Regularity of elliptic systems

Michael Pingen

Universidad de Granada
Facultad de Ciencias
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Lecture 1:

Hölder continuity of weak solutions of certain quasilinear elliptic systems

1 Introduction

We consider weak solutions of systems of elliptic partial differential equations of second order in diagonal form

$$Au = -D_\alpha\left(a^{\alpha\beta}(x,u,\nabla u)D_\beta u^i\right) = f^i(x,u,\nabla u) \quad (i=1,\dots,m) \tag{1}$$

in a domain $\Omega \subset \mathbb{R}^n$.

To give a definition of a weak solution of (1) we define the Sobolev space $H_2^1(\Omega,\mathbb{R}^m) := \{u : \Omega \to \mathbb{R}^m; u \in L_2(\Omega,\mathbb{R}^m), Du \in L_2(\Omega,\mathbb{R}^m)\}$, where Du denotes the weak derivative of u. Furthermore let
$\mathring{H}_2^1(\Omega,\mathbb{R}^m) := \{u : \Omega \to \mathbb{R}^m; u \in L_2, Du \in L_2, u|_{\partial\Omega} = 0\}$.

Definition 1 *u is called a weak solution of $Au = f$, if u satisfies the relation*

$$\int_\Omega a^{\alpha\beta}(x,u,\nabla u)D_\alpha u \cdot D_\beta\phi\,dx = \int_\Omega f(x,u,\nabla u)\cdot\phi\,dx \tag{2}$$

for alle $\phi \in \mathring{H}_2^1 \cap L_\infty(\Omega,\mathbb{R}^m)$.

We assume that $f^i(x,u,\nabla u)$ and $a^{\alpha\beta}(x,u,\nabla u)$ are measurable and require that u, f und $a^{\alpha\beta}$ satisfy the following structure conditions

i) $\sup_\Omega |u| \le \mathrm{M} < \infty$.

ii) $\lambda|\xi|^2 \le a^{\alpha\beta}\xi_\alpha\xi_\beta \le \mu|\xi|^2$ for $\mu > \lambda > 0$ and for all $\xi \in \mathbb{R}^n$.

iii) $|f(x,u(x),p)| \le aQ(x,p)$
$u(x)\cdot f(x,u(x),p) \le a^*Q(x,p)$
for all $x \in \Omega$ and for all $p \in \mathbb{R}^{n\times m}$ with some $a \ge 0$ and $a^* \in \mathbb{R}$, where $Q(x,p) := a^{\alpha\beta}(x)p_\alpha^i p_\beta^i$.

Remark:

- The second condition of iii) is always satisfied with $a^* = aM$, therefore we assume w.l.o.g. $a^* \le aM$.
- The notations $\sup_\Omega |u|$, $\inf_\Omega |u|$ etc. are to understand as essential Supremum etc., i.e. $\sup_\Omega |u| = ess\sup_\Omega |u|$ etc.
- In the whole course we use the summation convention, i.e. we sum over greek indices $\alpha, \beta, \dots$ from 1 to n and over latin indices $i, j, \dots$ from 1 to m.

We will also use the notations $B_r(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < r\}$ and

$$\overline{u} = \frac{1}{|B_r|} \int\limits_{B_r} u\,dx = \frac{1}{\omega_n r^n} \int\limits_{B_r} u\,dx = \fint\limits_{B_r} u\,dx \quad \text{with } \omega_n = |B_1|,\ B_1 \subset \mathbb{R}^n.$$

We will show that every bounded, weak solution of (1) under the condition

$$a^* + aM < 2$$

is Hölder continuous and fulfills an a priori estimate of the Hölder norm, more precisely we will prove:

Theorem 1 *Let u be a bounded weak solution of (1) in Ω. Let the above structure conditions i)-iii) be fulfilled and assume $a^* + aM < 2$. Then $u \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ which depends on $n, a, a^*, M, \lambda, \mu$. For every $\Omega' \subset\subset \Omega$ we have the estimate*

$$\sup_{\substack{x,y \in \Omega' \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C(n, a, a^*, M, \lambda, \mu, \Omega, \Omega').$$

This result has been established by Wiegner [WI1], [WI2] and Hildebrandt-Widman [HW2] in the years 1975-1977. All these authors proved this regularity result with the use of Green's functions. A more simple proof of Theorem 1 was given by Luis Caffarelli [CA] in 1982. He uses a weak Harnack inequality for supersolutions of a linear elliptic equation due to Moser [MO] (or [GT], Thm. 8.18) and completely avoids the use of Green's functions. Section 2 handles with his proof of Theorem 1.

Remarks:

1) In the case $n = 2$ a result due to Wiegner [WI3] shows that the condition $a^* + a \sup\limits_\Omega |u| < 2$ in Theorem 1 can be replaced by the weaker condition $a^* < 1$. In the case $n = 2$ this condition is the best possible. Frehse [FR] constructs a system and a bounded, weak solution u with $a^* = 1$ such that u is not continuous.

2) If $n \geq 3$ it's not possible to weaken the condition $a^* + a \sup\limits_\Omega |u| < 2$ through $a^* < 1$, as a counterexample of Struwe [ST] shows.

3) The condition $a^* + a \sup\limits_\Omega |u| < 2$ in Theorem 1 is the best possible for $n \geq 3$. If $a^* + a \sup\limits_\Omega |u| \leq 2$ there exist bounded, weak and discontinuous solutions, e.g. the function $u^i(x) = \frac{x^i}{|x|}$ $(i = 1, ..., n, x \in \mathbb{R}^n, n \geq 3)$ is a weak solution of the system $-\Delta u^i = u^i |\nabla u|^2$ (cf.[HW1], p.68).

For this system we have $a^* = 1, a = 1$ and $\sup|u| = 1 \to a^* + a\sup|u| = 2$.

4) With the idea of Caffarelli it's also possible to show a *Liouville theorem* for bounded, weak solutions of (1) in $\mathbb{R}^n$. This means, that every bounded, weak solution of a system (1) with the property $a^* + aM < 2$ in the whole $\mathbb{R}^n$ is a constant. This result was originally proven by Hildebrandt-Widman [HW3] with the use of Green's functions. Meier [ME2] found a proof of the Liouville theorem which bases on Caffarelli's idea to use a weak Harnack inequality for supersolutions of a elliptic equation and so he completely avoids the use of Green's functions. It is not possible to replace the condition $a^* + aM < 2$ by $a^* \leq 1$ if $n \geq 3$ (cf. [ME1]).

2 Proof of the Main Result

Before we will start with the proof of Theorem 1 we will define what we understand under a weak supersolution and state the crucial Harnack inequality.

Definition 2 *$u \in H_2^1(\Omega)$ is a weak subsolution of the elliptic equation $Lu = D_\alpha\left(a^{\alpha\beta}(x)D_\beta u\right) = 0$ if*

$$\int_\Omega a^{\alpha\beta}(x)D_\beta u D_\alpha\varphi\, dx \leq 0 \qquad \forall\, \varphi \in \mathring{H}_2^1 \cap L_\infty(\Omega), \varphi \geq 0.$$

u is a weak supersolution of $Lu = 0$ if $-u$ is a weak subsolution.

Lemma 1 *Let u be a weak supersolution of the equation $Lu = D_\alpha(a^{\alpha\beta}(x)D_\beta u) = 0$ in Ω, the coefficients $a^{\alpha\beta}(x)$ are required to be uniformly elliptic. Furthermore we assume that u is non-negative in a ball $B_{4R}(y) \subset \Omega$. Then we have the estimate*

$$R^{-n}||u||_{1,B_{2R}(y)} \leq C(n,\lambda,\mu)\inf_{B_R(y)} u.$$

Proof. see [GT, p.195-198].

We have (cf. [CA, Lemma1]):

Lemma 2 *Let u be a bounded, weak solution of (1) in B_{4R}, and let the above structure conditions i)-iii) be fulfilled and assume $a^* + aM = 2l < 2$. Then there is a constant $\delta(n, a, a^*, M, \lambda, \mu) \in (0,1)$, such that the estimate*

$$u(B_R) \subset B_{M(1-\delta)}(\delta\overline{u})$$

holds.

Proof. Calculate $A\left(\frac{1}{2}|u|^2\right)$. In the weak formulation we find:

$$\int\limits_\Omega a^{\alpha\beta}(x,u\nabla u)D_\alpha\left(\frac{1}{2}\sum_{i=1}^n (u_i)^2\right)D_\beta\phi\,dx = -\int\limits_\Omega a^{\alpha\beta}(x,u\nabla u)D_{\alpha\beta}\left(\frac{1}{2}\sum_{i=1}^n (u_i)^2\right)\phi\,dx$$

$$= -\int\limits_\Omega a^{\alpha\beta}(x,u,\nabla u)D_\alpha\left(u_i\cdot D_\beta u_i\right)\phi\,dx = -\int\limits_\Omega a^{\alpha\beta}(x,u,\nabla u)\left[D_\alpha u_i\cdot D_\beta u_i + u_i\cdot D_{\alpha\beta}u_i\right]\phi\,dx$$

$$= -\int\limits_\Omega a^{\alpha\beta}(x,u,\nabla u)D_\alpha u_i\cdot D_\beta u_i\phi\,dx - \int\limits_\Omega a^{\alpha\beta}(x,u,\nabla u)u_i D_{\alpha\beta}u_i\phi\,dx$$

$$= -\int\limits_\Omega Q(x,\nabla u)\phi\,dx + \int\limits_\Omega u_i a^{\alpha\beta}(x,u,\nabla u)D_\alpha u_i D_\beta\phi\,dx$$

$$= -\int\limits_\Omega Q(x,\nabla u)\phi\,dx + \int\limits_\Omega u_i\cdot f_i(x,u,\nabla u)\phi\,dx,$$

therefore: $A\left(\frac{1}{2}|u|^2\right) = u\cdot f - Q(x,\nabla u)$.

Now let $\xi\in\mathbb{R}^m$ with $|\xi|\le\frac{1-l}{a}$, it follows:

$$A\left(\frac{1}{2}|u|^2+\xi\cdot u\right)\le 0 \tag{3}$$

because:

a) $A\left(\frac{1}{2}|u|^2\right) = u\cdot f - Q(x,\nabla u)\le(a^*-1)Q(x,\nabla u)\le(l-1)Q(x,\nabla u)$, since $a^*\le l$ ($a^*\le aM$ and $a^*+aM\le 2l$)

b) $A(\xi\cdot u) = \xi\cdot Au\le|\xi||f|\le a|\xi|Q(x,\nabla u)\le a\frac{1-l}{a}Q(x,\nabla u)$ $= (1-l)Q(x,\nabla u)\Rightarrow A(\xi\cdot u)\le(1-l)Q(x,\nabla u)$

(3) implies that the scalar function $h(x) := \frac{1}{2}M^2+\frac{1-l}{a}M-\frac{1}{2}|u|^2-\xi\cdot u$ is a non-negative supersolution of A, since

$$Ah = A\left(-\frac{1}{2}|u|^2-\xi\cdot u\right) = -A\left(\frac{1}{2}|u|^2+\xi\cdot u\right)\ge 0.$$

For the mean value of h we have the estimate $\overline{h}\ge\frac{1-l}{a}M-\xi\cdot\overline{u}$. Now the weak Harnack inequality (Lemma 1) implies

$$h(x)\ge\inf h(x)\ge\delta_1\left(n,\lambda,\mu\right)\overline{h}(x)\ge\delta_1\left[\frac{1-l}{a}M-\xi\cdot\overline{u}(x)\right]\quad\forall\,x\in B_R.$$

We infer

$$h(x) = \frac{1}{2}\left(M^2-|u(x)|^2\right)+\frac{1-l}{a}M-\xi\cdot u(x)\ge\delta_1\left[\frac{1-l}{a}M-\xi\cdot\overline{u}(x)\right] \tag{4}$$

Choose ξ in the direction of u and let $|\xi| = \frac{1-l}{a}$; define θ as the angel between u and $\overline{u}$, set $r := \frac{|u|}{M}$. With these notations we have the relations:

$$\xi\cdot u = |\xi||u|\cos 0 = |\xi||u|\quad\text{and}\quad\xi\cdot\overline{u} = |\xi||\overline{u}|\cos\theta$$

Together with (4) it follows

$$\begin{aligned}
&\tfrac{1}{2}\left(M-|u|\right)\left(M+|u|\right)+\tfrac{1-l}{a}M-|\xi|rM \\
&=\tfrac{1}{2}\left(M-|u|\right)\left(M+|u|\right)+\tfrac{1-l}{a}(1-r)M \qquad \left(|\xi|=\tfrac{1-l}{a}\right)\\
&=\tfrac{1}{2}M(1-r)\left(M+|u|\right)+\tfrac{1-l}{a}(1-r)M \\
&=M(1-r)\left[\tfrac{1}{2}\left(M+|u|\right)+\tfrac{1-l}{a}\right] \\
&\geq \delta_1\left[\tfrac{1-l}{a}M-\xi\cdot\overline{u}\right] \qquad \text{(here we use (4))}\\
&=\delta_1\left[\tfrac{1-l}{a}M-\tfrac{1-l}{a}|\overline{u}|\cos\theta\right] \\
&=M\delta_1\tfrac{1-l}{a}\left(1-\tfrac{|\overline{u}|}{M}\cos\theta\right),
\end{aligned}$$

this means we arrive at:

$$M(1-r)\left[\frac{1}{2}(M+|u|)+\frac{1-l}{a}\right]\geq M\delta_1\frac{1-l}{a}\left(1-\frac{|\overline{u}|}{M}\cos\theta\right) \tag{5}$$

$$\Rightarrow \qquad (1-r)\frac{1-l}{a}\left[\frac{M+|u|}{2\frac{1-l}{a}}+1\right]\geq\delta_1\frac{1-l}{a}\left(1-\frac{|\overline{u}|}{M}\cos\theta\right)$$

We infer

$$1-\frac{|u|}{M}=1-r\geq\frac{\delta_1}{\frac{M+|u|}{2\frac{1-l}{a}}+1}\left(1-\frac{|\overline{u}|}{M}\cos\theta\right)$$

$$\geq\frac{\delta_1}{\frac{aM}{1-l}+1}\left(1-\frac{|\overline{u}|}{M}\cos\theta\right)\geq\delta_2\left(n,a,a^*,M,\lambda,\mu\right)\left(1-\frac{|\overline{u}|}{M}\cos\theta\right) \tag{6}$$

W.l.o.g. let $\delta_2<1$, (6) tells us that u lies in a ball with center $\overline{u}$ and a radius which is strictly smaller than M. Furthermore (6) implies

$$\begin{aligned}
1-r+r(1-r)&\geq r\delta_2\left(1-\tfrac{|\overline{u}|}{M}\cos\theta\right)+(1-r)\\
\Leftrightarrow 1-r^2&\geq r\delta_2+1-r-\delta_2 r\tfrac{|\overline{u}|}{M}\cos\theta\\
&\geq 1-r(1-\delta_2)-\delta_2 r\tfrac{|\overline{u}|}{M}\cos\theta\\
&\geq\delta_2\left(1-r\tfrac{|\overline{u}|}{M}\cos\theta\right) \qquad (r\leq 1)
\end{aligned}$$

Thus

$$r^2-\delta_2 r\frac{|\overline{u}|}{M}\cos\theta\leq 1-\delta_2 \tag{7}$$

$$\Rightarrow r^2-\delta_2 r\tfrac{|\overline{u}|}{M}\cos\theta+\left(\tfrac{1}{2}\delta_2\tfrac{|\overline{u}|}{M}\right)^2\leq 1-\delta_2+\left(\tfrac{1}{2}\delta_2\right)^2 \qquad \left(\tfrac{|\overline{u}|}{M}\leq 1\right)$$

With $\delta:=\frac{1}{2}\delta_2$ it follows from the last inequality:

$$r^2-2\delta r\tfrac{|\overline{u}|}{M}\cos\theta+\left(\delta\tfrac{|\overline{u}|}{M}\right)^2\leq 1-2\delta+\delta^2=(1-\delta)^2$$

$$\Rightarrow \tfrac{|u|^2}{M^2}-2\delta\tfrac{|u||\overline{u}|}{M^2}\cos\theta+\delta^2\left(\tfrac{|\overline{u}|}{M}\right)^2=\tfrac{1}{M^2}(|u-\delta\overline{u}|^2)\leq(1-\delta)^2,$$ therefore:

$$|u-\delta\overline{u}|^2\leq M^2(1-\delta)^2$$

That's the statement of Lemma 2. □

The next Corollary shows us what happens if the balls B_R become smaller:

Corollary 1 *Let u be a bounded weak solution of (1) in B_{4R}, let the above structure conditions i)-iii) be fulfilled and assume that $a^* + aM = 2l < 2$. Then there exist points $\xi_k \in \mathbb{R}^n$ and radii M_k, such that:*

i) $M_k \leq M(1-\delta)^k$

ii) $|\xi_k| \leq \left[1 - (1-\delta)^k\right] M$

iii) $|\xi_k| + M_k \leq M$, $|u - \xi_k| \leq M_k$ *a.e. in* $B_{4^{1-k}R}$

iv) $u\left(B_{4^{1-k}R}\right) \subset B_{M_k}(\xi_k)$,

δ is the constant appearing in Lemma 2.

Proof. We prove i)-iii) with induction:
induction start: k=0 $\rightarrow$ Choose ξ_0 as center of the ball B_{4R} and $M_0 = M$.
induction step: Assume that for an integer k ξ_k and M_k with the properties i)-iii) have already been constructed. Then $u^{(k)} := u - \xi_k$ is a weak solution of a system of type (1) in $B_{4^{1-k}R}$ with the properties:

$$|f| \leq aQ(x, \nabla u) \;\text{ and }\; u^{(k)} \cdot f = (u - \xi_k) \cdot f \leq (a^* + a|\xi_k|)\, Q(x, \nabla u) \tag{8}$$

Define $a_k^* := a^* + a|\xi_k|$
With iii) ($|\xi_k| + M_k \leq M$) it follows

$$a_k^* + aM_k \leq a^* + a\left(|\xi_k| + M_k\right) \leq a^* + aM = 2l < 2 \tag{9}$$

Furthermore we infer with iii): $u^{(k)} \cdot f \leq |u - \xi_k||f| \leq aM_k Q(x, \nabla u)$. Thus, with (8) and (9): $a_k^* \leq aM_k \Rightarrow a_k^* \leq l < 1$. With Lemma 2 we find:

$$\left|u^{(k)} - \delta\overline{u}^{(k)}\right| \leq (1-\delta)M_k \qquad \text{a.e. in} \quad B_{4^{-k}R} \tag{10}$$

Define now $\xi_{k+1} := \xi_k + \delta\overline{u}^{(k)} = \xi_k + \delta(\overline{u} - \xi_k) = (1-\delta)\xi_k + \delta\overline{u}$ and
$M_{k+1} := \sup\limits_{B_{4^{-k}R}} \left|u^{(k+1)}\right|$. We get:

$$\begin{aligned}
M_{k+1} &= \sup_{B_{4^{-k}R}} \left|u^{(k+1)}\right| = \sup_{B_{4^{-k}R}} |u - \xi_{k+1}| = \sup_{B_{4^{-k}R}} \left|u - \delta\overline{u}^{(k)} - \xi_k\right| \\
&= \sup_{B_{4^{-k}R}} \left|u^{(k)} - \delta\overline{u}^{(k)}\right| \leq (1-\delta)M_k \qquad \text{(with (10))} \\
&\leq (1-\delta)^{k+1} M \qquad \text{(with i))},
\end{aligned}$$

i.e. we have proven i). From the definition of M_k it follows:

$$|u - \xi_{k+1}| = |u^{(k+1)}| \leq M_{k+1} \;\text{ a.e. in } B_{4^{-k}R}$$

this proves the second part of iii).
Since $|\overline{u}| \leq M$ we come with ii) to:

$$|\xi_{k+1}| \leq (1-\delta)|\xi_k| + \delta M \leq (1-\delta)\left[1-(1-\delta)^k\right] M + \delta M$$
$$= (1-\delta)M + \delta M - (1-\delta)^{k+1} M$$
$$= \left[1-(1-\delta)^{k+1}\right] M, \qquad \text{thus ii) is proved.}$$

The first part of iii) we get with i) and ii) as follows:

$$|\xi_{k+1}| + M_{k+1} \leq \left[1-(1-\delta)^{k+1}\right] M + (1-\delta)^{k+1} M = M$$

iv) follows directly from the second part of iii). □

Lemma 3 describes how the oscillation of u behaves in a small ball.

Lemma 3 *Let u be a bounded weak solution of (1) in B_{4R}, let the above structure conditions i) - iii) be fulfilled and assume $a^* + aM = 2l < 2$. Then for any $r \in (0, R]$ we have the estimate*

$$osc_{B_r} u \leq 4M \left(\frac{\rho}{4R}\right)^\alpha,$$

α is defined by $4^{-\alpha} = max\left(1-\delta, \frac{1}{2}\right)$.

Proof. Since $|u| \leq M$, it's obvious that $osc_{B_R} u := \sup\limits_{x,y \in B_R} |u(x) - u(y)| \leq 2M$, furthermore we infer from Corollary 1

$$osc_{B_{4^{1-k}R}} u \leq 2M_k \leq 2M(1-\delta)^k \leq \theta^k 2M$$

with $\theta = 4^{-\alpha} := max\left(1-\delta, \frac{1}{2}\right) \Rightarrow \theta^k = \left(\frac{1}{4^k}\right)^\alpha$, and therefore

$$osc_{B_{4^{1-k}R}} u \leq 4^\alpha \left(\frac{1}{4^{k+1}}\right)^\alpha 2M \leq 4M \left(\frac{1}{4^{k+1}}\right)^\alpha, \tag{11}$$

because $4^\alpha \leq 2$.
Now let $0 < \rho \leq 1$. Then there is a $k \in \mathbb{N}$ with the property $\frac{1}{4^k} < \rho \leq \frac{1}{4^{k-1}}$. Since the oscillation is increasing in ρ, we derive from (11):

$$osc_{B_{\rho R}} u \leq osc_{B_{4^{1-k}R}} u \leq 4M \left(\tfrac{1}{4^{k+1}}\right)^\alpha \leq 4M \left(\tfrac{\rho}{4}\right)^\alpha \qquad \forall\ 0 < \rho \leq 1.$$

With $r := \rho R$ we find the desired estimate $osc_{B_r} u \leq 4M \left(\frac{r}{4R}\right)^\alpha$. □

As a result of Lemma 3 it follows that $osc_{B_r} u \to 0$, $r \to 0$; this means that u is continuous. More precisely: u has got a continuous representative, namely $\overline{u}(x) := \lim\limits_{\rho \to 0} \frac{1}{|B_\rho|} \int\limits_{B_\rho(x)} u(y)\, dy$. It's possible to show $\overline{u} = u$ a.e. in B_R.
(cf. [EG] ,p.43).

Now we are able to prove Theorem 1:

Proof of Theorem 1. Let $B_{9r}(x_0) \subset \Omega$.

First, we will prove the statement for the special case $\Omega' = B_r(x_0)$.

Let $x, y \in B_r(x_0)$ be arbitrary and fix them, set $r' := |x-y| \in [0, 2r)$.

Since $B_{4(2r)}(x) \subset B_{9r}(x_0) \subset \Omega$ we infer from Lemma 3:

$|u(x) - u(y)| \leq osc_{B_{r'}(x)} u \leq 4M \left(\frac{r'}{8r}\right)^{\alpha} \Rightarrow \frac{|u(x)-u(y)|}{|x-y|\alpha} \leq C(n, a, a^*, M, r, \lambda, \Lambda)$.

Next, we prove the assertion for an arbitrary $\Omega' \subset\subset \Omega$. With the theorem of Heine-Borel it follows that there exist finitely many balls $(B_{r_i}(\alpha_i))_{i=1,\ldots,N}$ ($\alpha_i \in \mathbb{R}^n$) with the property $\Omega' \subset \bigcup_{i=1}^{N} B_{r_i}(\alpha_i)$ und $B_{9r_i}(\alpha_i) \subset \Omega$, set $r(\Omega', \Omega) := \min_{i=1,\ldots,N} r_i$ and let $x, y \in \Omega'$ be arbitrary points.

1st case: $|x-y| < \frac{r}{2}$, then there is a $i \in \{1, \ldots, N\}$, such that $x, y \in B_{r_i}(\alpha_i)$. From the first step of the proof we infer $\frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq C(n, a, l, M, \Omega, \Omega', \lambda, \Lambda)$.

2nd case: $|x-y| \geq \frac{r}{2}$, here it's possible to estimate as follows:

$$\frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq \frac{2M}{\left(\frac{r}{2}\right)^{\alpha}} \leq C(n, a, l, M, \Omega', \Omega, \lambda, \Lambda)$$

As a result we have for any $\Omega' \subset\subset \Omega$:

$$\sup_{\substack{x,y \in \Omega' \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq C(n, a, l, M, \Omega', \Omega, \lambda, \Lambda) \Rightarrow u \in C^{0,\alpha}(\Omega').$$

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Lecture 2:

A priori estimtes for harmonic mappings between Riemannian manifolds

1 Introduction

Let $\mathcal{H}$ be a compact Riemannian manifold of dimension $n \geq 3$ with a regular boundary and let $\mathcal{M}$ be a complete Riemannian manifold of dimension m without boundary. Local coordinates on $\mathcal{H}$ and $\mathcal{M}$ are denoted by $(x^1, \ldots, x^n)$ and $(v^1, \ldots, v^m)$ respectively. The fundamental tensors of $\mathcal{H}$ and $\mathcal{M}$ will be denoted by $(\gamma_{\alpha\beta})$ and (g_{ik}) resp.; let $(\gamma^{\alpha\beta})$ and (g^{ik}) be the inverses of $(\gamma_{\alpha\beta})$ and (g_{ik}) resp., furthermore let $\det(\gamma_{\alpha\beta}) =: \gamma$. The Christoffel symbols for (g_{ik}) are given by

$$\Gamma^l_{ik} = g^{lj}\Gamma_{ijk} \qquad\qquad \Gamma_{ijk} = \frac{1}{2}\left(g_{jk,i} - g_{ik,j} + g_{ij,k}\right).$$

To give a precise statement of our results, we need the definition of a regular ball.

Definition 1 *Let $\boldsymbol{B}_M(Q) = \{P \in \mathcal{M}; dist_{\mathcal{M}}(P,Q) \leq M\}$ be a geodesic ball in $\mathcal{M}$ of radius M. $C(Q)$ will denote the cut locus of its center Q. If*

- $M < \frac{\pi}{2\sqrt{\kappa}}$
- $C(Q) \cap \boldsymbol{B}_M(Q) = \emptyset$

we call $\boldsymbol{B}_M(Q)$ a regular ball in $\mathcal{M}$, where $\kappa = \max\{0, \sup\limits_{\boldsymbol{B}_M(Q)} \boldsymbol{K}\}$ is an upper bound on the sectional curvature $\boldsymbol{K}$ of $\mathcal{M}$ on $\boldsymbol{B}_M(Q)$.

Examples: i) If $\mathcal{M}$ is simply connected and $\mathbf{K} \leq 0$, then every ball in $\mathcal{M}$ is a regular ball. (cf. [doC], p.149)

ii) $\mathcal{M} = S^m$, let $\mathcal{N}$ be the open upper hemisphere of S^m. Every ball in $\mathcal{N}$ is a regular ball.

iii) Let $\mathcal{M}$ be a connected manifold with $0 < \frac{\kappa}{4} < \mathbf{K} \leq \kappa$. Then every ball in $\mathcal{M}$ with radius $M < \frac{\pi}{2\sqrt{\kappa}}$ is a regular ball (cf. [GKM], p.254).

The energy of a C^1-map $U : \mathcal{H} \to \mathcal{M}$ is given in local coordinates v by

$$E(U) = \int\limits_{\mathcal{H}} e(v)\, d\mathcal{H}$$

with the energy density

$$e(v) = \frac{1}{2}\gamma^{\alpha\beta}(x) g_{ik}(v) D_\alpha v^i D_\beta v^k.$$

It is well known that the Euler-Lagrange equations for E are given by the elliptic system

$$\frac{1}{\sqrt{\gamma}} D_\alpha\{\sqrt{\gamma}\gamma^{\alpha\beta} D_\beta v^l\} + \gamma^{\alpha\beta}\Gamma^l_{ik}(v) D_\alpha v^i D_\beta v^k = 0 \quad (l = 1, \ldots, m). \tag{1}$$

C^2-Solutions of (1) are called harmonic mappings. If $v \in H^1_2$ solves this system in a weak sense, v is called a weakly harmonic mapping. Here we show regularity of weakly harmonic mappings U from $\Omega \subset \mathcal{H}$ into a regular ball $\mathbf{B}_M(Q) \subset \mathcal{M}$, more precisely we derive a priori estimates for the Hölder norm of U. These regularity results are well known from works of Hildebrandt, Jost and Widman [HJW] (p.274) and Giaquinta and Hildebrandt [GH] (p.144, p.148), who use Green's functions as an essential feature in their argument. In contrast to this we here completely avoid the use of Green's functions by an iteration process using a weak Harnack inequality for supersolutions of a linear elliptic equation. This idea was introduced by Caffarelli [CA] to show a priori estimates of solutions of certain quasilinear elliptic systems. Let $U : \mathcal{H} \to \mathcal{M}$ be a harmonic mapping which maps $\Omega \subset \mathcal{H}$ into a regular ball $\mathbf{B}_M(Q) \subset \mathcal{M}$ and let $\omega := \min\{0, \inf_{\mathbf{B}_M(Q)} \mathbf{K}\}$ denote a lower bound on the sectional curvature $\mathbf{K}$ of $\mathcal{M}$ on $\mathbf{B}_M(Q)$. The interior regularity result reads as follows:

Theorem 1 *Let u be the representation of the harmonic map U with respect to a normal chart centered at Q. Assume that*
$B_{4d} = \{x \in \mathbb{R}^n; |x| < 4d\}$ is a coordinate patch for Ω such that the components $\gamma_{\alpha\beta}(x)$ satisfy

$$\lambda|\xi|^2 \leq \gamma_{\alpha\beta}(x)\xi^\alpha\xi^\beta \leq \mu|\xi|^2, \quad 0 < \lambda \leq \mu \tag{2}$$

for all $\xi \in \mathbb{R}^n$ and all $x \in B_{4d}$. Then there exist a constant
$C = C(n, \lambda, \mu, M, \omega, \kappa) > 0$ and a number $\alpha = \alpha(n, \lambda, \mu, M, \omega, \kappa) > 0$ such that for every $d' < 4d$ the estimate

$$[u]_{\alpha, \overline{B_{d'}}} \leq C(4d - d')^{-\alpha}$$

holds.

Remarks:
1) This result is optimal for $n \geq 3$, because for $M = \frac{\pi}{2\sqrt{\kappa}}$ and $n \geq 3$ there exist bounded, discontinuous weakly harmonic maps $U : \mathcal{H} \to S^N \subset \mathbb{R}^{N+1}$ (cf. [HKW], p.14/15): Choose as one chart on S^N the steoreographic projection σ of S^N from the north pole $P = (0, \ldots, 0, 1)$ onto the equator plane
$\{u \in \mathbb{R}^{n+1}, u^{N+1} = 0\}$. Set $u = \sigma(u)$ and, $u = \sigma^{-1}(u) = \tau(u)$, then we have

$$\tau^i(u) = \tfrac{2u^i}{1+|u|^2} \qquad i = 1, \ldots, N$$

$$\tau^{N+1}(u) = 1 - \tfrac{2}{1+|u|^2}$$

As a discontinuous weakly harmonic mapping $U : \mathcal{H} \to S^N$ we find

$$U(x) = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}, \ldots, \frac{x_n}{|x|}, 0 \right)$$

2) For $n = 2$ every bounded, weakly harmonic mapping is Hölder continuous (cf. [HE]).
3) The same technique as here can be used to prove a-priori estimates for weakly harmonic mappings from Finsler manifolds into a regular ball in a Riemannian manifold, see a paper of von der Mosel and Winklmann [vMW].

It's also possible to prove a boundary estimate using a weak Harnack inequality at the boundary. For this we assume that a neighbourhood Ω of $\partial\mathcal{H}$ is mapped into a regular ball $\mathbf{B}_M(Q) \subset \mathcal{M}$. The boundary values are required to be Hölder continuous with Hölder exponent α_Θ. Then we can prove

Theorem 2 *Let u be the local representation of U with respect to normal coordinates around Q. Assume that*

$$\Sigma_{5R} = \{x = (x'_n, x^n) \in \mathbb{R}^n; |x'_n| < 5R, 0 < x^n < 5R\}$$

is a coordinate patch for Ω such that $\partial\mathcal{H} \cap \overline{\Omega}$ is mapped onto

$$\Sigma^0_{5R} = \{x = (x'_n, 0) \in \mathbb{R}^n; |x'_n| \le 5R\}$$

and the components $\gamma_{\alpha\beta}(x)$ satisfy the estimate (2). Then there exist positive constants $C = C\left(n, \lambda, \mu, M, \omega, \kappa, r, [u]_{\alpha_\Theta, \Sigma^0_{5R}}\right)$ and $\tau = \tau(n, \lambda, \mu, M, \omega, \kappa, \alpha_\Theta)$, such that for every $r \le R$

$$[u]_{\tau, \overline{\Sigma}_r} \le C.$$

2 Interior regularity

First we mention two results of Jost [JO] and Hildebrandt-Kaul [HK] respectively which were already used in [GH]. Lemma 1 permits the construction of normal coordinates around each point P of $\mathbf{B}_M(Q)$.

Lemma 1 *Let $\boldsymbol{B}_M(Q)$ be a regular ball in $\mathcal{M}$. Then any two points in $\boldsymbol{B}_M(Q)$ can be connected by a geodesic contained in $\boldsymbol{B}_M(Q)$. This geodesic is shorter than any curve in $\boldsymbol{B}_M(Q)$ connecting the two points and it contains no pair of conjugate points.*

Proof. See [JO].

Lemma 1 yields that the exponential map $exp_Q : \mathbf{B}_M(Q) \to \mathcal{M}$ is a diffeomorphism und it follows that we can construct normal coordinates on $\mathbf{B}_M(Q)$ around any point $P \in \mathbf{B}_M(Q)$.

The Jacobi field estimates of Lemma 2 will be used in the proof of regularity. To state the result of this lemma we need the following functions:

$$a_\nu(t) = \begin{cases} t\sqrt{\nu}\, \cot\left(t\sqrt{\nu}\right), & \nu > 0, 0 \le t < \frac{\pi}{\sqrt{\nu}} \\ t\sqrt{-\nu}\, \coth\left(t\sqrt{-\nu}\right), & \nu \le 0, 0 \le t < \infty \end{cases},$$

$$b_\nu(t) = \begin{cases} \frac{\sin\left(t\sqrt{\nu}\right)}{t\sqrt{\nu}}, & \nu > 0, 0 \le t < \frac{\pi}{\sqrt{\nu}} \\ \frac{\sinh\left(t\sqrt{-\nu}\right)}{t\sqrt{-\nu}}, & \nu \le 0, 0 \le t < \infty \end{cases}$$

Lemma 2 *Let $\boldsymbol{B}_M(Q)$ be a regular ball in $\mathcal{M}$ and let $v = (v^1, \ldots, v^m)$ be normal coordinates on $\boldsymbol{B}_M(Q)$. Then for all $\xi \in \mathbb{R}^m$ the following estimates hold:*

$$\{\delta_{ik} - a_\omega(|v|)g_{ik}(v)\}\xi^i\xi^k \le \Gamma^l_{ik}(v)v^l\xi^i\xi^k \le \{\delta_{ik} - a_\kappa(|v|)g_{ik}(v)\}\xi^i\xi^k \qquad (3)$$

$$b^2_\kappa(|v|)\xi^i\xi^i \le g_{ik}(v)\xi^i\xi^k \le b^2_\omega(|v|)\xi^i\xi^i. \qquad (4)$$

Proof. see [HK], p.212-214.

For any chart $\psi : \mathbf{B}_M(Q) \to \mathbb{R}^m$ we can define the local representation $v(x)$ of U by $v = \psi \circ U \circ \chi^{-1}$. We use the following abbreviations: $a^{\alpha\beta} := \sqrt{\gamma}\gamma^{\alpha\beta}$, $f^l(v) := a^{\alpha\beta}\Gamma^l_{ik}(v)D_\alpha v^i D_\beta v^k$. Hence, the elliptic system (1) can also be written in the form

$$-D_\beta\left(a^{\alpha\beta}D_\alpha v^l\right) = f^l(v) \qquad (l = 1, \ldots, m)$$

or in the weak form

$$\int_{B_{4d}} a^{\alpha\beta}D_\alpha v^i D_\beta \varphi^i\, dx = \int_{B_{4d}} f^l(v)\varphi^l\, dx \qquad \forall\, \varphi \in H^1_2 \cap L_\infty\left(B_{4d}, \mathbb{R}^m\right). \qquad (5)$$

The coefficients $a^{\alpha\beta}(x)$ are uniformly elliptic, $\lambda^*|\xi|^2 \le a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \le \mu^*|\xi|^2$ a.e. in Ω and for all $\xi \in \mathbb{R}^n$ with $\lambda^* = \frac{\lambda^{\frac{n}{2}}}{\mu}$ and $\mu^* = \frac{\mu^{\frac{n}{2}}}{\lambda}$.

Introducing the functions

$$\mathcal{L}(v) = a^{\alpha\beta}D_\alpha v^i D_\beta v^i,\ \mathcal{E}(v) = a^{\alpha\beta}g_{ik}(v)D_\alpha v^i D_\beta v^k,\ \mathcal{P}(v) = \mathcal{L}(v) - v^l f^l(v)$$

on B_{4d} Lemma 2 implies

$$a_\kappa(|v|)\mathcal{E}(v) \le \mathcal{P}(v) \le a_\omega(|v|)\mathcal{E}(v). \qquad (6)$$

We shall write $u = u(x)$ if we use normal coordinates on $\mathbf{B}_M(Q)$ with center Q. The fact that $a_\kappa(t)$ is decreasing yields

$$a_\kappa(|u|) \geq a_\kappa(M) > a_\kappa\left(\frac{\pi}{2\sqrt{\kappa}}\right) = 0.$$

To prove Theorem 1 we need some further lemmata.

Lemma 3 *Let v be a representation of the harmonic map U with respect to a normal chart centered at a point $P \in \boldsymbol{B}_M(Q)$ such that $|v| < \frac{\pi}{2\sqrt{\kappa}}$. Then $-D_\alpha\left(a^{\alpha\beta}(x)D_\beta|v|^2\right) \leq 0$.*

Proof. Use the test function $\varphi = \eta v$ with $\eta \in C_c^\infty(B_{4d}), \eta \geq 0$. We obtain

$$\int\limits_{B_{4d}} a^{\alpha\beta} D_\alpha v^i D_\beta v^i \eta\, dx - \int\limits_{B_{4d}} f^l v^l \eta\, dx = -\frac{1}{2}\int\limits_{B_{4d}} a^{\alpha\beta} D_\alpha |v|^2 D_\beta \eta\, dx$$

$$\Rightarrow \qquad \int\limits_{B_{4d}} \mathcal{P}(v)\eta\, dx = -\frac{1}{2}\int\limits_{B_{4d}} a^{\alpha\beta} D_\alpha |v|^2 D_\beta \eta\, dx.$$

With (6) and $a_\kappa(|v|) \geq 0$ we infer $\int\limits_{B_{4d}} a^{\alpha\beta} D_\alpha |v|^2 D_\beta \eta\, dx \leq 0$, i.e. $|v|^2$ is a subsolution. □

The next two lemmata have been proven by Meier [ME].

Lemma 4 *Let $v \in H_2^1 \cap L_\infty(B_R)$ be a solution of $-D_\alpha\left(a^{\alpha\beta}(x)D_\beta v\right) \leq 0$ in $B_R \subset \mathbb{R}^n$. The coefficients $a^{\alpha\beta}(x)$ are required to be uniformly elliptic with constants $0 < \lambda \leq \mu$. Then there is a constant $\delta_0 = \delta_0(n, \lambda, \mu) \in (0,1)$ with the property*

$$\sup_{B_{\frac{R}{4}}} v \leq (1-\delta_0)\sup_{B_R} v + \delta_0 \fint\limits_{B_{\frac{R}{4}}} v\, dx \tag{7}$$

.

Proof. The function $u := \sup\limits_{B_R} v - v$ is a non-negative supersolution of $-D_\alpha\left(a^{\alpha\beta}(x)D_\beta u\right) = 0$ in B_R. A weak Harnack inequality ([GT], Thm. 8.18) yields

$$R^{-n}\int\limits_{B_{\frac{R}{4}}}\left(\sup_{B_R} v - v\right) dx \leq C(n,\lambda,\mu)\inf_{B_{\frac{R}{4}}}\left(\sup_{B_R} v - v\right).$$

$$\Rightarrow \qquad \omega_n 4^{-n}\sup_{B_R} v - \omega_n 4^{-n}\fint\limits_{B_{\frac{R}{4}}} v\, dx \leq C\left(\sup_{B_R} v - \sup_{B_{\frac{R}{4}}} v\right)$$

The desired estimate follows with $\delta_0 := \frac{\omega_n}{C4^n}$. □

Lemma 5 *Let* $v \in H_2^1 \cap L_\infty(B_R)$ *be a solution of* $-D_\alpha\left(a^{\alpha\beta}(x)D_\beta v\right) \le 0$ *in* $B_R \subset \mathbb{R}^n$*, again* $a^{\alpha\beta}(x)$ *being uniformly elliptic with constants* $0 < \lambda \le \mu$*. Let* $\epsilon \in \left(0, \frac{1}{2}\right)$ *and* $m \in \mathbb{N}$ *be the smallest integer with* $(1-\delta_0)^m \le \epsilon^2$*. Then for* $s = 4^{-m}$ *the estimate*

$$\sup_{B_{sR}} v \le 2\epsilon^2 \sup_{B_R} v + (1-\epsilon^2) \fint_{B_{\overline{R}}} v \, dx \tag{8}$$

holds, where $\overline{R}$ *is some radius in* $\left[sR, \frac{R}{4}\right]$ *and* δ_0 *is the constant of Lemma 4.*

Proof. Iterating (7) one gets with $t_i = \{1-(1-\delta_0)^m\}^{-1}\delta_0(1-\delta_0)^{m-i}$ for any $m \in \mathbb{N}$ the inequality

$$\sup_{B_{4^{-m}R}} v \le (1-\delta_0)^m \sup_{B_R} v + \{1-(1-\delta_0)^m\} \sum_{i=1}^{m} t_i \fint_{B_{4^{-i}R}} v \, dx.$$

Set $\overline{v}_{\overline{R}} := \max\limits_{i=1,\dots,m} \fint_{B_{4^{-i}R}} v \, dx$, where m is the smallest integer with $(1-\delta_0)^m \le \epsilon^2$. Note that $\sum\limits_{i=1}^{m} t_i = 1$; with $s := 4^{-m}$ we deduce

$$\begin{aligned}
\sup_{B_{sR}} v &\le \epsilon^2 \sup_{B_R} v + \{1-(1-\delta_0)^m\}\overline{v}_{\overline{R}} \\
&\le \epsilon^2 \sup_{B_R} v + \left\{1-(1-\delta_0)\epsilon^2\right\} \overline{v}_{\overline{R}} \\
&\le \epsilon^2 \sup_{B_R} v + \delta_0\epsilon^2 \sup_{B_R} v + (1-\epsilon^2)\overline{v}_{\overline{R}} \\
&\le 2\epsilon^2 \sup_{B_R} v + \left(1-\epsilon^2\right) \overline{v}_{\overline{R}}.
\end{aligned}$$

□

The following result is already contained in [GH], Proposition 1.

Lemma 6 *Let* v *be a representation of the harmonic map* U *with respect to a normal chart, such that* $|v| \le M$*. Then there exists a constant* $C = C(n, \lambda, \mu, M, \kappa) > 0$ *such that for all* $B_R(x_0) \subset B_{4d}$ *with* $B_{4R}(x_0) \subset B_{4d}$ *the estimate*

$$R^{2-n} \int_{B_R(x_0)} \mathcal{E}(v) \, dx \le C\left[M^2(4R) - M^2(R)\right] \tag{9}$$

holds, where $M(R) := \sup\limits_{B_R(x_0)} |v|$*.*

Proof. Test the weak formulation (5) again with $\varphi = \eta v$, $\eta \in \mathring{H}_2^1 \cap L_\infty(B_{4R}(x_0)), \eta \ge 0$ $(4R \le 4d - |x_0|)$. Thus

$$2 \int_{B_{4R}(x_0)} \left[a^{\alpha\beta} D_\alpha v^i D_\beta v^i - f^l v^l\right] \eta \, dx = - \int_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha |v|^2 D_\beta \eta \, dx.$$

With (6) and the fact that $a_\kappa(|v|) \geq a_\kappa(M) > 0$ we infer

$$2a_\kappa(M) \int\limits_{B_{4R}(x_0)} \mathcal{E}(v)\eta\, dx \leq - \int\limits_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha |v|^2 D_\beta \eta\, dx. \qquad (10)$$

With $M(t) := \sup\limits_{B_t(x_0)} |v|$ $(t \leq 4d - |x_0|)$ and $z := M^2(4R) - |v|^2$ estimate (10) yields

$$0 \leq \int\limits_{B_{4R}(x_0)} \mathcal{E}(v)\eta\, dx \leq \frac{1}{2a_\kappa(M)} \int\limits_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha z D_\beta \eta\, dx. \qquad (11)$$

We see that z is a non-negative supersolution of $-D_\beta\left(a^{\alpha\beta} D_\alpha\right) = 0$ in $B_{4R}(x_0)$ and a weak Harnack-inequality gives us the estimate

$$R^{-n} \int\limits_{B_{2R}(x_0)} z\, dx \leq C \inf_{B_R(x_0)} z \qquad (12)$$

with a constant $C = C(n, \lambda, \mu, M, \kappa)$.
Now let $w \in \mathring{H}^1_2(B_{2R}(x_0))$ be a solution of

$$\int\limits_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha \varphi D_\beta w\, dx = R^{-2} \int\limits_{B_{4R}(x_0)} \varphi\, dx \quad \forall\, \varphi \in \mathring{H}^1_2 \cap L_\infty(B_{4R}(x_0)) \qquad (13)$$

From a weak minimum principle for supersolutions (eg. [GT], Thm. 8.1) we infer $\inf\limits_{B_{4R}(x_0)} w \geq 0$. Since $w \neq 0$ we can once again use the weak Harnack inequality to estimate

$$0 < R^{-n} \int\limits_{B_{2R}(x_0)} w\, dx \leq C \inf_{B_R(x_0)} .$$

It follows that there exists a constant $C_1 > 0$ which depends only on n, λ and μ with $0 < C_1 \leq w$ in $B_R(x_0)$. In addition, there is a constant $C_2 > 0$ with the property $0 \leq w \leq C_2$ in $B_{4R}(x_0)$ (cf. [HW1], Lemma 2.1). By using wz as a test function in (13) we infer

$$\frac{1}{2} \int\limits_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha z D_\beta w^2\, dx + \int\limits_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha D_\beta w z\, dx = R^{-2} \int\limits_{B_{2R}(x_0)} zw\, dx.$$

Since w is bounded by C_2 and $z \geq 0$ (12) yields

$$\int\limits_{B_{4R}(x_0)} a^{\alpha\beta} D_\alpha z D_\beta w^2\, dx \leq CR^{-2} \int\limits_{B_{2R}(x_0)} z\, dx \leq CR^{n-2} \inf_{B_R(x_0)} z.$$

With $\eta = w^2$, $\inf\limits_{B_R(x_0)} z = M^2(4R) - M^2(R)$ and in view of (11) the proof of Lemma 6 is complete. $\square$.

For the iteration process we need some notations:
Choose $J \in \mathbb{N}$ such that $M(1+J^{-1}) < \frac{\pi}{2\sqrt{\kappa}}$ and set $K := \sqrt{2 + \frac{b_\omega^2(M)}{4}}$. Furthermore set $\epsilon := \frac{1}{2KJ}$ and let $s = 4^{-m}$ denote the constant of Lemma 5 belonging to this ϵ. Now we can prove

Corollary 1 *Let v be a representation of U in normal coordinates such that $|v| \leq M$. Then there is an integer i_0 with the property*

$$\fint_{B_{R_0}(x_0)} |v - \overline{v}_{R_0}|^2 \, dx \leq M^2 \epsilon^4 s^{nJ} \tag{14}$$

for $R_0 = 4^{-i_0} R$, $\overline{v}_{R_0}$ is the average of v on $B_{R_0}(x_0)$.

Proof. Since $|v| \leq M$ we have $0 < b_\kappa^2(M) \leq b_\kappa^2(|v|) \leq 1$. Then there exists a constant $C = C(n, \lambda, \mu, M, \kappa)$ with the property $C \leq \lambda^* b_\kappa^2(|v|)$. By virtue of this estimate, relation (4), the ellipticity condition
$\lambda^* |\xi|^2 \leq a^{\alpha\beta} \xi_\alpha \xi_\beta$ and Lemma 6 we conclude

$$R^{2-n} \int_{B_R(x_0)} |\nabla v|^2 \, dx \leq C(n, \lambda, \mu, M, \kappa) \left[M^2(4R) - M^2(R) \right]. \tag{15}$$

Using the Poincaré inequality we arrive for $R_0 = 4^{-i_0} R$ (i_0 an arbitrary integer) at the estimate

$$\fint_{B_{R_0}(x_0)} |v - \overline{v}_{R_0}|^2 \, dx \leq C \left[M^2(4^{-i_0+1} R) - M^2(4^{-i_0} R) \right].$$

With $p := \left[\frac{C}{\epsilon^4 s^{nJ}} \right] + 1$ we infer

$$\begin{aligned} M^2 \geq M^2(R) - M^2(4^{-p} R) &= \sum_{i=1}^{p} (M^2(4^{-i+1} R) - M^2(4^{-i} R)) \\ &\geq p \left[M^2(4^{-i_0+1} R) - M^2(4^{-i_0} R) \right] \quad \text{for some } i_0 \in \{1, \ldots, p\}. \end{aligned}$$

Therefore $\fint_{B_{R_0}} |v - \overline{v}_{R_0}|^2 \, dx \leq C M^2 \frac{1}{p} \leq M^2 \epsilon^4 s^{nJ}$. □

The following geometric lemma will be needed at some steps in the proof of Theorem 1.

Lemma 7 *Let P_1, P_2 be two points in $\boldsymbol{B}_M(Q)$ with coordinates p_1 and p_2. Then*

$$b_\kappa(M) |p_1 - p_2| \leq dist_{\mathcal{M}}(P_1, P_2) \leq b_\omega(M) |p_1 - p_2|.$$

Proof. Consider for $t \in [0,1]$ the connecting line

$$C(t) = \exp_Q((1-t)p_1 + t p_2) =: \exp_Q(c(t)).$$

Consequently (with (4)and b_ω is increasing)

$$\begin{aligned} dist_{\mathcal{M}}(P_1,P_2) &\le \int\limits_0^1 \sqrt{g_{ik}(c(t))\dot{c}^i(t)\dot{c}^k(t)} \\ &\le \int\limits_0^1 b_\omega(|c(t)|)|\dot{c}(t)|\,dt \\ &\le b_\omega(M)|p_1-p_2|. \end{aligned}$$

For the estimate from below we let $C_1(t) = \exp_Q(c_1(t))$ $(0 \le t \le 1)$ be the geodesic in $\mathbf{B}_M(Q)$ of length $dist_{\mathcal{M}}(P_1,P_2)$, which connects P_1 and P_2 (Lemma 1 guarantees the existence of such a geodesic). As a consequence of (4) and since b_κ is decreasing we infer

$$\begin{aligned} dist_{\mathcal{M}}(P_1,P_2) &= \int\limits_0^1 \sqrt{g_{ik}(c_1(t))\dot{c}_1^i(t)\dot{c}_1^k(t)}\,dt \\ &\ge \int\limits_0^1 b_\kappa(|c_1(t)|)|\dot{c}_1(t)|\,dt \\ &\ge b_\kappa(M)\left|\int\limits_0^1 \dot{c}_1(t)\,dt\right| = b_\kappa(M)|p_1-p_2|. \qquad \square \end{aligned}$$

Now we can prove Theorem 1:

Proof of Theorem 1. For $0 \le j \le J$ set $R_j = s^j R_0$, $\tau_j = \frac{j}{J}\overline{u}_{R_0}$, $P_j = \exp_Q(\tau_j)$. Let $v^{(j)}$ be the representation of U with respect to normal coordinates around P_j and set $M_0 := M$, $M_j = \left(1+\frac{1}{J}-\frac{j}{J}\right)M$ $(1 \le j \le J)$ (note that $M_j \le M$).

It will be shown by induction that

$$|v^{(j)}| \le M_j \quad \text{in } B_{R_j} \qquad (j=0,\dots,J). \tag{16}$$

Since $P_0 = Q$ the start of the induction is obvious. Suppose that (16) has been shown up to $j-1$, then we get with the triangle inequality

$$|v^{(j)}| = dist_{\mathcal{M}}(U,P_j) \le |v^{(j-1)}| + dist_{\mathcal{M}}(P_{j-1},P_j).$$

Since (16) holds for $j-1$ we infer $|v^{(j-1)}| \le M_{j-1}$ in $B_{R_{j-1}}$. Using the fact that $dist_{\mathcal{M}}(P_{j-1},P_j) = \frac{1}{J}|\overline{u}_{R_0}|$ (P_j and P_{j-1} are both on the geodesic that connects Q and P_J) we have

$$|v^{(j)}| \le M_{j-1} + MJ^{-1} \le M(1+J^{-1}) < \frac{\pi}{2\sqrt{\kappa}} \quad \forall\, j=0,\dots J \quad \text{in } B_{R_{j-1}}.$$

Applying Lemma 3 we obtain

$$-D_\alpha\left(a^{\alpha\beta}(x)D_\beta|v^{(j)}|^2\right) \le 0 \quad \text{in } B_{R_{j-1}}.$$

From Lemma 5 and $|v^{(j)}| \leq M(1+J^{-1})$ in $B_{R_{j-1}}$ we infer for some radius $\overline{R} \in \left[sR_{j-1}, \frac{R_{j-1}}{4}\right]$:

$$\sup_{B_{sR_{j-1}}} |v^{(j)}|^2 \leq 2\epsilon^2 \sup_{B_{R_{j-1}}} |v^{(j)}|^2 + (1-\epsilon^2) \fint_{B_{\overline{R}}} |v^{(j)}|^2\, dx \leq 2\epsilon^2 M^2 (1+J^{-1})^2 + (1-\epsilon^2) \fint_{B_{\overline{R}}} |v^{(j)}|^2\, dx. \tag{17}$$

Lemma 7 yields

$$\begin{aligned} |v^{(j)}| &\leq dist_{\mathcal{M}}(U, P_J) + dist_{\mathcal{M}}(P_J, P_j) \\ &\leq dist_{\mathcal{M}}(U, P_J) + (1 - \tfrac{j}{J})|\overline{u}_{R_0}| \leq b_\omega(M)|u - \overline{u}_{R_0}| + \left(1 - \tfrac{j}{J}\right) M. \end{aligned}$$

Therefore we obtain from Young's inequality

$$|v^{(j)}|^2 \leq (1+\epsilon^{-2}) b_\omega^2(M) |u - \overline{u}_{R_0}|^2 + (1+\epsilon^2) \left(1 - \frac{j}{J}\right)^2 M^2. \tag{18}$$

We have $s^J R_0 \leq s s^{j-1} R_0 = sR_{j-1} \leq \overline{R}$, with this estimate we derive

$$\fint_{B_{\overline{R}}} |u - \overline{u}_{R_0}|^2\, dx \leq s^{-nJ} \fint_{B_{R_0}} |u - \overline{u}_{R_0}|^2\, dx. \tag{19}$$

Combining (18), (19) and Corollary 1 we can estimate relation (17) by recalling that $K = \sqrt{2 + \frac{b_\omega^2(M)}{4}}$ and $\epsilon = \frac{1}{2KJ}$ as follows:

$$\begin{aligned} \sup_{B_{sR_{j-1}}} |v^{(j)}|^2 &\leq 2\epsilon^2 (1+J^{-1})^2 M^2 + (\epsilon^{-2} - \epsilon^2) b_\omega^2(M) s^{-nJ} \fint_{B_{R_0}} |u - \overline{u}_{R_0}|^2\, dx \\ &\quad + (1-\epsilon^4) \left(1 - \frac{j}{J}\right)^2 M^2 \\ &\leq 2\epsilon^2 (1+J^{-1})^2 M^2 + (1-\epsilon^4) b_\omega^2(M) M^2 \epsilon^2 + (1-\epsilon^4)\left(1 - \frac{j}{J}\right)^2 M^2 \\ &\leq 2\epsilon^2 (1+J^{-1})^2 M^2 + b_\omega^2(M) M^2 \epsilon^2 + \left(1 - \frac{j}{J}\right)^2 M^2 \\ &\leq M^2 \left[8\epsilon^2 + \epsilon^2 b_\omega^2(M) + \left(1 - \frac{j}{J}\right)^2\right] \\ &\leq M^2 \left(2K\epsilon + 1 - \frac{j}{J}\right)^2 = M_j^2. \end{aligned}$$

This proves relation (16).

Now set $P = P_J$, from (16) we obtain for $j = J$:

$$dist_{\mathcal{M}}(U, P) = |v^{(J)}| \leq \frac{M}{J} \quad \text{in } B_{R_J} = B_{s^J 4^{-i_0} R}.$$

Applying Lemma 7 this leads to the following estimate of the oscillation:

$$osc_{B_{R_J}} u \leq \frac{1}{b_\kappa(M)} osc_{B_{R_J}} U \leq \frac{2}{b_\kappa(M)} \sup_{B_{R_J}} dist_{\mathcal{M}}(U, P) \leq \frac{2M}{b_\kappa(M) J}.$$

Since $J \to \infty$ if $R_J \to 0$, it has been shown that u is continuous.

Now there exists an integer $i_1 = i_1(n, \lambda, \mu, M, \omega, \kappa)$ such that for all balls $B_{4R}(x_0) \subset B_{4d}$ we have

$$osc_{B_{\tilde{R}}} u < \frac{M}{2b_\omega(M)},$$

with $\tilde{R} = 4^{-i_1} R$.

Let u' be the representation of U with respect to normal coordinates around $U((\chi^{-1}(x_0))$ and set $\omega'(\varrho) := \sup_{B_{\varrho(x_0)}} |u'|^2$, where $0 < \varrho \le \tilde{R}$.

From Lemma 7 it follows

$$|u'| = dist_{\mathcal{M}}(U, U(\chi^{-1}(x_0))) \le b_\omega(M)|u - u(x_0)| \le b_\omega(M) osc_{B_\varrho} u \le M.$$

With $R = \frac{\varrho}{4}$ relation (15) yields

$$\varrho^{2-n} \int_{B_{\frac{\varrho}{4}}} |\nabla u'|^2 \, dx \le C(n, \lambda, \mu, \omega, \kappa) \left[\omega'(\varrho) - \omega'\left(\frac{\varrho}{4}\right) \right]. \qquad (20)$$

Now let $P \in \mathbf{B}_M(Q)$ be the point which corresponds to $\overline{u}_{\frac{\varrho}{4}}$ under $\exp_Q$ and denote by v the representation of U with respect to normal coordinates around P. Then

$$|v| = dist_{\mathcal{M}}(U, P) \le b_\omega(M)|u - \overline{u}_{\frac{\varrho}{4}}| \le M < \tfrac{\pi}{2\sqrt{\kappa}}.$$

Consequently we infer from Lemma 3 $-D_\alpha \left(a^{\alpha\beta} D_\beta |v|^2 \right) \le 0$ in B_ϱ. Applying Lemma 5 we get for an arbitrary $\epsilon \in \left(0, \frac{1}{2}\right]$ and $s = 4^{-m}$, where m is the smallest integer such that $(1 - \delta_0)^m < \epsilon^2$ (δ_0 being the constant of Lemma 4), the estimate

$$\sup_{B_{s\varrho}} |v|^2 \le 2\epsilon^2 \sup_{B_\varrho} |v|^2 + (1 - \epsilon^2) \fint_{B_{\overline{\varrho}}} |v|^2 \, dx \qquad (21)$$

with some $\overline{\varrho} \in \left[s\varrho, \frac{\varrho}{4} \right]$.

The Poincaré inequality yields $s^{-n} \fint_{B_{\frac{\varrho}{4}}} |u - \overline{u}_{\frac{\varrho}{4}}|^2 \, dx \le C(n, \lambda, \mu, M, \kappa, s) \varrho^{2-n} \int_{B_{\frac{\varrho}{4}}} |\nabla u|^2 \, dx$.

Since $\fint_{B_{\overline{\varrho}}} |u - \overline{u}_{\frac{\varrho}{4}}|^2 \, dx \le \frac{1}{\omega_n s^n \left(\frac{\varrho}{4}\right)^n} \int_{B_{\frac{\varrho}{4}}} |u - \overline{u}_{\frac{\varrho}{4}}|^2 \, dx \le s^{-n} \fint_{B_{\frac{\varrho}{4}}} |u - \overline{u}_{\frac{\varrho}{4}}|^2 \, dx$

we infer with $|v| \le b_\omega(M)|u - \overline{u}_{\frac{\varrho}{4}}|$ (see above) the estimate

$$\fint_{B_{\overline{\varrho}}} |v|^2 \, dx \le C_1(n, \lambda, \mu, \omega, \kappa, M, \epsilon) \varrho^{2-n} \int_{B_{\frac{\varrho}{4}}} |\nabla u|^2 \, dx.$$

According to (21) it follows

$$\sup_{B_{s\varrho}} |v|^2 \le 2\epsilon^2 \sup_{B_\varrho} |v|^2 + C_1 \varrho^{2-n} \int_{B_{\frac{\varrho}{4}}} |\nabla u|^2 \, dx. \tag{22}$$

From the Jacobi field estimate (4) and $|u| \le M$ we deduce

$$\frac{1}{\mu} b_\kappa^2(M) |\nabla u|^2 \le \gamma^{\alpha\beta}(x) g_{ik}(u) D_\alpha u^i D_\beta u^k \le \frac{1}{\lambda} b_\omega^2(M) |\nabla u|^2.$$

The energy density $e(U) = \frac{1}{2}\gamma^{\alpha\beta}(x) g_{ik}(v) D_\alpha v^i D_\beta v^k$ doesn't depend on the choice of the coordinates, so we have

$$\frac{2\lambda}{b_\omega^2(M)} e(U) \le |\nabla u|^2 \le \frac{2\mu}{b_\kappa^2(M)} e(U).$$

Since $|u'| \le M$ the same estimate holds for u' instead of u, this means that we can compare $|\nabla u|^2$ with $|\nabla u'|^2$, i.e.

$$|\nabla u|^2 \le \frac{2\mu}{b_\kappa^2(M)} e(U) \le \frac{\mu b_\omega^2(M)}{\lambda b_\kappa^2(M)} |\nabla u'|^2.$$

Combining this with (20) and (22) we can conclude

$$\sup_{B_{s\varrho}} |v|^2 \le 2\epsilon^2 \sup_{B_\varrho} |v|^2 + C_2(n, \lambda, \mu, M, \omega, \kappa, \epsilon) \left[\omega'(\varrho) - \omega'(s\varrho) \right]. \tag{23}$$

From the triangle inequality and Lemma 7 we derive the following two estimates:

1) $|v| \le b_\omega(M) |u - \overline{u}_{\frac{\varrho}{4}}| \le 2 b_\omega(M) \sup_{B_\varrho} |u - u(x_0)| \le 2 \frac{b_\omega(M)}{b_\kappa(M)} \sup_{B_\varrho} |u'|$ in B_ϱ.

2) $|u'| \le b_\omega(M) |u - u(x_0)| \le 2 b_\omega(M) \sup_{B_{s\varrho}} |u - \overline{u}_{\frac{\varrho}{4}}| \le 2 \frac{b_\omega(M)}{b_\kappa(M)} \sup_{B_{s\varrho}} |v|$ in $B_{s\varrho}$.

Inserting this in (23) we observe

$$\omega'(s\varrho) \le C(n, \lambda, \mu, \omega, \kappa, M) \epsilon^2 \omega'(\varrho) + C_2 \left[\omega'(\varrho) - \omega'(s\varrho) \right].$$

Choosing now $\epsilon = \frac{1}{\sqrt{2C}}$ we arrive at

$$\omega'(s\varrho) \le \theta \omega'(\varrho) \quad \text{with } \theta = \frac{C_2 + \frac{1}{2}}{C_2 + 1} < 1.$$

By a standard iteration lemma (e.g. [GT], Lemma 8.23) we discover the following growth estimate

$$\omega'(\varrho) \le C_3^2 \left(\frac{\varrho}{\tilde{R}} \right)^{2C_4} \omega'(\tilde{R}) \quad \forall\, \varrho \le \tilde{R}$$

where C_3 and C_4 both only depend on $n, \lambda, \mu, M, \omega, \kappa$. This yields $\sqrt{\omega'(\varrho)} \leq C_3 \left(\frac{\varrho}{\tilde{R}}\right)^{C_4} \sqrt{\omega'(\tilde{R})}$. Since $\sqrt{\omega'(\varrho)} \leq osc_{B_\varrho} u \leq 2\sqrt{\omega'(\varrho)}$ one obtains for all $\varrho \leq \tilde{R} = 4^{-i_1} R$ (if $B_{4R}(x_0) \subset B_{4d}$)

$$osc_{B_\varrho} u \leq 2C_3 \left(\frac{\varrho}{\tilde{R}}\right)^{C_4} osc_{B_{\tilde{R}}} u \leq C \left(\frac{\varrho}{4R}\right)^{\alpha}. \tag{24}$$

To show that (24) is valid for all $\varrho \leq R$ we need a covering argument. We will only sketch it (see also the proof of Theorem 1, Lecture 1).

Let $x, y \in B_\varrho(x_0)$, $x \neq y$.
If $\varrho \in \left(0, \frac{\tilde{R}}{4}\right]$ we find easily $|u(x) - u(y)| \leq C|x - y|^\alpha \varrho^{-\alpha}$.
If $\varrho \in \left(\frac{\tilde{R}}{4}, R\right]$ we cover $\overline{B_\varrho(x_0)}$ with finite many balls $B_{r_i}(\alpha_i)$ ($r_i \leq \frac{\tilde{R}}{4}$, $i = 1, \ldots, L$) such that $\overline{B_\varrho(x_0)} \subset \bigcup_{i=1}^{L} B_{r_i(\alpha_i)}$ and $B_{4r_i}(\alpha_i) \subset B_{4d}$. It follows $|u(x) - u(y)| \leq C|x - y|^\alpha \varrho^{-\alpha}$, too.
We have derived (24) for all $\varrho \leq R$.
It remains to show the estimate appearing in Theorem 1. For any $d' < 4d$ we set $4R = 4d - d'$. Let x_1, x_2 be two different points in $\overline{B_{d'}}$, then $x_0 := \frac{x_1+x_2}{2} \in B_{d'}$ and $B_{4R}(x_0) \subset B_{4d}$.
If $|x_1 - x_2| \leq \frac{R}{2}$ we infer directly $|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\alpha (4R)^{-\alpha}$.
If $|x_1 - x_2| > \frac{R}{2}$ we use $1 \leq 2^\alpha R^{-\alpha} |x_1 - x_2|^\alpha$ and $|u| \leq M$ to show $|u(x_1) - u(x_2)| \leq C(4R)^{-\alpha} |x_1 - x_2|^\alpha$. In both cases we have the desired estimate $|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\alpha (4d - d')^{-\alpha}$. □

3 Boundary regularity

Now we prove a priori estimates at the Dirichlet boundary. Besides the assumptions made in the introduction we let $\Theta := U|_{\partial\mathcal{H}}$ be the boundary value mapping; for $x_0 \in \Sigma_R^0$ set $P := \Theta(\chi^{-1}(x_0))$. Since $U(\Omega) \subset \mathbf{B}_M(Q)$ one can introduce normal coordinates on $\mathbf{B}_M(Q)$ around P, let v be the local representation of U in these coordinates.
To prove a priori estimates we need the following auxiliary result (cf. [ME], p.6):

Lemma 8 *Let Υ be a domain in $\mathbb{R}^n$ and $v \in H_2^1 \cap L_\infty(\Upsilon \cap B_{4R})$ be a solution of $-D_\alpha \left(a^{\alpha\beta}(x) D_\beta v\right) \leq 0$ in $\Upsilon \cap B_{4R}$, the coefficients $a^{\alpha\beta}(x)$ are required to be uniformly elliptic with constants $0 < \lambda \leq \mu$. If $|B_R - \Upsilon| \geq \gamma |B_R|$ for some*

$\gamma > 0$, *there exists a constant* $\delta_0(n,\lambda,\mu,\gamma) \in (0,1)$ *such that*

$$\sup_{\Upsilon\cap B_R} v \le (1-\delta_0) \sup_{\Upsilon\cap B_{4R}} v + \delta_0 \sup_{\partial\Upsilon\cap B_{4R}} v. \tag{25}$$

Proof. Define $u := \sup_{\Upsilon\cap B_{4R}} v - v$, then u is a non-negative supersolution in $\Upsilon\cap B_{4R}$ and a weak Harnack inequality at the boundary ([GT], Thm. 8.26) implies

$$R^{-n} \int_{B_{2R}} |u_m^-|\, dx \le C(n,\lambda,\mu) \inf_{B_R} u_m^- \tag{26}$$

with $m = \inf_{\partial\Upsilon\cap B_{4R}} u = \sup_{\Upsilon\cap B_{4R}} v - \sup_{\partial\Upsilon\cap B_{4R}} v$ and $u_m^-(x) = \begin{cases} \inf\{u(x), m\}, & x \in \Upsilon \\ m, & x \notin \Upsilon \end{cases}$.

(26) yields

$$R^{-n} \int_{B_{2R}-\Upsilon} \left(\sup_{\Upsilon\cap B_{4R}} v - \sup_{\partial\Upsilon\cap B_{4R}} v \right) dx + R^{-n} \underbrace{\int_{B_{2R}\cap\Upsilon} |\inf\{u(x),m\}|\, dx}_{\ge 0} \le C \inf_{B_R} u_m^-.$$

Thus

$$R^{-n}|B_{2R}-\Upsilon| \left(\sup_{\Upsilon\cap B_{4R}} v - \sup_{\partial\Upsilon\cap B_{4R}} v \right) \le C \inf_{B_R\cap\Upsilon} \{u,m\}.$$

$$\Rightarrow \qquad \gamma\omega_n \left(\sup_{\Upsilon\cap B_{4R}} v - \sup_{\partial\Upsilon\cap B_{4R}} v \right) \le C \left(\sup_{\Upsilon\cap B_{4R}} - \sup_{\Upsilon\cap B_R} v \right).$$

By setting $\delta_0 := \frac{\gamma\omega_n}{C}$ we arrive at (25). □

For $x_0 \in \Sigma_R^0$ define $\mathcal{S}_R(x_0) := \{x \in \mathbb{R}^n; |x-x_0| < R, x^n > 0\}$. As a cornerstone for Theorem 2 we will prove the following a priori estimate:

Lemma 9 *Let u be the local representation of U with respect to normal coordinates around Q, let $\sigma(R) := osc_{\Sigma_R^0} u < \frac{M}{b_\omega(M)}$ and $2M + b_\omega(M)\sigma(R) < \frac{\pi}{\sqrt{\kappa}}$. Then there exists a radius $R^* \le R$, which depends on $n, \lambda, \mu, M, \omega, \kappa$ and $\sigma(R)$, such that for all $\varrho \in (0, R^*)$ the estimate*

$$osc_{\mathcal{S}_\varrho(x_0)} u \le C \left[\left(\frac{\varrho}{R^*}\right)^{\tilde\alpha} osc_{\mathcal{S}_R(x_0)} u + \sigma\left(\sqrt{\varrho R}\right) \right] \tag{27}$$

holds, where $C = C(n,\lambda,\mu,M,\omega,\kappa) > 0$ and $\tilde\alpha = \tilde\alpha(n,\lambda,\mu) > 0$.

Proof. Choose $\tau \in \mathbb{R}^N$ with $|\tau| \le M$ in a way that for $P := \exp_Q(\tau)$ the inequality $m_\tau(R) := \sup_{\Sigma_R^0} dist_{\mathcal{M}}(U,P) \le b_\omega(M)\sigma(R)$ is satisfied.
Determine an integer $J \ge 2$ with the property

$$2M + m_\tau + 3MJ^{-1} < \frac{\pi}{\sqrt{\kappa}}$$

where $m_\tau = m_\tau(R)$.

Let m be the smallest integer which satisfies $(1-\delta_0)^m \le \dfrac{M^2}{J^2(M+|\tau|)^2}$ where δ_0 is the constant from Lemma 8 (with the same $a^{\alpha\beta}$ as in section 2). For $j=0,\dots,J$ set $\tau_j = \frac{j}{J}\tau$ and $R_j = 4^{-jm}R$. Furthermore, define $P_j := \exp_Q(\tau_j)$ and let $v^{(j)}$ be the local representation of U with respect to a normal chart centered at P_j. Moreover, we set $M_0 := M$ and $M_j := J^{-1}M + m_{\tau_j}$ $(1 \le j \le J)$. We will show by induction

$$|v^{(j)}| \le M_j \quad \text{in } \mathcal{S}_j := \mathcal{S}_{R_j}(x_0) \quad (j=0,\dots,J). \tag{28}$$

For $j=0$ the assertion is obvious, suppose that (28) holds for the index $j-1$. Remember that P_j and P_{j-1} both lie on the geodesic line which connects Q and $P = P_J$. Thus

$$m_{\tau_{j-1}} \le \sup_{\Sigma_R^0} dist_{\mathcal{M}}(U, P_J) + \sup_{\Sigma_R^0} dist_{\mathcal{M}}(P_{j-1}, P_J) \le m_\tau + \tfrac{J-(j-1)}{J} M$$

Since (28) holds for $j-1$ we have for $x \in \mathcal{S}_{j-1}$

$$|v^{(j)}| \le dist_{\mathcal{M}}(U(\chi^{-1}(x), P_{j-1}) + dist_{\mathcal{M}}(P_j, P_{j-1}) \le M_{j-1} + MJ^{-1}.$$

We have chosen J in such a way that

$$\begin{aligned} &M + |\tau_j| + M_{j-1} + MJ^{-1} \\ \le\ & M + \tfrac{j}{J}M + 2MJ^{-1} + m_{\tau_{j-1}} \\ \le\ & 2M + 3MJ^{-1} + m_\tau \\ <\ & \tfrac{\pi}{\sqrt{\kappa}}. \end{aligned}$$

Since $|v^{(j)}| \le M + |\tau_j|$ and $|v^{(j)}| \le M_{j-1} + MJ^{-1}$ in $\mathcal{S}_{j-1}$ we infer

$$|v^{(j)}| \le \frac{1}{2}\left[M + |\tau_j| + M_{j-1} + MJ^{-1}\right] < \frac{\pi}{2\sqrt{\kappa}} \qquad \text{in } \mathcal{S}_{j-1}.$$

Lemma 3 yields now that $|v^{(j)}|^2$ is a subsolution in $\mathcal{S}_{j-1}$.

Applying Lemma 8 repeatedly we infer

$$\begin{aligned} \sup_{\mathcal{S}_{4^{-m}R_{j-1}}} \left|v^{(j)}\right|^2 &\le (1-\delta_0) \sup_{\mathcal{S}_{4^{-(m-1)}R_{j-1}}} \left|v^{(j)}\right|^2 + \delta_0 \sup_{\Sigma^0_{4^{-(m-1)}R_{j-1}}} \left|v^{(j)}\right|^2 \\ &\le (1-\delta_0) \left[(1-\delta_0) \sup_{\mathcal{S}_4^{-(m-2)}R_{j-1}} |v^{(j)}|^2 + \delta_0 \sup_{\Sigma^0_{4^{-(m-2)}R_{j-1}}} |v^{(j)}|^2 \right] \\ &\quad + \delta_0 \sup_{\Sigma^0_{4^{-(m-1)}R_{j-1}}} |v^{(j)}|^2 \end{aligned}$$

$$\le \ldots \le (1-\delta_0)^m \sup_{\mathcal{S}_{j-1}} \left|v^{(j)}\right|^2 + \sum_{\nu=0}^{m-1} (1-\delta_0)^{m-1-\nu}\delta_0 \sup_{\Sigma^0_{4^{-\nu}R_{j-1}}} \left|v^{(j)}\right|^2$$

$$\le (1-\delta_0)^m (M+|\tau|)^2$$

$$+ [1-(1-\delta_0)^m] \underbrace{\sum_{\nu=0}^{m-1} [1-(1-\delta_0)^m]^{-1}\, \delta_0 (1-\delta_0)^{m-1-\nu}}_{=1} \sup_{\Sigma^0_{4^{-\nu}R_{j-1}}} dist^2_{\mathcal{M}}(U, P_j)$$

$$\le (1-\delta_0)^m (M+|\tau|)^2 + [1-(1-\delta_0)^m]\, m^2_{\tau_j}$$

$$\le J^{-2}M^2 + m^2_{\tau_j}.$$

Since $\mathcal{S}_{4^{-m}R_{j-1}} = \mathcal{S}_j$ we have

$$\left|v^{(j)}\right| \le J^{-1}M + m_{\tau_j} = M_j \text{ in } \mathcal{S}_j.$$

This proves (28); inserting $j = J$ implies

$$\left|v^{(J)}\right| = dist_{\mathcal{M}}(U, P_J) \le J^{-1}M + m_\tau \le J^{-1}M + M < \frac{\pi}{2\sqrt{\kappa}}$$

in $\mathcal{S}_J$. On account of Lemma 3 we conclude that $\left|v^{(J)}\right|^2$ is a subsolution in $\mathcal{S}_J$.

For an arbitrary $x \in \mathcal{S}_{R_J}$ set $M_\tau(R_J) := \sup_{\mathcal{S}_{R_J}} dist_{\mathcal{M}}(U(\chi^{-1}(x)), P_J)$. By virtue of Lemma 8 we infer for $0 < 4\varrho < R_J$

$$M^2_\tau(\varrho) \le (1-\delta_0)M^2_\tau(4\varrho) + \delta_0 m^2_\tau(4\varrho).$$

A standard iteration lemma (e.g. [GT], Lemma 8.23) yields

$$M^2_\tau(4\varrho) \le K^2 \left(\left(\frac{4\varrho}{R^*}\right)^{2\tilde{\alpha}} M^2_\tau(R^*) + m^2_\tau\left(2\sqrt{\varrho R^*}\right) \right) \tag{29}$$

where $R^* := R_J$ (note that R^* depends on $n, \lambda, \mu, M, \omega, \kappa$ and $\sigma(R)$) and $K = K(n, \lambda, \mu)$, $\tilde{\alpha} = \tilde{\alpha}(n, \lambda, \mu)$. For $\varrho \in (0, R^*)$ we arrive at

$$M_\tau(\varrho) \le K \left(\left(\frac{\varrho}{R^*}\right)^{\tilde{\alpha}} M_\tau(R) + m_\tau\left(\sqrt{\varrho R^*}\right) \right). \tag{30}$$

Choose $x_0 \in \Sigma^0_\varrho$ in a way that $\tau = u(x_0)$ and choose a sequence $x_n \in \mathcal{S}_R$ with $u(x_n) \to u(x_0), n \to \infty$. Then the following estimates hold:

1) $M_\tau(R) = \sup_{\mathcal{S}_R} dist_{\mathcal{M}}(U, P_J) \le b_\omega(M) osc_{\mathcal{S}_R} u$

and 2) $m_\tau(\varrho) = \sup_{\Sigma^0_R} dist_{\mathcal{M}}(U, P_J) \le b_\omega(M)\sigma(\varrho)\ \forall\ \varrho \le R$

On account of $\frac{1}{2b_\omega(M)} M_\tau(\varrho) \le osc_{\mathcal{S}_\varrho} u \le \frac{2}{b_\kappa(M)} M_\tau(\varrho)$ we finally conclude for every $\varrho \leq R$

$$osc_{\mathcal{S}_\varrho(x_0)} u \leq C(n,\lambda,\mu,M,\omega,\kappa)\left[\left(\frac{\varrho}{R^*}\right)^{\tilde{\alpha}} osc_{\mathcal{S}_R(x_0)} u + \sigma\left(\sqrt{\varrho R}\right)\right]. \qquad \square$$

Proof of Theorem 2. Let $\Theta = U|_{\partial\mathcal{H}}$ be Hölder continuous with Hölder exponent α_Θ, i.e. $[u]_{\alpha_\Theta,\Sigma^0_{5R}} \leq C$. Then for every $\varrho \leq R$ the estimate $\sigma(\varrho) = osc_{\Sigma^0_\varrho} u \leq C\left(\frac{\varrho}{R}\right)^{\alpha_\Theta}$ holds, so there exists a radius $R_1 \leq R$ with the property $\sigma(R_1) = osc_{\Sigma^0_{R_1}} u < \dfrac{M}{b_\omega(M)}$ and $2M + b_\omega(M)\sigma(R_1) < \frac{\pi}{\sqrt{\kappa}}$.

Choose now $y \in \Sigma_R$ fixed and determine $x_0 \in \Sigma^0_R$ with the property $d := dist(y, \Sigma^0_R) = |y - x_0|$. To estimate the oscillation of u on $\overline{\Sigma_R} \cap B_\varrho(y)$ for $0 < \varrho \leq r \leq \frac{R_1}{2}$ we distinguish three cases:

1) $d \geq r$. Then $B_\varrho(y) \subset \Sigma_{5R}$ and we use Theorem 1 to observe

$$osc_{\overline{\Sigma_R}\cap B_\varrho(y)} u \leq C\left(\tfrac{\varrho}{r}\right)^{\alpha}.$$

2) $r \geq \varrho \geq d$. Lemma 9 and the Hölder continuity of u at the boundary imply

$$osc_{\overline{\Sigma_R}\cap B_\varrho(y)} u \leq osc_{\overline{\Sigma_R}\cap B_{2\varrho}(x_0)} u \leq osc_{\mathcal{S}_{2\varrho}(x_0)\cup\Sigma^0_{2\varrho}(x_0)} u$$
$$\leq C\left[\left(\tfrac{\varrho}{r}\right)^{\tilde{\alpha}} osc_{\mathcal{S}_{2r}} u + \left(\tfrac{\varrho}{r}\right)^{\alpha_\Theta}\right] \leq C\left(\tfrac{\varrho}{r}\right)^{\nu} \text{ with } \nu := \min\{\tilde{\alpha}, \alpha_\Theta\}.$$

3) $r \geq d \geq \varrho$. The interior regularity yields

$$osc_{\overline{\Sigma_R}\cap B_\varrho(y)} u \leq C\left(\tfrac{\varrho}{d}\right)^{\alpha} osc_{\overline{\Sigma_R}\cap B_{2d}(x_0)} u \leq C\left(\tfrac{\varrho}{d}\right)^{\alpha} osc_{\overline{\mathcal{S}_{2d}}(x_0)} u.$$

With Lemma 9 it follows that

$$osc_{\overline{\Sigma_R}\cap B_\varrho(y)} u \leq C\left(\tfrac{\varrho}{d}\right)^{\alpha}\left[\left(\tfrac{2d}{2r}\right)^{\tilde{\alpha}} osc_{\mathcal{S}_{2r}} u + \left(\tfrac{d}{r}\right)^{\alpha_\theta}\right]$$
$$\leq C\left(\tfrac{\varrho}{d}\right)^{\alpha} \max\left\{\left(\tfrac{d}{r}\right)^{\tilde{\alpha}}, \left(\tfrac{d}{r}\right)^{\alpha_\Theta}\right\} \leq C\left(\tfrac{\varrho}{r}\right)^{\tau} \text{ with } \tau \in \{\alpha, \tilde{\alpha}, \alpha_\Theta\}.$$

In all three cases we have

$$osc_{\overline{\Sigma_R}\cap B_\varrho(y)} u \leq C\left(\frac{\varrho}{r}\right)^{\tau}$$

with constants $C = C(n,\lambda,\mu,M,\omega,\kappa,[u]_{\alpha_\Theta,\Sigma^0_{5R}})$ and $\tau = \tau(n,\lambda,\mu,M,\omega,\kappa,\alpha_\Theta)$.

To prove the estimate of the Hölder seminorm we use a covering argument like the one in section 2 and distinguish the cases when $|x_1 - x_2|$ is "small" and when $|x_1 - x_2|$ is greater than a constant with depends on the number and the radii of the balls used for the covering. $\square$

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Lecture 3:

Regularity of weak solutions of degenerate elliptic systems

1 Introduction

We consider weak solutions of degenerate elliptic systems of the form

$$D_\alpha \left(a^{\alpha\beta}(x,u,\nabla u) D_\beta u^i \right) = f^i(x,u,\nabla u) \quad (i = 1, \ldots, m) \tag{1}$$

in a domain $\Omega \subset \mathbb{R}^n$. We assume there exists a locally integrable weight $w(x) > 0$ a.e. in Ω with the property

$$w(x)|\xi|^2 \leq a^{\alpha\beta}(x,u,\nabla u)\xi_\alpha \xi_\beta \leq C w(x) |\xi|^2 \; \forall \, \xi \in \mathbb{R}^n. \tag{2}$$

To prove regularity of weak solutions of (1) the weight w has to satisfy the "A_2-condition", which means that w belongs to the Muckenhouptclass A_2. The Muckenhouptclasses A_p are defined as follows:

Definition 1 *Let $1 < p < \infty$. The weight w is an element of A_p, if*

$$\sup_{B_R \subset \mathbb{R}^n} \left(\frac{1}{|B_R|} \int_{B_R} w(x)\, dx \right) \left(\frac{1}{|B_R|} \int_{B_R} w(x)^{\frac{-1}{p-1}}\, dx \right)^{p-1} =: C_p < \infty. \tag{3}$$

w is to be said of class A_∞, if for every $\epsilon > 0$ there exists a $\delta > 0$ with the property that for every measurable $E \subset B_R$ with $|E| < \delta |B_R|$ the inequality $w(E) \leq \epsilon w(B_R)$ holds, where $w(E) = \int_E w(x)\, dx$.

Example: $w(x) = |x|^\alpha$ is an element of A_2, if $\alpha \in (-n, n)$.

Optimal regularity results for weak solutions of uniformly elliptic systems of type (1) are well known due to Hildebrandt-Widman [HW], Wiegner [WI1], [WI2] and Caffarelli [CA] (see Lecture 1). For the case of a single elliptic equation Fabes, Kenig and Serapioni [FKS] have proven Hölder continuity for weak solutions. For degenerate elliptic systems only a few regularity results and a priori estimates are known. Baldes [BAL] and Baoyao [BAO] proved some results, e.g. weak solutions of systems with bounded weights $w \in A_2$ are under a smallness condition Hölder continous. We will show an a priori estimate of bounded, weak solutions which contains the results of [BAL] and [BAO].
Our proof uses an idea of L.Caffarelli [CA] to show a priori estimates for weak solutions of certain uniformly elliptic sytems. His main tool was a weak Harnack inequality for supersolutions of a uniformly elliptic linear equation; we will prove such a Harnack inequality for solutions of degenerate (in the above sense) elliptic equations. The proof of this Harnack inequality is based upon a work

of Trudinger [TR] in which a Harnack inequality for solutions of some "mildly" degenerate elliptic equations was shown. To state our regularity result exactly we have first to define the spaces $H_2^1(\Omega,\omega)$ and $\overset{\circ}{H}{}_2^1(\Omega,\omega)$:

Definition 2 *i)* $H_2^1(\Omega,\omega)$ *is the closure of* $C^\infty(\overline{\Omega})$ *with respect to the norm* $\int_\Omega |u(x)|^2 w\,dx + \int_\Omega |\nabla u(x)|^2 w\,dx$.
ii) $\overset{\circ}{H}{}_2^1(\Omega,\omega)$ *is the closure of* $C_c^\infty(\Omega)$ *with respect to the norm* $\int_\Omega |\nabla u(x)|^2 w\,dx$.

Now we can precise the term *weak solution*:

Definition 3 $u \in H_2^1(\Omega,\omega,\mathbb{R}^m)$ *is called a* *weak solution* *of (1) if*

$$\int_\Omega a^{\alpha\beta}(x) D_\beta u^i D_\alpha \phi^i\,dx = \int_\Omega f(x,u,\nabla u)^i f^i\,dx \tag{4}$$

holds for all $\phi \in \overset{\circ}{H}{}_2^1(\Omega,\omega,\mathbb{R}^m) \cap L_\infty(\Omega,\mathbb{R}^m)$.

Our regularity result reads as follows:

Theorem 1 *Let* u *be a bounded, weak solution of (1) in* $\Omega \subset \mathbb{R}^n$. *The coefficients* $a^{\alpha\beta}$ *are required to fulfill (2), under the assumption* $a^* + aM < 2$ u *is locally Hölder continuous and there exist constants* $C = C(n,a,a^*,M,\Omega,\Omega',C_2) > 0$ *and* $\alpha = \alpha(n,a,a^*,M,C_2) > 0$, *such that*

$$[u]_{\alpha,\Omega'} \leq C. \tag{5}$$

2 Properties of A_2-weights

Let w be an A_2-weight. We will state some properties of weight functions $w \in A_p$ which will be needed in the proof of the weak Harnack inequality.

Combining two results due to Muckenhoupt ([MU2], p.104) and Coifman-Fefferman ([CF], p.244]) we infer:

Lemma 1 $w \in A_p$ *for some* $p > 1$ *if and only if* $w \in A_\infty$, *i.e.* $A_\infty = \bigcup_{p>1} A_p$.

Lemma 2 *(cf. [MW, p.223]) Let* $w \in A_\infty$, *then there are positive constants* c_1 *and* c_2, *such that for* $c > 0$ *we have:*

$$c_1 \leq \frac{\int_{B_{cR}} w\,dx}{\int_{B_R} w\,dx} \leq c_2 \tag{6}$$

In particular, the <u>*doubling property*</u> $w(B_{2R}) \leq Kw(B_R)$ *with some* $K > 0$ *is valid.*

Lemma 3 *(cf. [FKS], p.90) Let* $w \in A_2$ *and* $\varphi_k \in C^\infty(\mathbb{R}^n)$ *be a sequence with the properties* $\int\limits_\Omega |\varphi_k|^2 w\,dx \to 0, k \to 0$ *and* $\int\limits_\Omega |\nabla\varphi_k - v|^2 w\,dx \to 0, k \to \infty$*. Then* $v \equiv 0$.

From Lemma 3 it follows that a gradient of a function $u \in H_2^1(\Omega, \omega)$ with $w \in A_2$ is unique.

The following two inequalities were proven by Fabes-Kenig-Serapioni ([FKS], Thm. 1.2 and Thm. 1.5). They generalize the Sobolev- and Poincaré inequlaities up to weight functions $w \in A_p$.

Lemma 4 *Let* $w \in A_p$*,* $1 < p < \infty$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Then there exist constants* $C > 0, \delta > 0$*, which depend only on* n *and* p*, such that for all balls* $B_R \subset \Omega$*, for all* $1 \leq k \leq \frac{n}{n-1} + \delta$ *and all* $u \in C_c^\infty(B_R)$ *the estimate*

$$\left(\frac{1}{w(B_R)} \int\limits_{B_R} |u|^{kp} w\,dx\right)^{\frac{1}{kp}} \leq CR \left(\frac{1}{w(B_R)} \int\limits_{B_R} |\nabla u|^p w\,dx\right)^{\frac{1}{p}} \tag{7}$$

holds.

Lemma 5 *Let* $w \in A_p$*,* $1 < p < \infty$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Then there are constants* $C > 0, \delta > 0$*, which depend only on* n *and* p*, such that for* $B_R \subset \Omega$*,* $u \in C^1(\overline{B_R})$ *and* $k \in [1, \frac{n}{n-1} + \delta]$ *the estimate*

$$\left(\frac{1}{w(B_R)} \int\limits_{B_R} \left|u(x) - \frac{1}{w(B_R)} \int\limits_{B_R} u(x) w\,dx\right|^{kp} w\,dx\right)^{\frac{1}{kp}} \leq CR \left(\frac{1}{w(B_R)} \int\limits_{B_R} |\nabla u|^p w\,dx\right)^{\frac{1}{p}} \tag{8}$$

holds.

<u>Remark:</u> It's possible to show that the weigth $w(x) = |detDf(x)|^{1-\frac{2}{n}}$, f quasiconform, posseses also these five properties, although it is in general not in A_2 (cf. [FKS], p.106-112). Since only the above properties of the A_2-weights are needed in the regularity proof, Theorem 1 is also valid for $w(x) = |detDf(x)|^{1-\frac{2}{n}}$. For example, $f(x) = |x|^\alpha x$, $\alpha > -1$ is a quasiconform mapping. This yields that $w(x) = |x|^\beta$ is for $\beta > 2 - n$ an admissible weight for Theorem 1.

3 A Harnack inequality for degenerate weights

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider the elliptic equation

$$Lu = D_\alpha \left(a^{\alpha\beta}(x) D_\beta u \right) = 0. \tag{9}$$

The symmetric coefficients $a^{\alpha\beta}(x)$ are required to satisfy (2) with $w(x) \in A_2$. We consider weak solutions in the weighted Sobolev space $H_2^1(\Omega, \omega)$. We mention that the estimates of Lemma 4 and Lemma 5 remain valid for
$w \in A_2$ if $u \in \overset{\circ}{H}{}_2^1(B_R, \omega)$ and $u \in H_2^1(B_R, \omega)$ resp. This can been seen by an approximation argument. Lemma 6 is a version of Lemma 7.8 in [GT] for weighted Sobolev spaces and shows that compositions of a Sobolev fucntion $u \in H_2^1(\Omega, \omega)$ and a piecewise smooth function are in the space $H_2^1(\Omega, \omega)$.

Lemma 6 *Let f be a piecewise smooth function (i.e. f is continuous and has a piecewise continuous derivative) with $f' \in L_\infty(\mathbb{R})$. If $u \in H_2^1(\Omega, w)$, then $f \circ u \in H_2^1(\Omega, w)$. Furthermore, in the points where f' exists we have $D(f \circ u) = f'(u)Du$, in the other points we set $D(f \circ u) = 0$.*

Definition 4 *A function $u \in H_2^1(\Omega, w)$ is called weak subsolution of (9) in Ω, if for every $\varphi \in \overset{\circ}{H}{}_2^1(\Omega, w), \varphi \geq 0$ we have*

$$\int_\Omega a^{\alpha\beta}(x) D_\beta u D_\alpha \varphi \, dx \leq 0. \tag{10}$$

$u \in H_2^1(\Omega, w)$ is a weak supersolution, if $-u$ is a weak subsolution.

As mentioned in the introduction it was shown [FKS] that weak solutions of (9) are locally Hölder continuous, this result was proven with the help of a weak Harnack inequality, which was proven with a generalized John-Nirenberg Lemma. We will use an idea due to Trudinger [TR] to prove this Harnack inequality without the use of a John-Nirenberg Lemma. Then we can modify the reglarity proof of Lecture 1 by using this weak Harnack inequality instead of the Harnack inequality for uniformly elliptic equations (see section 4). We will prove the following weak Harnack inequality:

Theorem 2 *Let u be a weak, non-negative supersolution of (9) in $\Omega \subset \mathbb{R}^n$. Then for every $B_R \subset \Omega$ and all $0 < \alpha < \beta < 1$, $0 < \gamma < k$ the estimate*

$$\left(\frac{1}{w(B_{\beta R})} \int_{B_{\beta R}} |u|^\gamma w \, dx \right)^{\frac{1}{\gamma}} \leq C(n, \alpha, \beta, \gamma, C_2) \inf_{B_{\alpha R}} u \tag{11}$$

holds, where $k > 1$ is the constant appearing in the Sobolev- and Poincaré inequalities (Lemmas 4 and 5).

The proof of Theorem 2 will be divided into three lemmas. The constants appearing in the proofs are generic constants and will only depend on the datas mention in the appropriate lemma.

Lemma 7 *Let u be a weak subsolution of (9). Then for any $B_R \subset \Omega$ the estimate*

$$\sup_{B_{\alpha R}} u \leq C(n, \alpha, \beta, C_2) \left(\frac{1}{w(B_{\beta R})} \int_{B_{\beta R}} |u^+|^2 w \, dx \right)^{\frac{1}{2}} \tag{12}$$

holds, where $\alpha < \beta < 1$, C_2 is the A_2-Konstante of w and $u^+ := \sup(u, 0)$.

Proof. For $\delta \geq 1$ and $0 < N < \infty$ define F by

$$F(u) = F_\delta^N(u) = \begin{cases} (u^+)^\delta, & u \leq N \\ \delta N^{\delta-1} u - (\delta - 1) N^\delta & u > N \end{cases}$$

Test the weak formulation $\int_\Omega a^{\alpha\beta}(x) D_\beta u D_\alpha \varphi \, dx \leq 0$ with $\varphi(x) = \eta^2(x) F(u)$, $\eta \geq 0, \eta \in C_c^1(B_R)$ (in view of Lemma 6 φ is an admissible testfunction), we arrive at:

$$\int_\Omega a^{\alpha\beta}(x) D_\beta u \left[2\eta(x) D_\alpha \eta F(u) + \eta^2(x) F'(u) D_\alpha u \right] dx \leq 0.$$

Applying (2) we infer

$$\int_\Omega \eta^2 F'(u) |\nabla u|^2 w \, dx \leq C \int_\Omega \eta |\eta_x| F |\nabla u| w \, dx. \tag{13}$$

With $F(u) \leq u^+ F'(u)$ and the help of the Hölder inequality we see

$$\int_\Omega \eta^2 F' |\nabla u|^2 w \, dx \leq C \left(\int_\Omega \eta^2 |\nabla u|^2 F' w \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \eta_x^2 (u^+)^2 F' w \, dx \right)^{\frac{1}{2}}, \text{ therefore:}$$

$$\int_\Omega \eta^2 F' |\nabla u|^2 w \, dx \leq C \int_\Omega \eta_x^2 \left(u^+\right)^2 F' w \, dx. \tag{14}$$

Define $G(u) := \int_0^u |F'(t)|^{\frac{1}{2}} \, dt = \begin{cases} \sqrt{\delta} \frac{2}{\delta+1} |u^+|^{\frac{\delta+1}{2}}, & u \leq N \\ \sqrt{\delta} N^{\frac{\delta-1}{2}} |u|, & u > N \end{cases}$. (14) yields

$$\int_\Omega \eta^2 |\nabla G|^2 w \, dx \leq C \int_\Omega \eta_x^2 \left(u^+ G'\right)^2 w \, dx. \tag{15}$$

With the Sobolev inequality (Lemma 4) and Young's inequality we find

$$\left(\frac{1}{w(B_R)}\int\limits_{B_R}|\eta G|^{2k}w\,dx\right)^{\frac{1}{2k}} \leq CR\left(\frac{1}{w(B_R)}\int\limits_{B_R}\left(\eta^2|\nabla G|^2+\eta_x^2G^2\right)w\,dx\right)^{\frac{1}{2}}$$

$$\overset{(15)}{\leq} CR\left(\frac{1}{w(B_R)}\int\limits_{B_R}\eta_x^2\left[\left(u^+G'\right)^2+G^2\right]w\,dx\right)^{\frac{1}{2}}.$$

Since $G \leq u^+G'$ we can summarize

$$\left(\frac{1}{w(B_R)}\int\limits_{B_R}|\eta G|^{2k}w\,dx\right)^{\frac{1}{2k}} \leq CR\left(\frac{1}{w(B_R)}\int\limits_{B_R}\eta_x^2\left(u^+G'\right)^2w\,dx\right)^{\frac{1}{2}}. \tag{16}$$

Set $q := \frac{\delta+1}{2} > 1$. By taking roots we obtain:

$$\left(\frac{1}{w(B_R)}\int\limits_{B_R}|\eta G|^{2k}w\,dx\right)^{\frac{1}{2kq}} \leq (CR)^{\frac{1}{q}}\left(\frac{1}{w(B_R)}\int\limits_{B_R}\eta_x^2\left(u^+G'\right)^2w\,dx\right)^{\frac{1}{2q}}.$$

Choose ρ and σ in a way that $\alpha \leq \rho < \sigma \leq \beta$ and η in a way that $\operatorname{supp}\eta \subset B_{\sigma R}$, $\eta \equiv 1$ in $B_{\rho R}$, $|\eta_x| \leq \frac{2}{(\sigma-\rho)R}$. By setting $N = \infty$ the definition of $F(u)$ yields $G(u) = \int\limits_0^u |F'(t)|^{\frac{1}{2}}\,dt = \int\limits_0^u |\delta(t^+)^{\delta-1}|^{\frac{1}{2}}\,dt = \sqrt{\delta}\int\limits_0^u (t^+)^{\frac{\delta-1}{2}}\,dt = \frac{\sqrt{\delta}}{q}(u^+)^q$.
With the "doubling property" (Lemma 2) and $w(B_{\beta R}) \leq w(B_R)$ we infer

$$\left(\frac{1}{w(B_{\rho R})}\int\limits_{B_{\rho R}}|\eta G|^{2k}w\,dx\right)^{\frac{1}{2kq}} \leq (CR)^{\frac{1}{q}}\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\sigma R}}\left(\eta_x u^+G'\right)^2w\,dx\right)^{\frac{1}{2q}}.$$

From $G(u) = \frac{\sqrt{\delta}}{q}\left(u^+\right)^q$ and the properties of η it follows

$$\left(\frac{1}{w(B_{\rho R})}\int\limits_{B_{\rho R}}\left(u^+\right)^{2kq}w\,dx\right)^{\frac{1}{2kq}} \leq \left(\frac{Cq}{\sigma-\rho}\right)^{\frac{1}{q}}\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\sigma R}}\left(u^+\right)^{2q}w\,dx\right)^{\frac{1}{2q}}. \tag{17}$$

Now we will iterate estimate (17). For this we set $q_0 = 1, q_i = kq_{i-1}$ $= k^i \to \infty, i \to \infty$, since $k > 1$. Furthermore, $\rho_i = \alpha + (\beta-\alpha)^{1+i}$, $\sigma_i = \rho_{i-1}$ $\to \rho_0 = \beta, \sigma_\infty = \alpha$ and $\rho_i - \rho_{i+1} = (\beta-\alpha)^{1+i}\left(1-(\beta-\alpha)\right)$. We obtain:

$$\lim_{i\to\infty}\left(\frac{1}{w(B_{\rho R})}\int\limits_{B_{\rho R}}\left(u^+\right)^{2kq_i} w\,dx\right)^{\frac{1}{2kq_i}} = \sup_{B_{\alpha R}} u^+.$$ In view of (17) we estimate

$$\sup_{B_{\alpha R}} u \le \sup_{B_{\alpha R}} u^+ \le \prod_{l=0}^{\infty}\left(\frac{Cq_l}{\rho_l-\rho_{l+1}}\right)^{\frac{1}{q_l}}\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}}\left(u^+\right)^2 w\,dx\right)^{\frac{1}{2}}. \tag{18}$$

It remains to show that the product appearing in (18) is finite.

$\prod_{l=0}^{\infty}\left(\frac{Cq_l}{(\beta-\alpha)^{1+l}(1-(\beta-\alpha))}\right)^{\frac{1}{q_l}} = \exp\left(\sum_{l=0}^{\infty}\left(\frac{1}{q_l}\log\left(\frac{Cq_l}{(\beta-\alpha)^{1+l}(1-(\beta-\alpha))}\right)\right)\right).$

Using the geometric sum we get

i) $\sum\limits_{l=0}^{\infty}\frac{1}{q_l} = \sum\limits_{l=0}^{\infty}\frac{1}{k^l} = \frac{1}{1-\frac{1}{k}} < \infty \quad (k>1)$

ii) $\sum\limits_{l=0}^{\infty}\frac{\log q_l}{q_l} = \sum\limits_{l=0}^{\infty}\frac{\log k^l}{k^l} = \log k\sum\limits_{l=0}^{\infty}\frac{l}{k^l} = \log k\left[\frac{1}{k\left(1-\frac{1}{k}\right)^2}\right] = \frac{k\log k}{(k-1)^2} < \infty$

iii) $\sum\limits_{l=0}^{\infty}\frac{l+1}{q_l} < \infty \quad$ (by using i) and ii))

Altogethter we arrive at

$\quad\exp\left(\sum_{l=0}^{\infty}\left(\frac{1}{q_l}\log\left(\frac{Cq_l}{(\beta-\alpha)^{1+l}(1-(\beta-\alpha))}\right)\right)\right)$

$= \exp\left(\frac{1}{1-\frac{1}{k}}\log C + \frac{k\log k}{(k-1)^2} - \frac{\log(1-(\beta-\alpha))}{1-\frac{1}{k}} - C\log(\beta-\alpha)\right) \le C(n,\alpha,\beta,C_2).$

The assertion $\sup\limits_{B_{\alpha R}} \le C\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}}|u^+|^2 w\,dx\right)^{\frac{1}{2}}$ follows with (18). □

Lemma 8 *We assume that the assumptions of Theorem 2 are satisfied. Then*

$$\frac{1}{\inf\limits_{B_{\alpha R}} u} \le \exp\left(C - \frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}}\log u\,w\,dx\right) \tag{19}$$

Proof. First we assume $u \ge \epsilon > 0$, choose as testfunction $\phi(x) = \eta(x)u^{-1}(x)$, $\eta \ge 0, \eta \in C_c^1(\Omega)$ (ϕ is admissible). Since u is a supersolution of (9) we infer

$$\int\limits_{\Omega} a^{\alpha\beta}(x)D_\beta u D_\alpha \eta u^{-1}\,dx - \int\limits_{\Omega} a^{\alpha\beta}(x)D_\beta u D_\alpha u\eta u^{-2}\,dx \ge 0.$$

Define $v := \log\left(\frac{t}{u}\right)$, $t = const. > 0 \to D_\beta v = -\frac{D_\beta u}{u}$.

Since $\int\limits_{\Omega} a^{\alpha\beta}(x)D_\beta u D_\alpha u\eta u^{-2}\,dx \ge \int\limits_{\Omega}|\nabla u|^2\eta u^{-2}w\,dx \ge 0$, it follows

$\int\limits_{\Omega} a^{\alpha\beta}(x)D_\beta v D_\alpha \eta\,dx \le 0$. This implies that v is a weak subsolution of (9) and

Lemma 7 yields

$$\sup_{B_{\alpha R}} v \leq C \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} |v^+|^2 w \, dx \right)^{\frac{1}{2}}. \tag{20}$$

Now we use $\phi(x) = \eta^2(x) u^{-1}(x)$, $\eta \in C_c^1(\Omega)$ as a testfunction in (10). Thus,

$$2 \int\limits_{\Omega} a^{\alpha\beta}(x) D_\beta u \eta D_\alpha \eta u^{-1} \, dx - \int\limits_{\Omega} a^{\alpha\beta}(x) D_\beta u D_\alpha u \eta^2 u^{-2} \, dx \geq 0.$$

Using (2) and the Hölder inequality we obtain

$$\begin{aligned} \int\limits_{\Omega} \eta^2 u^{-2} |\nabla u|^2 w \, dx &\leq C \int\limits_{\Omega} \eta |\eta_x| |\nabla u| u^{-1} w \, dx \\ &\leq C \left(\int\limits_{\Omega} |\eta_x|^2 w \, dx \right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\nabla u|^2 \eta^2 u^{-2} w \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore $\int\limits_{\Omega} \eta^2 u^{-2} |\nabla u|^2 w \, dx \leq C \int\limits_{\Omega} \eta_x^2 w \, dx$.

Choose η in a way that $\eta \equiv 1$ in $B_{\beta R}$, supp $\eta \subset B_R$, $|\eta_x| \leq \frac{2}{(1-\beta)R}$ and notice $\nabla v = u^{-1} \nabla u \to |\nabla v|^2 = u^{-2} |\nabla u|^2$, with this the last estimate yields

$$\int\limits_{B_{\beta R}} |\nabla v|^2 w \, dx \leq C \left(\frac{1}{R^2} \int\limits_{B_R} w \, dx \right).$$

Determine t in such a way that $\int\limits_{B_{\beta R}} v w \, dx = 0$, i.e. $\log t = \frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} \log u w \, dx$.

The Poincaré inequality (Lemma 5) an the above estimate yield

$$\begin{aligned} \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} |v|^{2k} w \, dx \right)^{\frac{1}{2k}} &\leq CR \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} |\nabla v|^2 w \, dx \right)^{\frac{1}{2}} \\ &\leq C \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} w \, dx \right)^{\frac{1}{2}} \leq C(n, \beta, C_2). \end{aligned}$$

Together with (20) this implies

$$\begin{aligned} \sup_{B_{\alpha R}} v = \log t - \inf_{B_{\alpha R}} (\log u) = \log t + \log \left(\frac{1}{\inf\limits_{B_{\alpha R}} u} \right) &\leq C \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} |v^+|^2 w \, dx \right)^{\frac{1}{2}} \\ \overset{Hölder}{\leq} C \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} |v|^{2k} w \, dx \right)^{\frac{1}{2k}} &\leq C. \end{aligned}$$

We see $\log\left(\frac{1}{\inf\limits_{B_{\alpha R}} u}\right) \le C - \log t$, using the definition of t we arrive at

$$\left(\inf_{B_{\alpha R}} u\right)^{-1} \le \exp\left(C - \frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}} \log u w\, dx\right).$$

If only $u \ge 0$ instead of $u \ge \epsilon$ we use Levi's Theorem to complete the proof. □

Lemma 9 *We assume that the assumptions of Theorem 2 are satisfied. Then*

$$\left(\frac{1}{w(B_{\alpha R})}\int\limits_{B_{\alpha R}} |u|^\gamma w\, dx\right)^{\frac{1}{\gamma}} \le \exp\left(C + \frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}} \log u w\, dx\right) \tag{21}$$

where $0 < \gamma < k$, $C = C(n, \alpha, \beta, \gamma, C_2)$.

Proof. W.l.o.g. we assume again $u \ge \epsilon > 0$, if only $u \ge 0$ we use Levi's Theorem. Consider $f = v^- = \left(\log \frac{u}{t}\right)^+$ and test the weak formulation with $\phi(x) = \eta^2(x)u^{-1}(x)\left(f^\delta(x) + (2\delta)^\delta\right)$ where $\delta \ge 1$, $\eta \in C_c^1(B_R), \eta \ge 0$ (ϕ is admissible). We infer
$D_\alpha\phi = 2\eta D_\alpha \eta u^{-1}\left(f^\delta + (2\delta)^\delta\right) + \eta^2 u^{-2}\left(\delta f^{\delta-1} - f^\delta - (2\delta)^\delta\right) D_\alpha u$,
a short calculation shows the inequality

$$\delta f^{\delta-1} \le \frac{1}{2}\left(f^\delta + (2\delta)^\delta\right). \tag{22}$$

The weak formulation (10) shows

$$\int\limits_\Omega a^{\alpha\beta}(x) D_\beta u \eta^2 u^{-2}\left(-\delta f^{\delta-1} + f^\delta + (2\delta)^\delta\right) D_\alpha u\, dx \le 2\int\limits_\Omega a^{\alpha\beta}(x) D_\beta u \eta D_\alpha \eta u^{-1}\left(f^\delta + (2\delta)^\delta\right) dx.$$

In view of (2) we can estimate

$$\int\limits_\Omega \eta^2 u^{-2}\left(f^\delta + (2\delta)^\delta - \delta f^{\delta-1}\right)|\nabla u|^2 w\, dx \le C\int\limits_\Omega \eta |\nabla u||\eta_x| u^{-1}\left(f^\delta + (2\delta)^\delta\right) w\, dx.$$

(22) yields

$$\int\limits_\Omega \eta^2 u^{-2}\left(f^\delta + (2\delta)^\delta\right)|\nabla u|^2 w\, dx \le C\int\limits_\Omega \eta|\eta_x| u^{-1}\left(f^\delta + (2\delta)^\delta\right)|\nabla u| w\, dx.$$

With $|\nabla f|^2 = u^{-2}|\nabla u|^2$ and the Hölder inequality we infer

$$\int\limits_\Omega \eta^2\left(f^\delta + (2\delta)^\delta\right)|\nabla f|^2 w\, dx \le C\int\limits_\Omega \eta_x^2\left(f^\delta + (2\delta)^\delta\right) w\, dx.$$

Using Young's inequality we have $f^\delta + (2\delta)^\delta \le 2\left(f^{\delta+1} + (2\delta)^\delta\right)$; with this and (22) we obtain

$$\delta \int\limits_\Omega \eta^2 f^{\delta-1} |\nabla f|^2 w\, dx \le C \int\limits_\Omega \eta_x^2 \left(f^{\delta+1} + (2\delta)^\delta\right) w\, dx. \tag{23}$$

By setting $q := \frac{\delta+1}{2} > 1$ the Sobolev inequality (Lemma 4) and Young's inequality imply for $\eta f^q \in \mathring{H}_2^1(B_R, w)$:

$$\left(\frac{1}{w(B_R)} \int\limits_{B_R} |\eta f^q|^{2k} w\, dx\right)^{\frac{1}{2k}} \le CR \left(\frac{1}{w(B_R)} \int\limits_{B_R} \left(\eta_x^2 f^{\delta+1} + \eta^2(\delta+1)^2 f^{\delta-1}|\nabla f|^2\right) w\, dx\right)^{\frac{1}{2}}.$$

With (23) we can estimate further

$$\begin{aligned} &CR \left(\frac{1}{w(B_R)} \int\limits_{B_R} \left(\eta_x^2 f^{\delta+1} + \eta^2(\delta+1)^2 f^{\delta-1}|\nabla f|^2\right) w\, dx\right)^{\frac{1}{2}} \\ &\le CR \left(\frac{1}{w(B_R)} \int\limits_{B_R} \left(q\eta_x^2 f^{\delta+1} + \frac{(\delta+1)^2}{\delta} \eta_x^2 \left(f^{\delta+1} + (2\delta)^\delta\right)\right) w\, dx\right)^{\frac{1}{2}} \\ &\le C\sqrt{q} R \left(\frac{1}{w(B_R)} \int\limits_{B_R} \left(\eta_x f^q\right)^2 w\, dx + (2\delta)^\delta \sup_{B_R} |\eta_x|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Altogether we have

$$\left(\frac{1}{w(B_R)} \int\limits_{B_R} |\eta f^q|^{2k} w\, dx\right)^{\frac{1}{2k}} \le C\sqrt{q} R \left(\frac{1}{w(B_R)} \int\limits_{B_R} \left(\eta_x f^q\right)^2 w\, dx + (2\delta)^\delta \sup_{B_R} |\eta_x|^2\right)^{\frac{1}{2}}. \tag{24}$$

Choose ρ and σ in such a way, that $\alpha \le \rho < \sigma \le \beta$ and η in a way, that supp $\eta \subset B_{\sigma R}, \eta \equiv 1$ in $B_{\rho R}, |\nabla \eta| \le \frac{2}{(\sigma-\rho)R}$. Taking roots yields:

$$\begin{aligned} &\left(\frac{1}{w(B_R)} \int\limits_{B_R} |\eta f^q|^{2k} w\, dx\right)^{\frac{1}{2kq}} \\ &\le (C\sqrt{q}R)^{\frac{1}{q}} \left\{\left(\frac{2}{(\sigma-\rho)R}\right)^2 \frac{1}{w(B_{\sigma R})} \int\limits_{B_{\sigma R}} f^{2q} w\, dx + (2\delta)^\delta \left(\frac{2}{(\sigma-\rho)R}\right)^2\right\}^{\frac{1}{2q}} \\ &\le (Cq)^{\frac{1}{q}} (\sigma-\rho)^{-\frac{1}{q}} \left\{(2\delta)^{\frac{\delta}{\delta+1}} + \left(\frac{1}{w(B_{\sigma R})} \int\limits_{B_{\sigma R}} f^{2q} w\, dx\right)^{\frac{1}{2q}}\right\}. \end{aligned}$$

Using the doubling property we find

$$\left(\frac{1}{w(B_{\rho R})}\int\limits_{B_{\rho R}} f^{2qk} w\,dx\right)^{\frac{1}{2kq}} \leq (Cq)^{\frac{1}{q}}(\sigma-\rho)^{-\frac{1}{q}}\left\{Cq+\left(\frac{1}{w(B_{\sigma R})}\int\limits_{B_{\sigma R}} f^{2q} w\,dx\right)^{\frac{1}{2q}}\right\}. \tag{25}$$

By taking $q_i = k^i \geq 1, \rho_i = \alpha + 2^{-i}(\beta-\alpha), \sigma_i = \rho_i + 2^{-i}(\beta-\alpha)$
$\rightarrow \sigma_i - \rho_i = 2^{-i}(\beta-\alpha)$ (notice $\sigma_i = \rho_{i-1}, \sigma_1 = \beta, \rho_\infty = \alpha$) (25) reads as follows:

$$\left(\frac{1}{w(B_{\rho_i R})}\int\limits_{B_{\rho_i R}} f^{2k^{i+1}} w\,dx\right)^{\frac{1}{2k^{i+1}}} \leq \left(Ck^i\right)^{\frac{1}{k^i}}\left(2^{-i}(\beta-\alpha)\right)^{-\frac{1}{k^i}}\left[Ck^i+\left(\frac{1}{w(B_{\sigma_i R})}\int\limits_{B_{\sigma_i R}} f^{2k^i} w\,dx\right)^{\frac{1}{2k^i}}\right]$$

$$\leq \left(Ck^i 2^i\right)^{\frac{1}{k^i}}\left[Ck^i+\left(\frac{1}{w(B_{\sigma_i R})}\int\limits_{B_{\sigma_i R}} f^{2k^i} w\,dx\right)^{\frac{1}{2k^i}}\right]$$

<u>Iteration of this estimate</u>

First step:

$$\left(\frac{1}{w(B_{\rho_i R})}\int\limits_{B_{\rho_i R}} f^{2k^{i+1}} w\,dx\right)^{\frac{1}{2k^{i+1}}}$$

$$\leq \left(Ck^i 2^i\right)^{\frac{1}{k^i}}\left[Ck^i + \left(Ck^{i-1}2^{i-1}\right)^{\frac{1}{k^{i-1}}}\left(Ck^{i-1}+\left(\frac{1}{w(B_{\sigma_{i-1} R})}\int\limits_{B_{\sigma_{i-1} R}} f^{2k^{i-1}} w\,dx\right)^{\frac{1}{2k^{i-1}}}\right)\right]$$

After $i-1$ iteration steps we obtain

$$\left(\frac{1}{w(B_{\rho_i R})}\int\limits_{B_{\rho_i R}} f^{2k^{i+1}} w\,dx\right)^{\frac{1}{2k^{i+1}}} \leq \sum_{j=1}^{i} Ck^j \prod_{l=j}^{i}\left(Ck^l 2^l\right)^{\frac{1}{k^l}} + \prod_{j=1}^{i}\left(Ck^j 2^j\right)^{\frac{1}{k^j}}\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}} f^{2k} w\,dx\right)^{\frac{1}{2k}}. \tag{26}$$

<u>Estimation of the sums and products in (26)</u>

1) $\prod\limits_{j=1}^{i}\left(Ck^j 2^j\right)^{\frac{1}{k^j}} \leq \exp\left(\sum\limits_{j=1}^{\infty}\frac{1}{k^j}\log\left(Ck^j 2^j\right)\right) = \exp\left(\sum\limits_{j=1}^{\infty}\left(\frac{1}{k^j}\log(C)+\frac{j}{k^j}\log(2k)\right)\right) < \infty,$

thus

$$\prod_{j=1}^{i}\left(Ck^j 2^j\right)^{\frac{1}{k^j}}\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}} f^{2k} w\,dx\right)^{\frac{1}{2k}} \leq C(n,\alpha,\beta,C_2)\left(\frac{1}{w(B_{\beta R})}\int\limits_{B_{\beta R}} f^{2k} w\,dx\right)^{\frac{1}{2k}}.$$

2a) If $1 < k < 2$, set $p := 2k^{m+1}, m \in \mathbb{N}$

$$\rightarrow \sum_{i=1}^{m+1} Ck^i \prod_{j=i}^{m+1} \left(Ck^j 2^j\right)^{\frac{1}{k^j}} \leq C \sum_{i=1}^{m+1} k^i \prod_{j=i}^{m+1} (2k)^{\frac{j}{k^j}} \leq C \sum_{i=1}^{m+1} k^i \prod_{j=i}^{m+1} k^j$$

$$= C \left[k^{1+\sum_{i=1}^{m+1} i} + k^{2+\sum_{i=2}^{m+1} i} + \ldots + k^{(m+1)+(m+1)} \right].$$

It applies $i + \sum_{l=i}^{m+1} l = \frac{(m+1)(m+2)}{2} + \frac{3i-i^2}{2}$, this term takes its maximum (for a fixed m) at $i = 1$ and $i = 2$, for these i we have: $i + \sum_{l=i}^{m+1} l = \frac{(m+1)(m+2)}{2} + 1$, therefore $C \left[k^{1+\sum_{i=1}^{m+1} i} + \ldots + k^{(m+1)+(m+1)} \right] \leq Ck^{2+\sum_{i=2}^{m+1} i} \leq C2k^{m+1} \leq Cp.$

2b) If $k \geq 2$, set $p := 2k^{m+1}$, then we infer

$$\sum_{i=1}^{m+1} Ck^i \prod_{j=i}^{m+1} \left(Ck^j 2^j\right)^{\frac{1}{k^j}} \leq C \sum_{i=1}^{m+1} k^i \prod_{j=1}^{\infty} (2k)^{\frac{j}{k^j}} \leq Ck^{m+1+\sum_{j=1}^{\infty} \frac{j}{2^j}} \leq Ck^{m+3}$$

$$\leq C(2k)^{m+1} \leq Cp.$$

From these inequalities we obtain by using the doubling property for every $p = 2k^{m+1}$

$$\left(\frac{1}{w(B_{\alpha R})} \int_{B_{\alpha R}} f^p w \, dx \right)^{\frac{1}{p}} \leq C \left[p + \left(\frac{1}{w(B_{\beta R})} \int_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \right]. \tag{27}$$

If $p = 2ak^{i+1}$ for some $i \in \mathbb{N}$ and $a \in (1, k)$, we can use the Hölder inequality to show

$$\left(\frac{1}{w(B_{\alpha R})} \int_{B_{\alpha R}} f^p w \, dx \right)^{\frac{1}{p}} \leq C \left(2k^{i+2} + \left(\frac{1}{w(B_{\beta R})} \int_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \right]$$

$$\leq C \left[p + \left(\frac{1}{w(B_{\beta R})} \int_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \right].$$

Altogether we have for any $p > 2k$ the estimate

$$\left(\frac{1}{w(B_{\alpha R})} \int_{B_{\alpha R}} f^p w \, dx \right)^{\frac{1}{p}} \leq C \left[p + \left(\frac{1}{w(B_{\beta R})} \int_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \right]. \tag{28}$$

Now let $p_0 \in \left(0, e^{-1}\right)$, by using Taylor series we infer

$$\frac{1}{w(B_{\alpha R})} \int_{B_{\alpha R}} e^{p_0 f} w \, dx = \sum_{l=0}^{\infty} \frac{{p_0}^l}{l!} \left(\frac{1}{w(B_{\alpha R})} \int_{B_{\alpha R}} f^l w \, dx \right)$$

$$\leq C \sum_{l=0}^{[2k]} \frac{{p_0}^l}{l!} \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{l}{2k}} + C \sum_{l=[2k]+1}^{\infty} \frac{{p_0}^l}{l!} \left(l + \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \right)^l .$$

Here, we used the Hölder inequality, the doubling property and (28).

Define $\Delta := \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}}$, with the help of the "Stirling formula"

$n! \cong \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ we obtain

$$C \sum_{l=[2k]+1}^{\infty} \frac{{p_0}^l}{l!} (l+\Delta)^l \leq C \sum_{l=[2k]+1}^{\infty} \frac{(p_0 e)^l \left(1 + \frac{\Delta}{l}\right)^l}{\sqrt{2\pi l}} \leq C \sum_{l=[2k]+1}^{\infty} (p_0 e)^l \left(1 + \frac{\Delta}{l}\right)^l$$

$$\leq C \sum_{l=[2k]+1}^{\infty} (p_0 e)^l e^{\Delta} \leq C \frac{1}{1 - p_0 e} e^{\Delta}.$$

In other words

$$C \sum_{l=[2k]+1}^{\infty} \frac{{p_0}^l}{l!} \left(l + \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \right)^l \leq C e^{\left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}}} .$$

Now we can estimate the sum $\sum_{l=0}^{[2k]} \frac{{p_0}^l}{l!} \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{l}{2k}}$ by

$C \sum_{l=0}^{\infty} \frac{{p_0}^l}{l!} \Delta^l = C e^{p_0 \Delta} \leq C e^{\Delta}$. For $0 < p_0 < e^{-1}$ it follows

$$\frac{1}{w(B_{\alpha R})} \int\limits_{B_{\alpha R}} e^{p_0 f} w \, dx \leq C e^{\left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}}} . \tag{29}$$

Since $f = \log \left(\frac{u}{t}\right)^+$ and $\left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}} \leq C$ (compare this with the proof of Lemma 8) we conclude

$$\left(\frac{1}{w(B_{\alpha R})} \int\limits_{B_{\alpha R}} e^{p_0 f} w \, dx \right)^{\frac{1}{p_0}} \overset{u \geq 0}{\geq} \left(\frac{1}{w(B_{\alpha R})} \int\limits_{B_{\alpha R}} \left| \frac{u}{t} \right|^{p_0} w \, dx \right)^{\frac{1}{p_0}} \leq C \left(e^{\left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} f^{2k} w \, dx \right)^{\frac{1}{2k}}} \right)^{\frac{1}{p_0}} \leq C. \tag{30}$$

At the end of this proof we estimate $||\frac{u}{t}||_{L_\gamma(w,B_{\alpha R})}$ in terms of $||\frac{u}{t}||_{L_{p_0}(w,B_{\alpha' R})}$ $(\alpha < \alpha' < \beta < 1)$. We assumed that $u \geq 0$ is a supersolution, therefore $-u \leq 0$ is a subsolution of (9). Replace the function $F(u)$ in the proof of Lemma 7 by

$$F(u) = \begin{cases} -u^\delta, & u \leq N \\ -\delta N^{\delta-1}u + (\delta-1)N^\delta & u > N \end{cases}, \quad \text{where } \delta \in (-1,0).$$

Analogously to the proof of Lemma 7 we infer (notations as in Lemma 7):

$$\left(\frac{1}{w(B_R)} \int\limits_{B_R} |\eta G|^{2k} w\,dx \right)^{\frac{1}{2k}} \leq CR \left(\frac{1}{w(B_R)} \int\limits_{B_R} |\eta_x u G'|^2 w\,dx \right)^{\frac{1}{2}}.$$

Repeating the arguments in the proof of Lemma 7 for $q \in \left(0, \frac{1}{2}\right)$ we arrive at

$$\left(\frac{1}{w(B_{\rho R})} \int\limits_{B_{\rho R}} u^{2kq} w\,dx \right)^{\frac{1}{2kq}} \leq \left(\frac{Cq}{\sigma - \rho} \right)^{\frac{1}{q}} \left(\frac{1}{w(B_{\beta R})} \int\limits_{B_{\sigma R}} u^{2q} w\,dx \right)^{\frac{1}{2q}}. \quad (31)$$

Iteration of (31):

Let $q_0 := \frac{\gamma}{2k} < \frac{1}{2}$, $q_i := \frac{1}{k} q_{i-1} = \frac{\gamma}{2k^{i+1}} \to 0$, $i \to \infty$. After finite many iteration steps we achieve $2q_i < p_0 < e^{-1}$.

As is easily seen (31) holds also for $\frac{u}{t}$ we infer with $2q_0 k = \gamma$, $\sigma_i =: \alpha' < \beta$ and the doubling property from (30) and (31)

$$\left(\frac{1}{w(B_{\alpha R})} \int\limits_{B_{\alpha R}} \left|\frac{u}{t}\right|^\gamma w\,dx \right)^{\frac{1}{\gamma}} \leq C \left(\frac{1}{w(B_{\alpha' R})} \int\limits_{B_{\alpha' R}} \left|\frac{u}{t}\right|^{p_0} w\,dx \right)^{\frac{1}{p_0}} \overset{(30),\alpha\leftrightarrow\alpha'}{\leq} C. \quad (32)$$

It follows $\frac{1}{t} \left(\frac{1}{w(B_{\alpha R})} \int\limits_{B_{\alpha R}} u^\gamma w\,dx \right)^{\frac{1}{\gamma}} \leq e^C$; with $\log t = \frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} \log u w\,dx$ we obtain

$$\left(\frac{1}{w(B_{\alpha R})} \int\limits_{B_{\alpha R}} u^\gamma w\,dx \right)^{\frac{1}{\gamma}} \leq exp \left(C + \frac{1}{w(B_{\beta R})} \int\limits_{B_{\beta R}} \log u w\,dx \right). \qquad \square$$

Proof of Theorem 2

Multiply (19) with (21), for suitable choices of α and β the statement of Theorem 2 follows. $\square$

4 The regularity result and examples

To prove Theorem 1 we use exactly the same proof as in the uniformly elliptic case, except one part: Instead of the weak Harnack inequality for uniformly elliptic equations (Lemma 1, Lecture 1) we use now the weak Harnack inequality derived in section 3 (Theorem 2). It's obvious, that we can prove a priori estimates for every system in diagonal form, for which a weak Harnack inequality for the corresponding equation is valid. In the authors thesis [PI] such a Harnack inequality is proved for coefficients which depend on two different weights. The idea of the proof is the same as in section 3, but due to the second weight the proof becomes more involved, we will only mention an example (see below).

Examples: 1) Consider the weight $w(x) = r^\alpha = |x|^\alpha$ with $\alpha \in (-n, n)$ in $B_R(0) \subset \mathbb{R}^n$. A short calculation shows $w \in A_2$. Form the theory of quasiconformal mappings we concluded at the end of section 2 that $|x|^{\alpha(n-2)}$, $\alpha > -1$ is also an admissible weight. Therefore every bounded, weak solution of the system

$$-D_i(r^\alpha D_i u^k) = 0 \qquad (k = 1, \ldots, m)$$

in $B_R(0)$ is locally Hölder continuous, provided $\alpha > -n$.

2) Let $w(x) = (\log r)^k = (\log |x|)^k$, $x \in B_{\frac{1}{2}}(0) \subset \mathbb{R}^n, k \in 2\mathbb{N}$. Then we can calculate that $w \in A_2$. It follows that every bounded, weak solution of the system

$$-D_\alpha((\log r)^k D_\alpha u^i) = 0 \quad (i = 1, \ldots, m)$$

with $k \in 2\mathbb{N}$ is locally Hölder continuous in $B_{\frac{1}{2}}(0)$.

3) Let $w(x) = r^\alpha (\log r)^2$, $r = |x|$, $x \in B_{\frac{1}{2}}(0) \subset \mathbb{R}^n, \alpha \in (-n, n)$. Then it's possible to show $w \in A_2$. We obtain that every bounded weak solution of the system

$$-D_i(r^\alpha (\log r)^2 D_i u^k) = 0 \quad (k = 1, \ldots, m)$$

is locally Hölder continuous in $B_{\frac{1}{2}}(0)$, if $\alpha \in (-n, n)$.

4) Consider the weights $v(x) = |x|$ and $w(x) = |x|^\tau$, $\tau \in (1, 2)$, $x \in B_1(0) \subset \mathbb{R}^n$. In [PI] it's shown that every bounded, weak solution of

$$-D_\alpha \left(a^{\alpha\beta}(x) D_\beta u^k \right) = 0 \qquad (k = 1, \ldots, m)$$

is locally Hölder continuous, provided the coefficients $a^{\alpha\beta}$ fulfill the estimate

$$|x|^\tau |\xi|^2 \le a^{\alpha\beta}(x) \xi_\alpha \xi_\beta \le |x| |\xi|^2 \ \ \forall \xi \in \mathbb{R}^n \qquad (\tau \in (1, 2))$$

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