily remove this difficulty as we do in the next example.

Example 5.2. Consider the following problem of the type introduced in Section 3 above,

$$\ell_1(y) = \ell_2(y) = -i\frac{dy}{dx}, \tag{5.14}$$

$$U_1(y) = y(-1) - y(1) = 0, \quad V_1(y) = y(0^-) - y(0^+) = 0. \tag{5.15}$$

Similar to the previous example it follows that $\lambda_0 = 0$ is an eigenvalue with a constant eigenfunction. A fundamental set of (5.14) is for $\lambda \neq 0$

$$y_{11}(x,\lambda) = e^{i\lambda x}, \quad -1 \le x < 0, \quad y_{21}(x,\lambda) = e^{i\lambda x}, \quad 0 < x \le 1. \tag{5.16}$$

Hence (also for $\lambda \neq 0$)

$$\triangle(\lambda) = \begin{vmatrix} U_{1,-1}(y_{11}) & U_{1,1}(y_{21}) \\ V_{1,0^-}(y_{11}) & V_{1,0^+}(y_{21}) \end{vmatrix} = \begin{vmatrix} e^{-i\lambda} & -e^{i\lambda} \\ 1 & -1 \end{vmatrix} = e^{i\lambda} - e^{-i\lambda} = 2i\sin\lambda. \tag{5.17}$$

Thus the eigenvalues of (5.14)–(5.15) are $\lambda_k = k\pi,\ k \in \mathbb{Z}$. The corresponding eigenfunctions are

$$\phi_k(x) = e^{i\lambda_k x} = e^{ik\pi x}, \quad -1 \le x \le 1, \quad k \in \mathbb{Z}. \tag{5.18}$$

Notice that, according to (5.17), the problem is strongly regular. Also it is not difficult to see that the problem is self adjoint. The kernel $\Phi(\cdot, \lambda)$ will be in this case

$$\Phi(x, \lambda) = e^{i\lambda x}, \quad -1 \le x \le 1, \tag{5.19}$$

and the sampling result associated with (5.14) - (5.15) will be for the Fourier transformation

$$F(\lambda) = \int_{-1}^{1} f(x)\, e^{i\lambda x}\, dx, \quad f(\cdot) \in L^2(-1, 1), \tag{5.20}$$

and has the form

$$F(\lambda) = \sum_{k=-\infty}^{\infty} f(k\pi)\ \frac{\sin(\lambda - k\pi)}{\lambda - k\pi}. \tag{5.21}$$

The last series is the celebrated sampling series of Whittaker and Shannon, [31, 28], a similar result is obtained if $V_1(y)$ has the more general form (5.3).

We would like to mention that the treatment of sampling associated with general first order problems is, to the best we know, due to Campbell, [10]. A more recent study based on the construction of Kramer analytic kernels, see [17, 13], is carried out by Everitt and Poulkou, [16].

Example 5.3. Let $a, b \in \mathbb{R}, \quad a \neq 0 \neq b$. We will find the problem adjoint to the following problem:

$$\ell_1(y) = \ell_2(y) = \frac{d^2 y}{dx^2}, \tag{5.22}$$

$$U_1(y) = y(-1) = 0, \quad U_2(y) = y(1) = 0, \tag{5.23}$$

$$V_1(y) = y(0^-) + a y(0^+) = 0, \quad V_2(y) = y'(0^-) + b y'(0^+) = 0. \tag{5.24}$$

We complement $U_1(\cdot)$ and $U_2(\cdot)$ by

$$U_3(y) := y'(1), \quad U_4(y) := y'(-1). \tag{5.25}$$

So we have the system $(U_i(\cdot))_{i=1}^4$ of linearly independent forms. Similarly, we complement $V_1(\cdot)$ and $V_2(\cdot)$ with

$$V_3(y) = y'(0^+), \quad V_4(y) = y(0^+). \tag{5.26}$$

Using integration by parts,

$$\begin{aligned} \int_{-1}^{1} y''(x)\overline{z}(x)\, dx &= y'(0^-)\overline{z}(0^-) - \overline{z}'(0^-)y(0^-) - y'(-1)\overline{z}(-1) \\ &\quad + \overline{z}'(-1)y(-1) + y'(1)\overline{z}(1) - y(1)\overline{z}'(1) - y'(0^+)\overline{z}(0^+) \\ &\quad + \overline{z}'(0^+)y(0^+) + \int_{-1}^{1} y(x)\overline{z}''(x)\, dx. \end{aligned} \tag{5.27}$$

Equation (5.27) leads to

$$\begin{aligned}\int_{-1}^{1} y''(x)\overline{z}(x)\,dx &= y(-1)\overline{z}'(-1) + y(1)(-\overline{z}'(1)) + y'(1)\overline{z}(1) + y'(-1)(-\overline{z}'(-1)) \\ &\quad +(y(0^-) + ay(0^+))(-\overline{z}'(0^-)) + (y'(0^-) + by'(0^+))\overline{z}(0^-) \\ &\quad +(-y'(0^+))b\overline{z}(0^-) + \overline{z}(0^+)) \\ &\quad +y(0^+)(a\overline{z}'(0^-) + \overline{z}(0^+)) \\ &\quad +\int_{-1}^{1} y(x)\overline{z}''(x)\,dx.\end{aligned} \tag{5.28}$$

This means that the problem adjoint to (5.22)–(5.23) is defined by

$$\ell_1^*(y) = \ell_2^*(y) = \frac{d^2y}{dx^2}, \tag{5.29}$$

$$U_1^*(y) = y(-1) = 0, \quad U_2^*(y) = y(1) = 0, \tag{5.30}$$

$$V_1^*(y) = y(0^-) + \frac{1}{b}y(0^+) = 0, \quad V_2^*(y) = y'(0^-) + \frac{1}{a}y'(0^+). \tag{5.31}$$

Hence the problem will be self adjoint only if $ab = 1$.

The above example indicates that it is not difficult to compute the adjoint problem since the classical technique is used with the necessary changes.

Example 5.4 Consider a boundary value problem of the type introduced in the previous example with $a = b = 1$, i.e. a self adjoint problem. Let $\lambda = \rho^2$. Then we have for $\lambda \neq 0$ the fundamental sets

$$y_{11}(x,\lambda) = e^{-\rho x}, \quad y_{12}(x,\lambda) = e^{\rho x}, \quad 1 \leq x < 0; \tag{5.32}$$

$$y_{21}(x,\lambda) = e^{-\rho x}, \quad y_{22}(x,\lambda) = e^{\rho x}, \quad 0 < x \leq 1. \tag{5.33}$$

Hence

$$\triangle(\lambda) = \begin{vmatrix} e^{\rho} & e^{-\rho} & 0 & 0 \\ 0 & 0 & e^{-\rho} & e^{\rho} \\ 1 & 1 & 1 & 1 \\ -\rho & \rho & -\rho & \rho \end{vmatrix} = 2\rho\left[-e^{2\rho} + e^{-2\rho}\right] = 4\rho\sinh 2\rho. \tag{5.34}$$

For $\rho \neq 0$ the zeros of $\sinh 2\rho$ are $\rho_k = i\frac{k\pi}{2}$, $k \in \mathbb{Z} \setminus \{0\}$, hence the nonzero eigenvalues are

$$\lambda_k = (\rho_k)^2 = -\frac{k^2\pi^2}{4}, \quad k = 1, 2 \ldots. \tag{5.35}$$

The case when $\lambda = 0$ has to be considered separately, when $\lambda = 0$ we have the fundamental sets

$$y^0_{11}(x,0) = x, \quad y^0_{12}(x,0) = 1, \quad -1 \le x < 0, \tag{5.36}$$

$$y^0_{21}(x,0) = x, \quad y^0_{22}(x,0) = 1, \quad 0 < x \le 1. \tag{5.37}$$

Zero will be an eigenvalue if $\triangle(0) = 0$. But

$$\triangle(0) = \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix} == - \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2 \neq 0. \tag{5.38}$$

Hence zero is not an eigenvalue and the eigenvalues are only $\lambda_k = -\frac{k^2\pi^2}{4}, \quad k = 1, 2, \ldots$. The corresponding eigenfunctions are

$$\phi_k(x) = \begin{cases} \phi_{k1}(x), & -1 \le x < 0, \\ \phi_{k2}(x), & 0 < x \le 1, \end{cases} \tag{5.39}$$

where

$$\phi_{k1}(x) = \begin{vmatrix} e^{-ik\frac{\pi}{2}x} & e^{ik\frac{\pi}{2}x} & 0 & 0 \\ 0 & 0 & e^{-ik\frac{\pi}{2}} & e^{ik\frac{\pi}{2}} \\ 1 & 1 & 1 & 1 \\ -ik\frac{\pi}{2} & ik\frac{\pi}{2} & -ik\frac{\pi}{2} & ik\frac{\pi}{2} \end{vmatrix} = c_k \sin\frac{k\pi(1-x)}{2}. \tag{5.40}$$

Similarly

$$\phi_{k2}(x) = -c_k \sin\frac{k\pi(1-x)}{2}, \quad k = 1, 2, \ldots. \tag{5.41}$$

Here

$$c_k = 2k\pi, \qquad k = 1, 2, \ldots. \tag{5.42}$$

Notice that the eigenfunctions corresponding to the eigenvalues $\lambda_{2k}, \ k \in I\!N$, are discontinuous in $x = 0$. Now we are ready for formulating the sampling series associated with this example. Let $\Phi(x,\lambda)$ be the function

$$\Phi(x,\lambda) = \begin{cases} \mathcal{K}_1(x,\lambda) & , \quad -1 \le x < 0, \\ \mathcal{K}_2(x,\lambda) & , \quad 0 < x \le 1, \end{cases} \tag{5.43}$$

where

$$\mathcal{K}_1(x,\lambda) = \begin{vmatrix} e^{-\rho x} & e^{\rho x} & 0 & 0 \\ 0 & 0 & e^{-\rho} & e^{\rho} \\ 1 & 1 & 1 & 1 \\ -\rho & \rho & -\rho & \rho \end{vmatrix} \tag{5.44}$$

$$= 2\rho(e^{-\rho(1-x)} - e^{\rho(1-x)})$$

$$= -4\rho \sinh(1-x)\rho, \qquad -1 \le x < 0.$$

Similarly

$$\mathcal{K}_2(x,\lambda) = 4\rho \sinh(1-x)\rho, \qquad 0 < x \le 1. \tag{5.45}$$

Hence the transformation

$$F(\lambda) = -\int_{-1}^{0} f(x) 4\rho \sinh(1-x)\rho\, dx + \int_{0}^{1} f(x) 4 \sinh(1-x)\rho\, dx, \tag{5.46}$$

where $f(x) \in L^2[-1,1]$ can be reconstructed via

$$F(\lambda) = \sum_{k=1}^{\infty} (-1)^{k+1} F\left(-\frac{k^2\pi^2}{4}\right) \frac{4\sqrt{\lambda}\sinh 2\sqrt{\lambda}}{4\lambda + k^2\pi^2}. \tag{5.47}$$

Example 5.5. This example shows that there are several problems where the above technique cannot be applied. These problems are not strongly regular problems. An example is the following:

$$\ell_1(y) = \ell_2(y) = y'' = \lambda y, \tag{5.48}$$

$$U_1(y) = y(-1) = 0, \quad U_2(y) = y'(-1) + y(1) = 0, \tag{5.49}$$

$$V_1(y) = y(0^-) + y(0^+) = 0, \quad V_2(y) = y'(0-) + y'(0+) = 0. \tag{5.50}$$

In this case if $\lambda = \rho^2$, then

$$\triangle(\lambda) = -2\rho^2 e^{-2\rho}[e^{2\rho} - 1]^2. \tag{5.51}$$

Thus

$$\theta_0 = 4, \quad \theta_1 = -2, \quad \theta_{-1} = -2, \tag{5.52}$$

i.e. $\theta_0^2 = 4\theta_1\,\theta_{-1}$. So the problem is regular since none of the θ's is zero and it is not strongly regular. Among the difficulties this problem has is that although we can compute the eigenvalues, namely $\rho_k = ik\pi$, these eigenvalues are simple geometrically but have multiplicity two as zeros of $\triangle(\lambda)$. This is one of the major properties of the eigenvalues of problem (3.1) and its adjoint which plays an important role in the derivation of the sampling theorem. Another property which has been lost in this problem is that the eigenfunctions are no more a basis. We will consider such problems separately.

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