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WEIL II

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Contents

Introduction

[1] "It is now nearly" 40 "years since Deligne, in the summer of 1973, proved the last of the Weil Conjectures, the Riemann Hypothesis for projective smooth varieties over finite fields. That proof was the subject of his article [Del74], often referred to as Weil I. In the fall of that same year, Deligne formulated and proved a vast generalization of his Weil I result. This generalization, which is the subject of Deligne's article [Del80], is referred to as Weil II. As marvelous an achievement as Weil I was, it is Weil II which has turned out to be the fundamental tool. For example, both the "hard Lefschetz theorem" for projective smooth varieties over finite fields and Deligne's general equidistribution theorem ("generalized Sato Tate conjecture in the function field case") require Weil II.

Deligne's proof of Weil I was based upon combining Grothendieck's ℓ-adic cohomological theory of L-functions, the monodromy theory of Lefschetz pencils, and Deligne's own stunning transposition to the function field case of Rankin's method [Ran39] of "squaring," developed by Rankin in 1939 to give the then best known estimates for the size of Ramanujan's $\tau(n)$. Deligne's proof of Weil II is generally regarded as being much deeper and more difficult than his proof of Weil I. In the

[1] [Kat01, Introduction]

spring of 1984, Laumon found a significant simplification [Lau87] of Deligne's proof of Weil II, based upon Fourier transform ideas."

In our seminar we will follow Katz's "Four Lectures on Weil II" [Kat01] to prove an important part of Weil II. The backbone of the proof is the same as in Weil I: in order to show the statement for a particular L-function one puts the L-function in a family of L-functions with good monodromy (in particular, it is geometrically irreducible, which is shown by using Fourier tranformations) and applies Rankin's method. In Katz's proof the family is constructed by twisting with wild (but very explicit) characters which come from Artin-Schreier sheaves.

1. Overview

Give an overview of the four lectures on Weil II by Katz [Kat01].

2. Review of ℓ-adic sheaves and ℓ-adic cohomology

Reference: [Kat01, p.83–88].

- Introduce lisse $\bar{\mathbb{Q}}_\ell$ sheaves as representaions of the étale fundamental group.
- Explain ℓ-adic cohomology, Tate twists and Poincare duality [Kat01, p.86].
- L-functions for constructible sheaves and rationality via cohomological interpretation (Grothendieck-Lefschetz trace formula).

3. Weights and ι-pure sheaves

- Weights, ι-pure lisse sheaves [Kat01, p.88–89].
- State the Target Theorem and the corollary [Kat01, p.89–90].
- Reduction of the Target Theorem to the case of a pure weight zero lisse sheaf on $\mathbb{A}^1$.

4. Twisting with wild characters

- Begin of [Kat01, Lecture II]: Introduce the Artin-Schreier sheaves.
- Define the Swan conductor [Kat83, p.212] and Swan polygon, prove the Wild twisting Proposition [Kat83, p.216].
- Use Grothendieck-Neron-Ogg-Shafarevic (GNOS) [Ray95] to compute the dimensions of $H^i_c(\mathbb{A}^1 \times_k \bar{k}, \mathcal{F} \otimes \mathcal{L}_{\psi(f)})$ [Kat01, p.96]. If time permits then sketch the proof of GNOS.
- Define the auxiliary sheaf [Kat01, p.96] and state the Purity Theorem [Kat01, p.97] (and Purity Corollary).

5. Degeneration and Weight drop Lemma

- Prove the Degeneration Lemma [Kat01, p.98] and the Weight drop Lemma [Kat01, p.100] in detail.
- Conclude that the Purity Theorem implies the Target Theorem [Kat01, p.98–101].

6. Monodromy Theorem implies Purity Theorem I

- Explain why the auxiliary sheaf $\mathcal{G}$ is lisse [Kat01, p.102].
- Introduce the geometric monodromy group and state the Monodromy Theorem [Kat01, p.103].
- Prove the Duality/Conjugacy Lemma [Kat01, p.104] and the Positive trace Corollary.

7. Monodromy Theorem implies Purity Theorem II

- After recalling the results from the previous talk start at [Kat01, p.106] "How to bring to bear the monodromy theorem?"; the main goal is that $H_c^2(C_d \times_k \bar{k}, (\mathcal{G} \otimes \mathcal{G}^\vee)^{\otimes a})$ is ι-pure of weight 2 [Kat01, p.108].
- Apply Rankin's method to deduce the Purity Theorem [Kat01, p.108–111].

8. Geometric irreducibility

- Geometric irreducibility of the auxiliary sheaf via Fourier transformation (need references)
- Geometrically finite order of the determinant [Kat01, p.112].

9. Moment criterion

- Prove the Moment criterion ("Larsen's Alternative") [Kat01, p.113].
- Try to cover as much as you can from [Kat01, p.114–119] "Step 4: Cohomological version of the Moment criterion."

10. The cohomological moment calculation

- "Step 5" [Kat01, p.120–124]

11. Application: Weil I and geometric semi-simplicity

- Recall that we have proved the Target Theorem [Kat01, p.124] and its Corollary.
- Prove the main theorem of Weil I [Kat01, p.125].
- Prove geometric semi-simplicity of lisse pure sheaves [Kat01, p.127].
- Explain "What we don't get from the Target Theorem" [Kat01, p.130].

12. Application: Hard Lefschetz Theorem

- Prove the Hard Lefschetz Theorem [Kat01, p.128], take the details from [Del80, 4.1.1–4].
- Prove the semi-simplicity of the geometric monodromy group [Kat01, p.128–129].

13. Application: Sato-Tate equidistribution and Ramanujan's Conjecture

- Show how Weil II implies Sato-Tate in the function field case [Del80, 3.5.3], [Kat88, §3].
- "Weil implies Ramanujan" [Kat01, p.129–130], [Del80, 3.7.1].

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Electronic versions of the references: `http://www.uni-due.de/~bm0065/teaching/ws1112/`