A Numerical Solution Method for an Infinitesimal Elasto-Plastic Cosserat Model

Patrizio Neff [1], Krzysztof Chełmiński [2],
Wolfgang Müller [3],Christian Wieners [3]

[1] Department of Mathematics, AG 6
Technische Universität Darmstadt

[2] Faculty of Mathematics,
Technical University Warsaw, Poland

[3] Institute of Applied and Numerical Mathematics
Universität Karlsruhe (TH)

Warsaw, December 5, 2006

A micro-polar extension to infinitesimal elasticity

- We present a geometrically linear generalized continua of Cosserat micro-polar type in the elasto-plastic case.
- Starting with linear elasticity we postulate independent infinitesimal micro-rotations of the material. Thus, as a consequence of balance of angular momentum, Cauchy stresses $\boldsymbol{\sigma}$ are not symmetric any more.
- Cosserat regularize the mesh size dependence of localization computations where shear failure mechanisms play a dominant role.
- Models are of engineering interest: Diebels/Ehlers, Iordache/William, Dietsche/Steinmann/William, Ristinmaa/Vecchi, de Borst, ...
- We restrict Cosserat micro-rotations to the elastic response of the material. Inelasticity is formulated as in Prandtl-Reuß plasticity. The elasto-plastic Cosserat problem is well-posed (Neff/Chełmiński Appl.Math.Opti06, PRSE05).

The Elastic Static Cosserat Model

Compute displacement **u** and infinitesimal rotations id $+\overline{A}$ with $\overline{A}^T = -\overline{A}$, such that

$$\int_\Omega W(D\mathbf{u}, \bar{A}, D\bar{A}) - \ell(\mathbf{u}) \longrightarrow \text{min} \quad \text{w.r.t. } (\mathbf{u}, \bar{A})$$

where the stored energy is

$$\begin{aligned} W(D\mathbf{u}, \bar{A}, D\bar{A}) \quad = \quad & \mu \, |\,\text{sym}(D\mathbf{u} - \bar{A})|^2 + \frac{\lambda}{2} \, \text{tr}(\text{sym}(D\mathbf{u} - \bar{A}))^2 \\ & + \mu_c \, |\,\text{skew}(D\mathbf{u} - \bar{A})|^2 + \mu L_c^2 \, |D\bar{A}|^2 . \end{aligned}$$

Remark:
Coercive without Korn‘s first inequality!

Linear Elastic Static Cosserat Model - Equations

We want to determine

$$\begin{array}{lll} \text{displacements} & \mathbf{u}: & \bar{\Omega}\times[0,T]\longrightarrow\mathbb{R}^d, \\ \text{infinitesimal micro-rotations} & \bar{A}: & \Omega\times[0,T]\longrightarrow\mathfrak{so}(d), \\ \textit{non-symmetric}\text{ stresses} & \boldsymbol{\sigma}: & \Omega\times[0,T]\longrightarrow\mathbf{R}^{d,d}, \end{array}$$

satisfying the essential boundary conditions, the equilibrium equations

$$\begin{array}{lcll} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t) & = & \mathbf{b}(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times[0,T]\,, \\ \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}(\mathbf{x}) & = & \mathbf{t}_N(\mathbf{x},t), & (\mathbf{x},t)\in\Gamma_N\times[0,T]\,, \\ -\mu L_c^2\Delta\bar{A}(\mathbf{x},t) & = & \mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t))-\bar{A}(\mathbf{x},t)\big), & (\mathbf{x},t)\in\Omega\times[0,T]\,, \\ D\bar{A}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) & = & \mathbf{0}, & (\mathbf{x},t)\in\Gamma_N\times[0,T], \end{array}$$

and the constitutive relation

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x},t) \;=\;& 2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x},t))\big)+\lambda\operatorname{div}(\mathbf{u})(\mathbf{x},t)\mathbf{I} \\ &+2\mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t))-\bar{A}(\mathbf{x},t)\big), \qquad (\mathbf{x},t)\in\Omega\times[0,T]\,. \end{aligned}$$

Infinitesimal Elastic Cosserat Model - Data

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary. We fix a time interval $[0, T]$.

Given data:

displacement vector	$\mathbf{u}_D :$	$\Gamma_D \times [0, T] \longrightarrow \mathbb{R}^d,$
suitable infinitesimal micro-rotations	$\bar{A}_D :$	$\Gamma_D \times [0, T] \longrightarrow \mathfrak{so}(d),$

where $\mathfrak{so}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} : \boldsymbol{\tau}^T = -\boldsymbol{\tau}\}$ is the Lie algebra of skew-symmetric matrices, and a load functional

$$\ell(t, \mathbf{v}) = \int_\Omega \mathbf{b}(t) \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v} \, d\mathbf{a}$$

depending on traction force densities $\mathbf{t}_N$ and body force densities $\mathbf{b}$.

The material is described by a linear elastic response depending on the Lamé constants $\lambda, \mu > 0$ and Cosserat constants $L_c > 0$ and $\mu_c \geq 0$.

Infinitesimal Elasto-Plastic Cosserat Model - Flow rule

Inelastic material behavior is modeled by a convex function

$$\phi\colon \ \mathrm{Sym}(d) \longrightarrow \mathbb{R}$$

determining the convex set of admissible symmetric stresses $\mathbf{K} = \{\boldsymbol{\tau} \in \mathrm{Sym}(d)\colon \phi(\boldsymbol{\tau}) \le 0\}$. We assume that ϕ is smooth for $\boldsymbol{\tau} \neq \mathbf{0}$, and we assume $\phi(\mathbf{0}) < 0$. In a first approach we choose the von Mises yield criterion $\phi(\boldsymbol{\tau}) = |\,\mathrm{dev}(\boldsymbol{\tau})| - K_0$ for a given yield stress $K_0 > 0$. Here $\mathrm{dev}\colon \ \mathrm{Sym}(d) \longrightarrow \mathrm{Sym}(d)$ with $\mathrm{dev}\,\boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}\,\mathrm{tr}\,\boldsymbol{\tau}$.

We have $P_{\mathbf{K}}(\boldsymbol{\theta}) = \boldsymbol{\theta} - \max\{0,\gamma\}\,\frac{\mathrm{dev}(\boldsymbol{\theta})}{|\,\mathrm{dev}(\boldsymbol{\theta})|}$.

We use the following realization $\mathbf{C}(\boldsymbol{\theta}) \in \partial P_{\mathbf{K}}(\boldsymbol{\theta})$, where $P_{\mathbf{K}}(\boldsymbol{\theta})$ is the multi-valued derivative of the projection defined by $\mathbf{C}(\boldsymbol{\theta}) = \mathrm{id}$ for $|\,\mathrm{dev}(\boldsymbol{\theta})| \le K_0$ and

$$\mathbf{C}(\boldsymbol{\theta}) = \frac{1}{d}\mathbf{I}\otimes\mathbf{I} + \frac{K_0}{|\,\mathrm{dev}(\boldsymbol{\theta})|}\Big(\big(\mathrm{id} - \frac{1}{d}\mathbf{I}\otimes\mathbf{I}\big) - \frac{\mathrm{dev}(\boldsymbol{\theta})}{|\,\mathrm{dev}(\boldsymbol{\theta})|} \otimes \frac{\mathrm{dev}(\boldsymbol{\theta})}{|\,\mathrm{dev}(\boldsymbol{\theta})|}\Big) \ \text{for } |\,\mathrm{dev}(\boldsymbol{\theta})| > K_0\,.$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

We want to determine

displacements	$\mathbf{u}:$	$\bar{\Omega} \times [0,T] \longrightarrow \mathbb{R}^d$,
infinitesimal micro-rotations	$\bar{A}:$	$\Omega \times [0,T] \longrightarrow \mathfrak{so}(d)$,
non-symmetric stresses	$\boldsymbol{\sigma}:$	$\Omega \times [0,T] \longrightarrow \mathbf{R}^{d,d}$,
symmetric plastic strains	$\varepsilon_p:$	$\Omega \times [0,T] \longrightarrow \mathrm{Sym}(d)$ with $\varepsilon_p(0) = \mathbf{0}$,
and a plastic multiplier	$\Lambda:$	$\Omega \times [0,T] \longrightarrow \mathbb{R}$,

satisfying the essential boundary conditions and the equilibrium equations

$$\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t) &= \mathbf{b}(\mathbf{x},t), && (\mathbf{x},t) \in \Omega \times [0,T]\,,\\
\boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x},t), && (\mathbf{x},t) \in \Gamma_N \times [0,T]\,,\\
-\mu L_c^2 \Delta \bar{A}(\mathbf{x},t) &= \mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t)) - \bar{A}(\mathbf{x},t)\big), && (\mathbf{x},t) \in \Omega \times [0,T]\,,\\
D\bar{A}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) &= \mathbf{0}, && (\mathbf{x},t) \in \Gamma_N \times [0,T],
\end{aligned}$$

the constitutive relation

$$\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x},t) &= 2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x},t)) - \varepsilon_p(\mathbf{x},t)\big) + \lambda \operatorname{div}(\mathbf{u})(\mathbf{x},t)\mathbf{I}\\
&\quad + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t)) - \bar{A}(\mathbf{x},t)\big), \qquad (\mathbf{x},t) \in \Omega \times [0,T]\,,
\end{aligned}$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

the complementary conditions for the yield criterion

$$\Lambda(\mathbf{x},t)\phi\big(T_E(\mathbf{x},t)\big) = 0, \quad \Lambda(\mathbf{x},t) \geq 0, \quad \phi\big(T_E(\mathbf{x},t)\big) \leq 0, \quad (\mathbf{x},t) \in \Omega \times [0,T]\,.$$

and the flow rule

$$\frac{d}{dt}\varepsilon_p(\mathbf{x},t) = \Lambda(\mathbf{x},t)D\phi\big(T_E(\mathbf{x},t)\big)\,, \qquad (\mathbf{x},t) \in \Omega \times [0,T]\,,$$

depending on $T_E(\mathbf{x},t) = 2\mu\big(\,\mathrm{sym}(D\mathbf{u}(\mathbf{x},t)) - \varepsilon_p(\mathbf{x},t)\big)$.

For given material history $\varepsilon_p(t)$ at fixed time t, the displacement and the micro-rotations are determined by minimizing the total elastic energy

$$\mathcal{I}(\mathbf{u},\bar{A},\varepsilon_p) = \mathcal{E}(\varepsilon(\mathbf{u}),\bar{A},\varepsilon_p) - \ell(t,\mathbf{u})\,,$$

$$\text{with } \mathcal{E}(\varepsilon,\bar{A},\varepsilon_p) \;=\; \mu\int_\Omega \big|\,\mathrm{sym}(\varepsilon) - \varepsilon_p\big|^2\,d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \mathrm{tr}(\varepsilon)^2\,d\mathbf{x} + \mu_c\int_\Omega |\,\mathrm{skew}(\varepsilon) - \bar{A}|^2\,d\mathbf{x} + \mu L_c^2\int_\Omega |D\bar{A}|^2\,d\mathbf{x}\,.$$

Infinitesimal Model - subdifferential formulation

$$\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma} &= \mathbf{b}\,, \\
-\mu L_c^2 \Delta \bar{A} &= \mu_c\big(\operatorname{skew}(D\mathbf{u}) - \bar{A}\big)\,, \\
\boldsymbol{\sigma} &= 2\mu\big(\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p\big) + \lambda\operatorname{div}(\mathbf{u})\mathbf{I} + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}) - \bar{A}\big)\,, \\
\tfrac{d}{dt}\boldsymbol{\varepsilon}_p &\in \partial\chi\big(T_E\big)\,, \\
T_E &= 2\mu\big(\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p\big)
\end{aligned}$$

where χ is the indicator function of the elastic domain **K**.

Discretization in space

Let h be a mesh size parameter, and let $\mathbf{V}_h \subset C^{0,1}(\Omega, \mathbb{R}^d)$ and $W_h \subset C^{0,1}(\Omega, \mathfrak{so}(d))$ be finite element spaces, and set

$$\begin{aligned} \mathbf{V}_h(\mathbf{u}_D) &= \{\mathbf{v} \in \mathbf{V}_h \colon \mathbf{v}(\mathbf{x}) = \mathbf{u}_D(\mathbf{x}) \text{ for } \mathbf{x} \in D_h\}, \\ W_h(A_D) &= \{B \in W_h \colon B(\mathbf{x}) = A_D(\mathbf{x}) \text{ for } \mathbf{x} \in D'_h\} \end{aligned}$$

where $D_h, D'_h \subset \Gamma_D$ are the sets of all nodal points on Γ_D for $\mathbf{u}$ and A.

Let $\mathbf{\Xi}_h \subset \Omega$ be quadrature points and let $\omega_{\boldsymbol{\xi}}$ be corresponding quadrature weights such that

$$\int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} = \sum_{\boldsymbol{\xi} \in \mathbf{\Xi}_h} \omega_{\boldsymbol{\xi}} \, \mathbf{v}(\boldsymbol{\xi}) \cdot \mathbf{w}(\boldsymbol{\xi}) \,, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{V}_h \ .$$

We set

$$\begin{aligned} \mathbf{\Lambda} \quad &= \quad \{\Lambda \colon \mathbf{\Xi}_h \longrightarrow \mathbb{R}\} \ , \\ \mathbf{\Sigma}_h \quad &= \quad \{\boldsymbol{\tau} \colon \mathbf{\Xi}_h \longrightarrow \mathbf{R}^{d,d}\} \ , \\ \text{and } \mathbf{E}_h^p \quad &= \quad \{\boldsymbol{\tau} \colon \mathbf{\Xi}_h \longrightarrow \mathfrak{sl}(d) \cap \mathrm{Sym}(d)\} \ , \end{aligned}$$

where $\mathfrak{sl}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} \colon \ \mathrm{tr}(\boldsymbol{\tau}) = 0\}$ is the Lie algebra of trace-free matrices.

Discretization in space

Determine

$$\begin{array}{ll} \text{displacements } \mathbf{u}: & [0,T] \longrightarrow \mathbf{V}_h, \\ \text{stresses } \boldsymbol{\sigma}: & [0,T] \longrightarrow \boldsymbol{\Sigma}_h, \\ \text{micro-rotations } \bar{A}: & [0,T] \longrightarrow W_h, \\ \text{plastic strains } \varepsilon_p: & [0,T] \longrightarrow \mathbf{E}_h^p, \\ \text{and a plastic multiplier } \Lambda: & [0,T] \longrightarrow \boldsymbol{\Lambda} \end{array}$$

satisfying

- the equilibrium equations,

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : D\mathbf{v}\, d\mathbf{x} &= \ell(\cdot,\mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0})\ , \\ \mu L_c^2 \int_\Omega D\bar{A} \cdot D\bar{B}\, d\mathbf{x} &= \mu_c \int_\Omega \left(\operatorname{skew}(D\mathbf{u}) - \bar{A} \right) : \bar{B}\, d\mathbf{x},\ \bar{B} \in W_h(\mathbf{0}) \end{aligned}$$

- the essential boundary conditions,
- the constitutive relation,
- the complementary conditions (Kuhn-Tucker),
- and the flow rule.

Discretization in time

The model of incremental infinitesimal plasticity is obtained by a decomposition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval and backward Euler scheme. For $n = 1, 2, 3, \ldots$ the next increment depends on the material history described by ε_p^{n-1}, the new load $\ell^n[\mathbf{v}] = \ell(t_n, \mathbf{v})$, and the new Dirichlet boundary values $\mathbf{u}_D^n = \mathbf{u}_D(t_n)$ and $\bar{A}_D^n = \bar{A}_D(t_n)$.

We compute the displacement vector $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, the stresses $\boldsymbol{\sigma}^n \in \boldsymbol{\Sigma}_h$, the micro-rotations $\bar{A} \in W_h(\bar{A}_D^n)$, the plastic strains $\varepsilon_p^n \in \mathbf{E}_h^p$, and the plastic multiplier $\Lambda^n \in \boldsymbol{\Lambda}$ satisfying additionally the discrete flow-rule:

$$\frac{1}{t_n - t_{n-1}} \Big(\varepsilon_p^n(\xi) - \varepsilon_p^{n-1}(\xi) \Big) = \Lambda^n(\xi) D\phi\big(T_E^n(\xi)\big) \ , \qquad \xi \in \boldsymbol{\Xi}_h \ ,$$

depending on $T_E^n(\xi) = 2\mu\big(\operatorname{sym}(D\mathbf{u}^n(\xi)) - \varepsilon_p^n(\xi)\big)$.

Since the problem is rate-independent, we define $\gamma^n = (t_n - t_{n-1})\Lambda^n \in \boldsymbol{\Lambda}$.

Fully discrete elasto-plastic problem

Together, we can state the fully discrete elasto-plastic Cosserat problem.

For given $\varepsilon_p^{n-1} \in \mathbf{E}_h^p$ find $\boldsymbol{\sigma}^n, T_E^n \in \boldsymbol{\Sigma}_h$, $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, $\bar{A} \in W_h(\bar{A}_D^n)$ and $\gamma^n \in \boldsymbol{\Lambda}$ such that

$$
\begin{aligned}
&T_E^n(\xi) = 2\mu\Big(\,\mathrm{sym}(D\mathbf{u}^n(\xi)) - \varepsilon_p^{n-1}(\xi) - \gamma^n(\xi) D\phi\big(T_E^n(\xi)\big)\Big)\,, && \xi \in \boldsymbol{\Xi}_h\,,\\
&\phi\big(T_E^n(\xi)\big) \le 0\,,\quad \gamma^n(\xi)\phi\big(T_E^n(\xi)\big) = 0\,,\quad \gamma^n(\xi) \ge 0\,, && \xi \in \boldsymbol{\Xi}_h\,,\\
&\boldsymbol{\sigma}^n(\xi) = T_E^n(\xi) + \lambda\,\mathrm{div}(\mathbf{u}^n)(\xi)\mathbf{I} + \ 2\mu_c(\mathrm{skew}(D\mathbf{u}^n(\xi)) - \bar{A}^n(\xi))\,, && \xi \in \boldsymbol{\Xi}_h\,,\\
&\int_\Omega \boldsymbol{\sigma}^n : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}]\,, && \mathbf{v} \in \mathbf{V}_h(\mathbf{0}),\\
&\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} = \mu_c \int_\Omega \big(\,\mathrm{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\, d\mathbf{x}\,, && \bar{B} \in W_h(\mathbf{0}).
\end{aligned}
$$

Discrete formulation of the Elasto-Plastic Model

Lemma:
The fully discrete elasto-plastic problem is equivalent to the following nonlinear variational problem. For given ε_p^{n-1} find $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n)$ such that

$$\int_\Omega P_{\mathbf{K}}\big(2\mu(\mathrm{sym}(D\mathbf{u}^n) - \varepsilon_p^{n-1})\big) : D\mathbf{v}\,d\mathbf{x} + \lambda \int_\Omega \mathrm{div}(\mathbf{u}^n)\,\mathrm{div}(\mathbf{v})\,d\mathbf{x}$$

$$+ 2\mu_c \int_\Omega \big(\mathrm{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : D\mathbf{v}\,d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \;,$$

$$\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\,d\mathbf{x} = \mu_c \int_\Omega \big(\mathrm{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\,d\mathbf{x}, \qquad \bar{B} \in W_h(\mathbf{0}) \;.$$

Variational Formulation of the discrete problem

Lemma:
Any minimizer $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}^n_D) \times W_h(\bar{A}^n_D)$ of the functional

$$\mathcal{I}^n_{\mathrm{incr}}(\mathbf{u}, \bar{A}) \quad = \quad \mathcal{E}_{\mathrm{incr}}(\varepsilon(\mathbf{u}), \bar{A}, \varepsilon_p^{n-1}) - \ell_n[\mathbf{u}]$$

solves the nonlinear variational update problem. Here $\mathcal{E}_{\mathrm{incr}}$ denotes the free energy of the incremental update problem defined by

$$\begin{aligned} \mathcal{E}_{\mathrm{incr}}(D\mathbf{u}, \bar{A}, \varepsilon_p) \quad &= \quad \frac{1}{2\mu} \int_\Omega \psi_{\mathbf{K}}\big(2\mu(\mathrm{sym}(D\mathbf{u}) - \varepsilon_p)\big)\, d\mathbf{x} + \frac{\lambda}{2} \int_\Omega \mathrm{tr}(D\mathbf{u})^2 \, d\mathbf{x} \\ &\quad + \mu_c \int_\Omega |\,\mathrm{skew}(D\mathbf{u}) - \bar{A}|^2 \, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\bar{A}|^2 \, d\mathbf{x} \, , \end{aligned}$$

$$\psi_{\mathbf{K}}(\boldsymbol{\theta}) \quad = \quad \begin{cases} \frac{1}{2}\, |\boldsymbol{\theta}|^2 & |\,\mathrm{dev}(\boldsymbol{\theta})| \le K_0 \, , \\ \frac{1}{2}\left(\frac{1}{d}\, \mathrm{tr}(\boldsymbol{\theta})^2 + 2K_0\, |\,\mathrm{dev}(\boldsymbol{\theta})| - K_0^2\right) & |\,\mathrm{dev}(\boldsymbol{\theta})| > K_0 \, . \end{cases}$$

If $\varepsilon_p^{n-1} = 0$ and $\mu_c = 0$ → classical Hencky-Problem.

The Dual Formulation

Lemma:
Let $\varepsilon_p^{n-1} \in L_2(\Omega, \mathrm{Sym}(3))$ with $\mathrm{tr}(\varepsilon_p^{n-1}) = 0$ be given.
Then, $(\boldsymbol{\sigma}^n, \bar{A}^n) \in L_2(\Omega, \mathbf{R}^{3,3}) \times W$ is uniquely determined by minimizing the dual functional

$$\begin{aligned}\mathcal{D}^n_{\mathrm{incr}}(\boldsymbol{\sigma}, \bar{A}) &= \int_\Omega \boldsymbol{\sigma} : \varepsilon_p^{n-1}\, d\mathbf{x} + \frac{1}{4\mu}\int_\Omega |\,\mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma}))|^2\, d\mathbf{x} \\ &+ \frac{1}{6(2\mu+3\lambda)}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})^2\, d\mathbf{x} + \frac{1}{4\mu_c}\int_\Omega |\,\mathrm{skew}(\boldsymbol{\sigma})|^2\, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\bar{A}|^2\, d\mathbf{x}\,.\end{aligned}$$

subject to the plastic constraint $\phi\big(\,\mathrm{sym}(\boldsymbol{\sigma}^n)\big) \le 0$ and the equilibrium constraint

$$\begin{aligned}&\int_\Omega \boldsymbol{\sigma}^n : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V} \\ &2\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} = \int_\Omega \mathrm{skew}(\boldsymbol{\sigma}^n) : \bar{B}\, d\mathbf{x}, \qquad \bar{B} \in W\,.\end{aligned}$$

The Dual Formulation

For the limit case $\mu_c = 0$, we have to assume that the convex set

$$\mathbf{K}^n = \left\{ \boldsymbol{\tau} \in L^2(\Omega, \mathrm{Sym}(3)) \colon \boldsymbol{\tau} \in \mathbf{K} \text{ a. e. in } \Omega,\ \int_\Omega \boldsymbol{\tau} : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}],\ \mathbf{v} \in \mathbf{V} \right\}$$

is not empty (weak safe-load assumption).

Lemma:
If $\boldsymbol{\eta}^n \in \mathbf{K}^n$ exists, we have

$$\| \operatorname{skew}(\boldsymbol{\sigma}^n) \|^2_{L^2(\Omega)} \quad \leq \quad 4\mu_c\, \mathcal{D}^n_{\mathrm{incr}}(\boldsymbol{\eta}^n, \mathbf{0})$$

and

$$\| D\bar{A}^n \|^2_{L^2(\Omega)} \quad \leq \quad \frac{C_1^2}{\mu^2 L_c^4}\, \mu_c\, \mathcal{D}^n_{\mathrm{incr}}(\boldsymbol{\eta}^n, \mathbf{0}) \, .$$

Theorem:
If $\boldsymbol{\eta}^n \in \mathbf{K}^n$ exists, we have

$$\lim_{\mu_c \longrightarrow 0} \| \operatorname{sym}(\boldsymbol{\sigma}^n_{\mu_c}) - \boldsymbol{\sigma}^n_0 \|_{L^2(\Omega)} = 0 \, .$$

The FEM convergence

Theorem:
For pure dirichlet data on **u** we have.

$$\|\mathbf{v}\|^2_{H^1(\Omega)} \leq \bar{C}\left(\|\operatorname{div}\mathbf{v}\|^2_{L^2(\Omega)} + \|\operatorname{curl}\mathbf{v}\|^2_{L^2(\Omega)}\right), \qquad \mathbf{v} \in \mathbf{V}\,.$$

Thus,

$$\|(\mathbf{u}-\mathbf{u}_h, \bar{A}-\bar{A}_h)\|_{\mathbf{V}\times W} \leq \frac{C}{\mu_c} \inf_{(\mathbf{v}_h,\bar{B}_h)\in\mathbf{V}_h\times W_h} \|(\mathbf{u}-\mathbf{v}_h, \bar{A}-\bar{B}_h)\|_{\mathbf{V}\times W}\ .$$

C is independent of $\mu_c \in (0,\mu]$.

Remark :
For $\mu_c > 0$ we observe $(\mathbf{u}_{\mu_c}, \bar{A}_{\mu_c}) \in H^2(\Omega)$ (Neff/Knees), thus

$$\|(\mathbf{u}_{\mu_c} - \mathbf{u}^h_{\mu_c}, \bar{A}_{\mu_c} - \bar{A}^h_{\mu_c})\|_{H^1(\Omega)} \leq \frac{C}{\mu_c}\, h\, \|(\mathbf{u}_{\mu_c}, \bar{A}_{\mu_c})\|_{H^2(\Omega)}$$

for isoparametric Q1 elements.
Balance μ_c with h!

Numerical Solution Algorithm

We formulate a semi-smooth Newton method for the nonlinear variational problem

$$(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n) : \qquad F^n(\mathbf{u}^n, \bar{A}^n) = 0$$

in every time step n, where F^n is the first variation of $\mathcal{I}_{\mathrm{incr}}^n$ defined by

$$F^n(\mathbf{u}, \bar{A})[\mathbf{v}, \bar{B}] = D\mathcal{I}_{\mathrm{incr}}^n(\mathbf{u}, \bar{A})[\mathbf{v}, \bar{B}], \qquad (\mathbf{v}, \bar{B}) \in \mathbf{V}_h(\mathbf{0}) \times W_h(\mathbf{0}) \; .$$

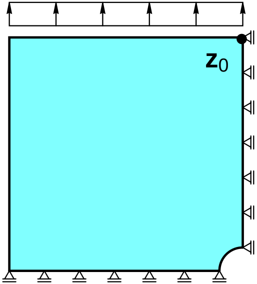
The functional F^n is semi-smooth, and the second variation $\partial^2\mathcal{I}_{\mathrm{incr}}^n = \partial F^n$ is multi-valued. Thus, the corresponding semi-smooth Newton method can be formally written as

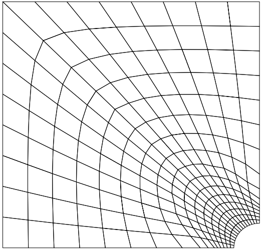
$$0 \in F^n(\mathbf{u}^{n,k}, \bar{A}^{n,k}) + \partial F^n(\mathbf{u}^{n,k}, \bar{A}^{n,k})[\mathbf{u}^{n,k+1} - \mathbf{u}^{n,k}, \bar{A}^{n,k+1} - \bar{A}^{n,k}] \; .$$

We consider the special case of the von Mises flow rule. The Newton increment is realized using the consistent linearization $\mathbf{C}(\theta)$.

Benchmark problem for parameter study of μ_c

Let $\Omega = (0,10) \times (0,10) \setminus B_1(10,0)$. We use Q1 discretization and present results for 198147 unknowns on uniform refinement level 4. We have chosen the parameters $K_0 = 450, \lambda \approx 1.1 \cdot 10^6, \mu \approx 8.0 \cdot 10^5$ and $L_c \approx 0.02$.
And apply traction force by Neumann boundary condition according to:
$\ell(t, v) = 100t \int_0^{10} \mathbf{v}(x_1, 10) dx_1$.

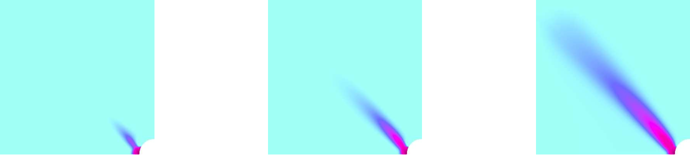




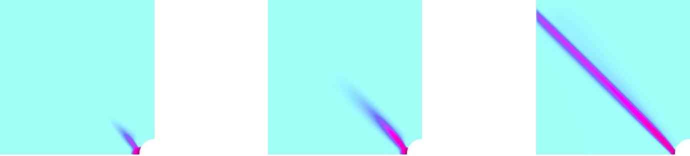
Geometry, boundary conditions and coarse mesh.

Numerical Experiment with M++

Cosserat Model ($\mu_c = \mu$) : Effective plastic strain



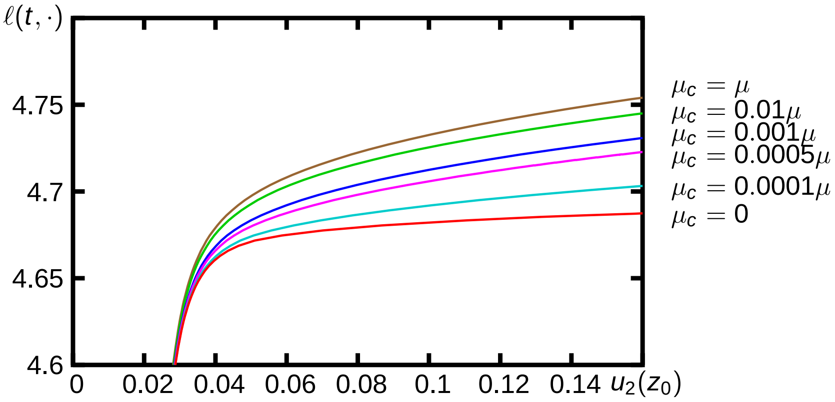
Prandtl-Reuß ($\mu_c = 0$) : Effective plastic strain



$t = 4.00$ $t = 4.40$ $t = 4.69$

no difference for $t < 4.50$

Numerical Experiment with M++



Load-displacement curve: displacement $\mathbf{u}$ is evaluated at special point $\mathbf{z}_0$.

Summary and Outlook

- The Elasto-Plastic Cosserat Model with pure Dirichlet data is well-posed. Solution exists globally in time.
- This Elasto-Plastic Cosserat Model is a regularization for classical perfect plasticity (shear failure mechanisms).
- Complete finite element analysis (depending on μ_c) is available: to appear in Math. Meth. Mod. Appl. Sci.
- Compared to gradient plasticity: no change of plasticity formulation needed! Classical variational concepts carry over.
- Future work will be the analysis and implementation of geometrically nonlinear elasto-plastic Cosserat Models.
- And a comparison with implementations of higher gradient plasticity models.
- μ_c measures rotational inhomogeneity
- no Cosserat effects in uniaxial tests (unlike hardening or viscosity)