FORSCHUNGSSEMINAR WINTERSEMESTER 2010/2011
ALGEBRAIC STACKS

ANDRE CHATZISTAMATIOU

Contents

Introduction

The hierarchy of spaces in algebraic geometry is as follows:

$$\text{(Schemes)} \subset \text{(Algebraic spaces)} \subset \text{(Algebraic-stacks)}.$$

Let us fix a base scheme S, it gives rise to the big fppf-site of S, which we denote by $(Sch/S)_{fppf}$. A scheme X over S defines a sheaf of sets $T \mapsto \mathrm{Hom}_S(T, X) = X(T)$ on $(Sch/S)_{fppf}$. Algebraic spaces, first introduced by Artin, form a full subcategory of the category of sheaves on $(Sch/S)_{fppf}$. By definition an algebraic space admits an étale, surjective morphism from a scheme. The idea is that étale locally an algebraic spaces looks like a scheme. Algebraic spaces appear naturally as moduli spaces when a descent datum is effective in the category of algebraic spaces but not in the category of schemes.

The interest of stacks derives from the theory of fine moduli. If the objects one attempts to classify have automorphisms the moduli space is represented by a stack. If X is an algebraic space (or a scheme) over S, then $X(T)$ is a set, for every object $T \in (Sch/S)_{fppf}$. If $\mathcal{X}$ is an algebraic stack over $(Sch/S)_{fppf}$, then $\mathcal{X}(T)$ is groupoid, i.e. a category whose morphisms are invertible. To make $\mathcal{X}$ algebraic one requires the existence of smooth surjective morphism from a scheme to $\mathcal{X}$ (étale surjective for Deligne-Mumford stacks). Algebraic stacks classifying objects without infinitesimal automorphisms were first introduced by Deligne and Mumford [DM69]. The restriction on the automorphism groups was later lifted by Artin [Art74].

The goal of the seminar is to learn about the basic definitions and properties of stacks, discuss some examples, and finally to give the proof of Behrend's trace formula [Beh93]. Behrend's trace formula is analogous to the Grothendieck-Lefschetz fixed point formula, and counts the number of $\mathbb{F}_q$-rational points of a stack as the trace over the cohomology. It is proved in Behrend's thesis for Deligne-Mumford stacks and quotient stacks by the action of a linear algebraic group. Quotient stacks are one of the main examples for stacks and are denoted by $[X/G]$, where X is a scheme (say over $S = \mathrm{Spec}(k)$) with a group action by G. If $X = \mathrm{Spec}(k)$ with trivial group action then the quotient stack is denoted by BG. Roughly speaking, one computes the cohomology of $[X/G]$ for $G = GL_n$ as follows. The G-principal bundle $X \to [X/G]$ defines a morphism $[X/G] \to BG$ which is a fibration with fibre X. Then the Leray spectral sequence computes the cohomology of $[X/G]$ in terms of the cohomology of X and BG. An interesting point is that the cohomology of a stack is not in general finite dimensional (for example $H^*(B\mathbb{G}_m, \mathbb{Q}_\ell) = \oplus_{i\geq 0}\mathbb{Q}_\ell(-i)$). Thus one has to prove the convergence of the trace. Behrend's trace formula for general (smooth) algebraic stacks is proved in [Beh03]. The proof in the general case involves stratifying a general stack such that the strata are quotient stacks and setting up an ℓ-adic formalism (derived categories and some of Grothendiecks six operations) for stacks.

Acknowledgements. I'm grateful to Jochen Heinloth for his help.

1. Sites and sheaves (14 Oct)

This talk should provide some basic facts on sites and sheaves. The reference is [spa, 8 and 14]. We will also need higher direct images and the Leray spectral sequence (use the material of [Mil08, 12]). Start with the definitions: sites, sheaves, morphism of sites, topoi. Give some examples of big sites: Zariski, étale, fppf site of a scheme [spa, 26]. Explain higher direct images and the Leray spectral sequence in detail.

2. Descent (21 Oct)

The main reference is [spa, 27]. Provide facts and proofs on properties of morphisms which are local on the base for the fpqc/fppf site. Important for us are the properties flat, smooth, étale, surjective, finite, of finite type. . .

3. Algebraic spaces (28 Oct)

The main reference is [spa, 33]. The definition in [spa] is apriori different from the standard definition, but turns out to be equivalent. Introduce étale equivalence relations and explain the relation with algebraic spaces [spa, 33.10]. Discuss properties of algebraic spaces and morphisms [spa, 35]. Example: Quotients by free group actions [spa, 33.14]. If time permits then prove Knutson's theorem [Knu71, II 6.7,p.131] that an algebraic space admits an open set which is a scheme.

4. 2-categories, fibred categories, stacks (4 Nov)

Basic definitions. For 2-categories we can use [spa, 4.25-26]. Explain the notion of 2-fibre products [spa, 4.27] (2-cartesian squares) and 2-cocartesian squares. Fibred categories [spa, 4.29] and fibred in groupoids [spa, 4.31]. For the definition of stacks we need to know about presheaves of morphisms associated to fibred categories [spa, 39.2] and descent data in fibred categories [spa, 39.3]. Finally, define a stack, morphisms between stacks. Definition of the inertia stack [spa, 39.7]. Give as many examples as possible [spa, 43].

5. Algebraic stacks (11 Nov)

Give the definition of an algebraic stack [spa, 45]. Properties of algebraic stacks [spa, 46.6] and (representable) morphisms. Points of stacks [spa, 46.4]. Explain stackification [spa, 39.9] (not in detail). As an example, define quotient stacks [spa, 40.16].

6. Groupoid representation of an algebraic stack and BG (18 Nov)

From algebraic stacks to groupoids [spa, 45.16]. From groupoids to algebraic stacks [spa, 45.17]. Recall the definition of quotient stacks [spa, 40.16]. Give the proof of [spa, Lemma 04M9], and explain the 2-cartesian diagram associated to the quotient construction (this is important for us). For an algebraic group G with an action on X we can take the quotient $[X/G]$. There is an explicit description [spa, 43.14]. Explain why the explicit description works. A special case is $BG = [S/G]$, we obtain

$$BG(T) = \{G - \text{principal bundles on } T\}.$$

7. Example: Moduli of abelian varieties (25 Nov)

Explain the two constructions of the moduli space of principally polarized abelian varieties [FC90, 4.11] in detail. The first method is via Hilbert schemes and quotients. The second method is via deformation theory. For this one should explain the results of [Art74]. This paper introduces Artin's criterion which allows one to prove algebraicity of a stack by verifying deformation-theoretic properties (cf. [spa, 50.5.1]).

8. The coarse moduli space for algebraic stacks (2 Dec)

Define the coarse moduli space attached to an algebraic stack [KM97] (this corresponds to [FC90, Theorem 4.10]). You can choose to explain the proof in [KM97] or [Con]. Keel and Mori use the language of groupoids, Conrad uses stacks.

9. Finiteness theorem and Leray spectral sequence (9 Dec)

The goal is to show the analog of Deligne's finiteness theorem for suitable stacks [Beh93, §1.1-1.2]. Start with the definition of the smooth site of an algebraic stack [LMB00] and constructible sheaves [Beh93, Definition 1.1.1]. There is a mistake in [LMB00] concerning the functoriality of the smooth site, this is corrected in [Ols07]. Follow [Beh93, §1.1] and prove [Beh93, Theorem 1.1.6]. Then [Beh93, §1.2] and [Beh93, Theorem 1.2.5].

10. The Leray spectral sequence for fibrations and purity (16 Dec)

The goal is to prove [Beh93, Theorem 1.4.3]. Let G be a connected linear algebraic group acting on a scheme X (over k algebraically closed). We get a G-principal bundle $X \to [X/G]$ and thus a morphism $[X/G] \to BG$. In this case the theorem states that the Leray spectral sequence can be written as

$$E_2^{i,j} = H^i(BG, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^j(X, \mathbb{Q}_\ell) \Rightarrow H^{i+j}([X/G], \mathbb{Q}_\ell).$$

For the proof we can follow [Beh93, §1.3-1.4]. If time permits show cohomological purity [Beh93, §2.1]

11. Cohomology of $B\mathrm{GL}_n$ and prove of the trace formula (13 Jan)

Recall/Prove the purity theorem. Show the trace formula for algebraic spaces [Beh93, §2.2]. Computation of the cohomology of $B\mathrm{GL}_n$ (this is essentially GL_n-equivariant cohomology) [Beh93, §2.3]. Proof of the trace formula [Beh93, §2.5].

12. The trace formula for Deligne-Mumford stacks (20 Jan)

Proof of the trace formula for Deligne-Mumford stacks [Beh93, §3.1]. The zeta function of an algebraic stack [Beh93, §3.2]. Applying the Lefschetz trace formula to the moduli stack of elliptic curves, Behrend proves a beautiful formula, expressing an infinite sum of traces of the Hecke operator T_p on cusp forms of all weights as a finite sum over the elliptic curves over the finite field $\mathbb{F}_p$ [Beh03, Proposition 6.4.11].

References

[Art74] M. Artin. Versal deformations and algebraic stacks. *Invent. Math.*, 27:165–189, 1974. `http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN002091542`.

[Beh93] Kai A. Behrend. The Lefschetz trace formula for algebraic stacks. *Invent. Math.*, 112(1):127–149, 1993. `http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN002110768`.

[Beh03] Kai A. Behrend. Derived l-adic categories for algebraic stacks. *Mem. Amer. Math. Soc.*, 163(774):viii+93, 2003. `http://www.math.ubc.ca/~behrend/ladic.pdf`.

[Con] Brian Conrad. The Keel-Mori theorem via stacks. `http://math.stanford.edu/~conrad/papers/coarsespace.pdf`.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969. `http://archive.numdam.org/article/PMIHES_1969__36__75_0.pdf`.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990. `/e_books/research/Faltings_G_Chai_C_Degeneration_of_abelian_varieties.djvu`.

[KM97] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997. `http://www.jstor.org/stable/2951828?origin=crossref`.

[Knu71] Donald Knutson. *Algebraic spaces*. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin, 1971. `/e_books/research/Knutson_Algebraic_Spaces.djv`.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2000. `/e_books/research/Laumon_Moret-Bailly-Champs_algebriques.pdf`.

[Mil08] James S. Milne. Lectures on etale cohomology (v2.10), 2008. Available at `http://www.jmilne.org/math/`.

[Ols07] Martin Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007. `http://dx.doi.org/10.1515/CRELLE.2007.012`.

[spa] The stacks project authors. Stacks project. `http://math.columbia.edu/algebraic_geometry/stacks-git/book.pdf`.