Workshop on Higher Dimensional Algebraic Geometry

National Taiwan University

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Invited Speakers:

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Andre Chatzistamatiou (Universität Duisburg-Essen)
Juan Cervino (Universität Duisburg-Essen)
Jungkai Alfred Chen (National Taiwan University)
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Radu Laza (State University of New York, Stony Brook)
Mircea Mustață (University of Michgan, Ann Arbor)
Yukinobu Toda (IPMU, University of Tokyo)
Hsian-Hua Tseng (Ohio State University)
Yu-Jong Tzeng (Stanford University)
Gerard van der Geer (Universiteit van Amsterdam)
Chin-Lung Wang (National Taiwan University)
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HIGHER DIRECT IMAGES OF THE STRUCTURE SHEAF IN POSITIVE CHARACTERISTIC

ANDRE CHATZISTAMATIOU (JOINT WORK WITH KAY RÜLLING)

Abstract. These are the notes for my talk at the conference "Higher Dimensional Algebraic Geometry, 2010" in Taipei (Taiwan). In this talk I present my joint work with Kay Rülling on the higher direct images of the structure sheaf under a proper birational morphism.

0.1. **Notation.** All schemes are assumed to be separated and of finite type over a perfect field k.

1. The story in characteristic zero

1.1. **Proper birational maps between smooth schemes.** In characteristic zero it is a well-known and frequently used fact that the higher direct images $R^i f_* \mathcal{O}_X$ of a proper birational morphism $f : X \to Y$ between smooth schemes vanish for $i > 0$. This statement was proved as a corollary of Hironaka's resolution of singularities.

Let us sketch the proof.

Proof. We resolve the indeterminacies of f^{-1} by successively blowing-up smooth subvarieties, and thus we get a commutative diagram

$$\begin{array}{ccc} & & \tilde{Y} \\ & \swarrow^{h} & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y. \end{array}$$

The natural morphism $\mathcal{O}_X \to Rh_* \mathcal{O}_{\tilde{Y}}$ has a section by the trace $Tr : Rh_* \mathcal{O}_{\tilde{Y}} \to \mathcal{O}_X$, and therefore

$$Rf_* \mathcal{O}_X \to Rf_* Rh_* \mathcal{O}_{\tilde{Y}} \to Rf_* \mathcal{O}_X$$

is the identity. Since p is the composition of blow-ups we get $Rf_* Rh_* \mathcal{O}_{\tilde{Y}} = Rp_* \mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$. ☐

1.2. **Rational singularities.** A special class of singularities which appear naturally in higher dimensional geometry are the *rational* singularities. Essentially, rational singularities do not affect the cohomological properties of the structure sheaf.

Definition 1.2.1. Let Y be a scheme and $f : X \to Y$ a resolution of singularities. We say that f is a *rational resolution* if

(1) $Rf_* \mathcal{O}_X = \mathcal{O}_Y$,

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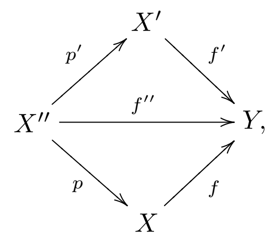
(2) $R^i f_* \omega_X = 0$ for $i > 0$ (this always holds if $\mathrm{char}(k) = 0$ by Grauert-Riemenschneider, but is needed in positive characteristic).

We say that Y has *rational singularities* if a rational resolution exists.

Suppose Y has rational singularities, then every resolution $f' : X' \to Y$ is rational. Indeed, let $f : X \to Y$ be a rational resolution then we can find a third resolution $f'' : X'' \to Y$ fitting into a commutative diagram



and since $Rp_* \mathcal{O}_{X''} = \mathcal{O}_X, Rp'_* \mathcal{O}_{X''} = \mathcal{O}_{X'}$, we obtain $Rf'_* \mathcal{O}_{X'} = Rf''_* \mathcal{O}_{X''} = Rf_* \mathcal{O}_X = \mathcal{O}_Y$.

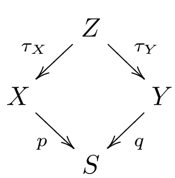
Important examples for rational singularities:

- Log terminal singularities which appear naturally in the minimal model program; this a theorem due to Elkik ([Elk81], see [KM98, Theorem 5.22]).
- Quotient singularities. This is a theorem due to Viehweg [Vie77].

2. The story in positive characteristic

Suppose that the characteristic of k is positive. Of course, the main problem is that resolution of singularities is not available yet. However, without using resolution we can show the following two results [CR09].

Theorem 1. *Let S be an arbitrary scheme and let X, Y be integral S-schemes. Assume that X, Y are smooth over k and properly birational over S, i.e. there exists an integral scheme Z and a commutative diagram*



such that τ_X, τ_Y are proper and birational (p and q being the fixed morphisms to S). Then for all i, there are isomorphisms of $\mathcal{O}_S$-modules

$$R^i p_* \mathcal{O}_X \cong R^i q_* \mathcal{O}_Y, \quad R^i p_* \omega_X \cong R^i q_* \omega_Y.$$

Corollary 2.0.2. *If S has rational singularities then every resolution of singularities is rational.*

Corollary 2.0.3. *If $S = \mathrm{Spec}(k)$ and X, Y proper, then $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y)$. In other words the dimensions $h^i(X, \mathcal{O}_X)$ are birational invariants.*

Theorem 2. *Consider a diagram*

$$\begin{array}{ccc} & & Y' \\ & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y, \end{array}$$

where Y' and X are connected smooth schemes, Y is integral and geometrically normal (e.g. k is perfect and Y is normal), p is surjective and finite such that $\deg(p) \in k^*$, and f is birational and proper. Then

$$Rf_*\mathcal{O}_X = \mathcal{O}_Y, \quad Rf_*\omega_X = \omega_Y,$$

where ω_Y is the dualizing sheaf of Y. In particular, Y has rational singularities.

By using Witt vector cohomology and Lefschetz trace formula we can show the following application.

Corollary 2.0.4. *Let k be a finite field. Let $p : Y' \to Y$ be a finite quotient as in Theorem 2, e.g. Y' is smooth and admits an action of a finite (abstract) group G of order prime to p and $Y = Y'/G$. Let $f : X \to Y$ be a resolution of singularities. For $y \in Y(k)$ the following formula holds*

$$\#f^{-1}(y)(k) = 1 \mod |k|.$$

As a corollary of Theorem 1 or Theorem 2 we obtain:

Corollary 2.0.5. *Let k be an arbitrary field. Let $f : X \to Y$ be a proper birational morphism between smooth schemes X, Y. Then $Rf_*(\mathcal{O}_X) = \mathcal{O}_Y$.*

For the remainder of the talk we will sketch the proof of Corollary 2.0.5. To this extend we introduce the notion of a cohomology theory with support. We discuss two examples for such a cohomology theory: the Chow groups and the Hodge cohomology.

Let (X, Φ) be a pair consisting of a smooth scheme X and a family of supports Φ on X, i.e. Φ is a set of closed subsets of X such that

(1) a finite union of elements in Φ is contained in Φ,
(2) if $W' \subset W \in \Phi$ for $W' \subset X$ closed then $W' \in \Phi$.

We define the Chow groups by

$$\mathrm{CH}(X, \Phi) := \varinjlim_{W \in \Phi} \mathrm{CH}(W).$$

We define the Hodge cohomology with support as follows

$$H(X, \Phi) := \bigoplus_{i,j} H^i_\Phi(X, \Omega^j_X) = \varinjlim_{W \in \Phi} \bigoplus_{i,j} H^i_W(X, \Omega^j_X).$$

These two "(co)homology" theories share the following functorialities. Let $F \in \{\mathrm{CH}, H\}$ and let $(X, \Phi), (Y, \Psi)$ be pairs of smooth schemes and families of supports.

(1) For a morphism $f : X \to Y$ such that $f\mid_\Phi$ is proper and $f(\Phi) \subset \Psi$ we have
$$f_* : F(X, \Phi) \to F(Y, \Psi).$$
(2) For a morphism $f : X \to Y$ such that $f^{-1}(\Psi) \subset \Phi$ we have
$$f^* : F(Y, \Psi) \to F(X, \Phi).$$

(3) A product

$$F(X,\Phi)\otimes_{\mathbb{Z}} F(Y,\Psi)\to F(X\times Y,\Phi\times\Psi).$$

Moreover, for a transversal cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

the equality $g_Y^*\circ f_* = f'_*\circ g_X^*$ holds.

These properties yield a calculus with correspondences. Given (X,Φ) and (Y,Ψ) there is natural family of supports $P(\Phi,\Psi)$ of $X\times Y$ and a map

$$F(X\times Y, P(\Phi,\Psi))\to \mathrm{Hom}(F(X,\Phi),F(Y,\Psi)),\quad c\mapsto [a\mapsto q_*(c\cup p^*(a))],$$

with the projections $p: X\times Y\to X, q: X\times Y\to Y$.

Our cohomology theories CH and H do not obviously satisfy the above functorialities. For CH one has to use Fulton's refined Gysin map to define f^*. For H we have to use Grothendieck duality to construct f_*.

The two examples are related by a functorial cycle class map

$$cl: \mathrm{CH}(X,\Phi)\to H(X,\Phi).$$

Proof of Corollary 2.0.5. If $f: X\to Y$ is a proper morphism between smooth schemes with $n=\dim X=\dim Y$ then the maps

$$f^*: H^*(Y,\mathcal{O}_Y)\to H^*(X,\mathcal{O}_X),\quad f_*: H^*(X,\mathcal{O}_X)\to H^*(Y,\mathcal{O}_Y),$$

are induced by $cl(\Gamma(f)^t)$ and $cl(\Gamma(f))$, respectively; here $\Gamma(f)\subset X\times Y$ denotes the graph of f (and $\Gamma(f)^t\subset Y\times X$ is the graph considered as closed set in $Y\times X$).

Then $f^*\circ f_*$ is induced by the correspondence $cl(\Gamma(f)^t\circ\Gamma(f))$. By using intersection theory we get

$$\mathrm{supp}(\Gamma(f)^t\circ\Gamma(f))\subset X\times_Y X.$$

Let $U\subset Y$ be an open set such that $f: f^{-1}(U)\to U$ is an isomorphism, and set $E := X\backslash f^{-1}(U)$. We have $X\times_Y X\subset \Delta_X\cup(E\times E)\subset(X\times X)$, and it is easy to see that

$$\Gamma(f)^t\circ\Gamma(f)=\Delta_X+Z$$

where Z is a cycle supported in $E\times E$.

Claim: $cl(Z)$ acts trivially on $H^*(X,\mathcal{O}_X)$. We may suppose that $Z=[W]$ with W irreducible. Now $cl([W])\in H^n_W(X\times X,\Omega^n_{X\times X})$ and we have a decomposition

$$\Omega^n_{X\times X}=\bigoplus_{i+j=n} p_1^*\Omega^i_X\otimes p_2^*\Omega^j_X \xrightarrow{pr} p_1^*\Omega^n_X.$$

The action on $H^*(X,\mathcal{O}_X)$ is trivial if $pr(cl([W]))=0$. Let η be the generic point of W then

$$H^n_W(X\times X,\Omega^n_{X\times X})\subset H^n_\eta(X\times X,\Omega^n_{X\times X}),$$

so that we may localise around η. Locally, write $W=p_2^{-1}(H_1)\cap H_2\cap\cdots\cap H_n$ for some hypersurfaces $H_2,\ldots,H_n$ in $X\times X$; since W does not map dominantly to X via p_2 we may choose H_1 to be a hypersurface in X. Since $cl([p_2^{-1}(H_1)]) \in H^1_{p_2^{-1}(H_1)}(X \times X, p_2^*\Omega^1_X)$ we obtain

$$\begin{aligned} cl([W]) &= cl([p_2^{-1}(H_1)]) \cup cl([H_2]) \cup \cdots \cup cl([H_n]) \\ &\in H^n_\eta(X \times X, \mathrm{image}(p_2^*\Omega_X \otimes \Omega^{n-1}_{X\times X})) = H^n_\eta(X \times X, \ker(pr)). \end{aligned}$$

□

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