Notes on strain-gradient plasticity. Finite strain invariant modelling and global existence in the infinitesimal rate-independent case.

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Plan of the talk

- Notation, Motivation
- Second law of thermodynamics with gradients of inner variables
- Higher order boundary conditions - insulation condition
- Invariance, dislocation density measure G
- Finite strain elasto-plastic model, linearized model, Legendre-transformation
- Reformulation as a variational inequality of mixed type
- Bilinear form and new Hilbert space: existence and uniqueness
- Open problems

Notation

- $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, deformation, Ω reference configuration
- $F = \nabla\varphi \in \mathrm{GL}^+(3, \mathbb{R})$, deformation gradient, $1\!\!1$ identity tensor
- $W(F)$ elastic free energy density, $S_1(F) = D_F W(F)$ first P. K. stresses
- $\|A\|$, $\langle A, B\rangle$, $\mathrm{tr}\,[A]$ norm, scalar product and trace
- $\mathrm{sym}\, X = \frac{1}{2}(X + X^T)$, $\mathrm{skew}\, X = \frac{1}{2}(X - X^T)$, $\mathrm{dev}\, X = X - \frac{1}{3}\mathrm{tr}\,[X] \cdot 1\!\!1$
- $F = F_e\, F_p$, multiplicative decomposition
- $\mathrm{Curl}\, F_p = \begin{pmatrix} \mathrm{curl}[F_p^1 - -] \\ \mathrm{curl}[F_p^2 - -] \\ \mathrm{curl}[F_p^3 - -] \end{pmatrix} \in \mathbb{M}^{3\times 3}$ for $F_p = \begin{pmatrix} F_p^1 - - - \\ F_p^2 - - - \\ F_p^3 - - - \end{pmatrix}$
- Note: $\mathrm{Curl}\, F_p$ transpose of Gurtin's definition!

Motivation

- Apparent size-dependence of yield stress: Hall-Petch: $\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0 + \frac{k^+}{\sqrt{\mathrm{grainsize}}}$ and Taylor scaling: $\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0 + k^+ \|\operatorname{Curl} F_p\|$

- Model proposals: Acharya, Fleck, Garikipati, Geers, Gurtin, Needleman, Svendsen, Steinmann, Tvergaard, ...

- Regularization of discrete time-step minimization problems: Ortiz, Mielke/Müller, Miehe ..., regularization of softening plasticity: Reddy, ...

- Here: purely phenomenological approach based on multiplicative decomposition $F = F_e\, F_p$, but no crystallographic model

- No plastic metric C_p, basic variable plastic distortion $F_p \in \mathrm{SL}(3)$. Full spatial, intermediate and referential invariance under superposed rigid rotations - based on F_p with plastic spin

- Simplifying assumption: normality of flow rule and rate-independency: $F_p \frac{\mathrm{d}}{\mathrm{dt}} F_p^{-1} \in -\partial\chi(\Sigma)$ with χ indicator function of elastic domain K

Thermodynamics with internal variables: global form

Free energy $W(\nabla\varphi, P, \nabla P)$ depends on inner variable P and gradient of inner variable ∇P. Second law of thermodynamics: extended principle of positive dissipation. For all "nice" test volumes $\mathcal{V} \subseteq \Omega$

$$\int_{\mathcal{V}} \underbrace{\langle S_1(F(t), P(t), \nabla P(t)), \nabla\varphi_t\rangle - \frac{\mathrm{d}}{\mathrm{dt}} W(\nabla\varphi(t), P(t), \nabla P(t))}_{\geq 0 \text{ classical 2. law}}$$
$$+ \int_{\partial\mathcal{V}} \langle q(\nabla\varphi, P, \dot{P}, \nabla P), N\rangle \,\mathrm{dV} \geq 0\,.$$

Here, $\int_{\partial\mathcal{V}} \langle q(\nabla\varphi, P, \dot{P}, \nabla P), N\rangle \,\mathrm{dV}$ working of the gradient of inner variable over the (internal) boundary $\partial\mathcal{V}$ (extra energy flux).

Constitutive assumption for flux q?

Thermodynamics with internal variables: local form

Free energy $W(\nabla\varphi, P, \nabla P)$ depends on inner variable P and gradient of inner variable ∇P. Second law of thermodynamics: extended principle of positive dissipation. For all "nice" test volumes $\mathcal{V} \subseteq \Omega$

$$\int_{\mathcal{V}} \langle S_1(F(t), P(t), \nabla P(t)), \nabla\varphi_t\rangle$$

$$-\frac{\mathrm{d}}{\mathrm{dt}} W(\nabla\varphi(t), P(t), \nabla P(t)) + \mathrm{Div}\, q(\nabla\varphi, P, \dot{P}, \nabla P)\, \mathrm{dV} \geq 0 \;\Leftrightarrow$$

$$\int_{\mathcal{V}} \frac{\mathrm{d}}{\mathrm{dt}} W(\nabla\varphi(t_0), P(t), \nabla P(t)) - \mathrm{Div}\, q(\nabla\varphi, P, \dot{P}, \nabla P)\, \mathrm{dV} \leq 0 \quad \text{at frozen } t_0 \in \mathbb{R}\,.$$

Introduce **extra energy flux** q suitably! (I. Müller, ARMA67)

Maugin99: "... select (the extra energy flux q) in such a form as to eliminate any true divergence term in the dissipation inequality".

Chosen energy W induces q!

Global energy inequality and insulation condition

Assume: Rate of change of the free energy is bounded by classical power of external forces, leads to a priori-energy inequality and motivates bc on inner variables:

$$\int_\Omega W(\nabla\varphi(t), P(t), \nabla P(t))\,\mathrm{dV} \leq \int_\Omega W(\nabla\varphi(0), P(0), \nabla P(0))\,\mathrm{dV}$$

$$+ \underbrace{\int_0^t \int_\Omega \langle f(t), \varphi_t(t)\rangle\,\mathrm{dV}\,\mathrm{dt} + \int_0^t \int_{\partial\Omega} \langle \varphi_t(t), S_1(\nabla\varphi(t), P(t), \nabla P(t)).\vec{n}\rangle\,\mathrm{dS}\,\mathrm{dt}}_{\text{classical external work of forces}}\,.$$

Sufficient (variational derivative)

$$\frac{\mathrm{d}}{\mathrm{dt}}P = \partial\chi\,(-[D_P W - \mathrm{Div}\, D_{\nabla P}W])\,, \quad \langle \partial\chi(X), X\rangle \geq 0\,,$$

$$\int_{\partial\Omega} \langle \frac{\mathrm{d}}{\mathrm{dt}}P, D_{\nabla P}W.\vec{n}\rangle\,\mathrm{dS} = \int_\Omega \mathrm{Div}\left(\underbrace{D_{\nabla P}W^T.\frac{\mathrm{d}}{\mathrm{dt}}P}_{q}\right)\mathrm{dV} = 0\,, \quad \textbf{insulation condition}$$

no nonlocal energy exchange across the external boundary is permitted.

Formulation in a variational setting

Multiplicative decomposition $F = F_e\, F_p$. Define thermodynamic potential $W(F_e, F_p, \operatorname{Curl} F_p)$

$$W(F_e, F_p, \operatorname{Curl} F_p) = W_{\mathrm{e}}(F_e) + W_{\mathrm{sh}}(F_p) + W_{\mathrm{curl}}(F_p, \operatorname{Curl} F_p)$$

How to choose $W_{\mathrm{e}}(F_e)$, $W_{\mathrm{sh}}(F_p)$ and $W_{\mathrm{curl}}(F_p, \operatorname{Curl} F_p)$?

Guiding principles:

- do not induce anisotropic behaviour unless explicitly accounted for (e.g. with structural tensors, invariant setting, crystallographic glide-planes).
- be consistent with established finite-strain formulations in the zero-plastic length scale limit $L_c \to 0$.
- ensure downwards compatibility with classical infinitesimal plasticity $(\varepsilon = \varepsilon_e + \varepsilon_p)$. External working only through φ, induce no Dirichlet boundary conditions on ε_p.

Postulate of referential, intermediate and spatial invariance

Multiplicative decomposition $F = F_e\, F_p$. **Elastic energy** $W_{\rm e}$.

Plastic indifference (Mielke, Miehe),
elastic isomorphy (Bertram) $\Rightarrow\; W_{\rm e} = W_{\rm e}(F_e)$

Frame-indifference: $W_{\rm e}(F_e) = W_{\rm e}(F_e^T F_e)$.

Elastic isotropy: $\forall\, Q \in {\rm SO}(3) :\; W_{\rm e}(F_e Q) = W_{\rm e}(F_e)$.

Invariance of local hardening: $\forall\, Q_2, Q_3 \in {\rm SO}(3) :\; W_{\rm sh}(Q_3^T F_p Q_2) = W_{\rm sh}(F_p)$.

Invariance of nonlocal hardening: $\forall\, Q_2, Q_3 \in {\rm SO}(3)$:
$W_{\rm curl}(Q_3^T X Q_2, Q_3^T Y Q_2) = W_{\rm curl}(X, Y)$ with $W_{\rm curl}(F_p, {\rm Curl}\, F_p)$.

Contrary to purely local model, these requirements do not imply that theory can be reduced to $C_p = F_p^T F_p$!

Similar conclusions based on more general invariance requirements:
Cermelli/Gurtin00, Parry/Silhavy99.

The referential dislocation density measure

Gurtin, Parry/Silhavy, Svendsen, Mielke/Müller, Menzel/Steinmann, Miehe: geometric dislocation density tensor in the intermediate configuration

$$G = \frac{1}{\det[F_p]}(\operatorname{Curl} F_p) F_p^T \,.$$

Rerential measure of dislocation density G_{R} (Gurtin00) given by

$$G_{\mathrm{R}} = F_p^{-1} \operatorname{Curl} F_p \,, \quad G = \frac{1}{\det[F_p]}(\operatorname{Curl} F_p) F_p^T = \frac{1}{\det[F_p]} F_p \, G_{\mathrm{R}} \, F_p^T \,.$$

Introduces in a natural way nonlocal kinematical backstress due to dislocation processes.

On the reference configuration, use $W_{\mathrm{curl}}(F_p, \operatorname{Curl} F_p) = W_{\mathrm{curl}}(G_{\mathrm{R}})$ and for control of plastic spin add: $sp_0 \, \|R_p^T \operatorname{Curl} R_p\|^2$, where $F_p = R_p \, U_p$.

The thermodynamic free energy

$$W(F_e, F_p, \mathrm{Curl}\, F_p) = W_\mathrm{e}(F_e) + W_\mathrm{sh}(F_p) + W_\mathrm{curl}(F_p, \mathrm{Curl}\, F_p)$$

$$W_\mathrm{e}(F_e) = \frac{\mu}{4} \| \frac{F_e^T F_e}{\det[F_e]^{2/3}} - 1\!\!1 \|^2 + \frac{\lambda}{4} \left((\det[F_e] - 1)^2 + (\frac{1}{\det[F_e]} - 1)^2 \right) ,$$

$$W_\mathrm{sh}(F_p) = \frac{\mu\, H_0}{4} \| \frac{F_p^T F_p}{\det[F_p]^{2/3}} - 1\!\!1 \|^2 ,$$

$$W_\mathrm{curl}(F_p^{-1}\, \mathrm{Curl}\, F_p) = \frac{\mu\, L_c^2}{2} \left(\|F_p^{-1}\, \mathrm{Curl}\, F_p\|^2 + sp_0\, \|R_p^T\, \mathrm{Curl}\, R_p\|^2 \right) .$$

μ, λ Lamé-parameters, H_0 dimensionless hardening modulus, L_c internal plastic length, $sp_0 \in [0,1]$. Energy is invariant w.r.t. reference, intermediate and spatial configuration and plastically indifferent.

Obtain flow rule by principle of maximal dissipation and thermodynamically conjugate forces: (for zero spin $sp_0 = 0$ only)

The finite strain gradient plasticity model (for $sp_0 = 0$)

$$\mathrm{Div}\, S_1(F_e) = -f\,, \quad S_1(F_e) = D_F[W_\mathrm{e}(F F_p^{-1})] = DW_\mathrm{e}(F_e)\cdot F_p^{-T}\,,$$

$$F_p \frac{\mathrm{d}}{\mathrm{dt}}[F_p^{-1}] \in -\partial \chi(\Sigma)\,, \quad \Sigma = \Sigma_\mathrm{e} + \Sigma_\mathrm{sh} + \Sigma_\mathrm{curl} \quad \text{(conjugate forces)}$$

$$\Sigma_\mathrm{e} = F_e^T DW_\mathrm{e}(F_e)\,, \quad \Sigma_\mathrm{sh} = -DW_\mathrm{sh}(F_p)\, F_p^T\,,$$

$$\Sigma_\mathrm{curl} = F_p^{-T} DW_\mathrm{curl}(F_p^{-1}\,\mathrm{Curl}\, F_p)\,(\mathrm{Curl}\, F_p)^T$$

$$- \mathrm{Curl}\left(F_p^{-T} DW_\mathrm{curl}(F_p^{-1}\,\mathrm{Curl}\, F_p)\right) F_p^T\,,$$

$$\varphi(x,t) = g_\mathrm{d}(x,t)\,, \quad F_p(x,t).\tau = F_p(x,0).\tau\,, \quad x \in \Gamma_D\,,$$

$$0 = [\mathrm{Curl}\, F_p(x,t)].\tau\,, \quad x \in \partial\Omega \setminus \Gamma_D\,, \quad F_p(x,0) = F_p^0(x)\,.$$

χ indicator of elastic domain K in stress space. τ: tangential directions on Γ_D.

Plastic indifference of dissipation (Mielke) ⇒ rate of plastic deformation given by $F_p \frac{\mathrm{d}}{\mathrm{dt}}[F_p^{-1}]$.

The quadratic energy for the geometrically linear model

Expand $F = 1\!\!1 + \nabla u,\ F_p = 1\!\!1 + p + \ldots,\ R_p = 1\!\!1 + \operatorname{skew} p + \ldots,\ F_e = 1\!\!1 + e + \ldots$.

Additive decomposition: $\quad \nabla u = e + p.$

Infinitesimal plastic distortion p need not be symmetric! Quadratic energy

$$W(\nabla u, p, \operatorname{Curl} p) = W_{\mathrm{e}}^{\mathrm{lin}}(\nabla u - p) + W_{\mathrm{sh}}(p) + W_{\mathrm{curl}}^{\mathrm{lin}}(\operatorname{Curl} p)\,,$$

$$W_{\mathrm{e}}^{\mathrm{lin}}(\nabla u - p) = \mu \|\operatorname{sym}(\nabla u - p)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\nabla u - p\right]^2,$$

$$W_{\mathrm{sh}}^{\mathrm{lin}}(p) = \mu\, H_0 \|\operatorname{dev}\operatorname{sym} p\|^2\,,$$

$$W_{\mathrm{curl}}^{\mathrm{lin}}(\operatorname{Curl} p) = \frac{\mu\, L_c^2}{2}\ \left(\|\operatorname{Curl} p\|^2 + sp_0\,\|\operatorname{Curl}\operatorname{skew} p\|^2\right).$$

extra energy flux: $q = \mu\, L_c^2\ \left(\frac{\mathrm{d}}{\mathrm{dt}} p \times \operatorname{Curl} p + sp_0\,\frac{\mathrm{d}}{\mathrm{dt}}[\operatorname{skew} p] \times \operatorname{Curl}\operatorname{skew} p\right)$
insulation condition: no extra energy flux across external boundary: $\langle q, \vec{n}\rangle_{\partial\Omega} = 0$ (locally)

Mielke: No net energy flux across the external boundary globally!

The geometrically linear model: strong form

$$\operatorname{Div}\sigma = -f\,, \quad \sigma = 2\mu\,\operatorname{sym}(\nabla u - p) + \lambda\,\operatorname{tr}[\nabla u - p]\,1\!\!1\,,$$

$$\dot p \in \partial\chi(\Sigma^{\mathrm{lin}})\,, \quad \Sigma^{\mathrm{lin}} = \Sigma^{\mathrm{lin}}_{\mathrm{e}} + \Sigma^{\mathrm{lin}}_{\mathrm{sh}} + \Sigma^{\mathrm{lin}}_{\mathrm{curl}}\,,$$

$$\Sigma^{\mathrm{lin}}_{\mathrm{e}} = 2\mu\,\operatorname{sym}(\nabla u - p) + \lambda\,\operatorname{tr}[\nabla u - p]\,1\!\!1 = \sigma\,,$$

$$\Sigma^{\mathrm{lin}}_{\mathrm{sh}} = -2\mu\,H_0\,\operatorname{dev}\operatorname{sym}p \quad \text{(local Prager backstress)}$$

$$\Sigma^{\mathrm{lin}}_{\mathrm{curl}} = -\mu\,L_c^2\,(\operatorname{Curl}\operatorname{Curl}p + s p_0\,\operatorname{skew}(\operatorname{Curl}\operatorname{Curl}\operatorname{skew}p))\,,$$

$$u(x,t) = u_{\mathrm{d}}(x)\,, \qquad p(x,0) = p^0(x)\,,$$

.... and nonlocal boundary conditions .

If elastic domain $K := \{\|\operatorname{dev}\operatorname{sym}\Sigma\| \le \sigma_{\mathrm{y}}\}$ then $\partial\chi_K(\Sigma) = \mathbb{R}^+\,\operatorname{dev}\operatorname{sym}\Sigma$ and $p \in \operatorname{Sym}(3)$ (irrotational). Suitable space for symmetric p is well-known $H_{\mathrm{curl}}(\Omega) := \{v \in L^2(\Omega)\,,\ \operatorname{Curl}v \in L^2(\Omega)\}$.

Higher order boundary conditions

extra energy flux: $q = \mu\, L_c^2\ \left(\frac{\mathrm{d}}{\mathrm{dt}} p \times \operatorname{Curl} p + s p_0 \frac{\mathrm{d}}{\mathrm{dt}}[\operatorname{skew} p] \times \operatorname{Curl} \operatorname{skew} p\right)$
insulation condition: no extra energy flux across external boundary: $\langle q, \vec{n}\rangle_{\partial\Omega} = 0$

Satisfy insulation condition (irrotational=no plastic spin= $p \in \operatorname{Sym}$). Sufficient:

$$0 = [\operatorname{Curl} p(x,t)].\tau\,, \qquad x \in \partial\Omega$$

Consistent with $L_c \to 0$: no boundary condition on plastic strain $\varepsilon_p := \operatorname{sym} p$!

Satisfy insulation condition ($p \notin \operatorname{Sym}$, with plastic spin). Sufficient:

$$\begin{aligned} 0 &= [\operatorname{Curl} p(x,t)].\tau\,, & x &\in \partial\Omega\,, \\ 0 &= [\operatorname{skew} p(x,t)].\tau\,, & x &\in \Gamma_D\,, \\ 0 &= [\operatorname{Curl} \operatorname{skew} p(x,t)].\tau\,, & x &\in \partial\Omega \setminus \Gamma_D \end{aligned}$$

τ: tangential directions at boundary.

Uniqueness of strong solutions with these boundary conditions but also with other choices.

Legendre Transformation of the flow rule

Legendre transformation of indicator function χ of elastic domain K

$$R(\xi) = \chi^*(\xi) = \begin{cases} \sigma_{\mathrm{y}}\ \|\xi\| & \operatorname{tr}[\xi] = 0 \\ \infty & \text{else} \end{cases},$$

with R the **dissipation potential** (Carstensen, Hackl, Mielke, Reddy, ...). **R is one-homogeneous due to rate-independency** of χ. Rate-independency is appropriate for homogenized samples.

Convex analysis: **flow rule can be formulated equivalently**

$$\dot p \in \partial\chi(\Sigma^{\mathrm{lin}}) \quad \Leftrightarrow \quad \Sigma^{\mathrm{lin}} \in [\partial\chi]^{-1}(\dot p) \quad \Leftrightarrow \quad \Sigma^{\mathrm{lin}} \in \partial R(\dot p) \quad \Leftrightarrow$$

$$R(q) \geq R(\dot p) + \langle \Sigma^{\mathrm{lin}}, q - \dot p\rangle \quad \forall\, q \in L^2([0,T], \mathfrak{sl}(3))\,,$$

$$\Sigma^{\mathrm{lin}} = \ldots -\mu\, L_c^2\,(\operatorname{Curl}\operatorname{Curl} p + s p_0 \operatorname{skew}(\operatorname{Curl}\operatorname{Curl}\operatorname{skew} p))$$

Weak reformulation: Variational inequality

Use insulation boundary conditions, partial integration and add weak form of elastic equilibrium. This yields mixed variational inequality: Strong solution (u,p) having "enough" regularity satisfies **weak primal problem of gradient-plasticity formulation**:

find $w=(u,p)\in Z=H^1((0,T);H_0^1(\Omega,\Gamma_D,\mathbb{R}^3))\times\mathcal{H}_\sharp)$ such that for almost all $t\in(0,T)$ it holds

$$\forall\,(v,q)\in H(\Omega):=H_0^1(\Omega,\Gamma_D,\mathbb{R}^3))\times\mathcal{H}_\sharp:$$
$$a\left(\begin{pmatrix}u\\ p\end{pmatrix},\begin{pmatrix}v-\dot u\\ q-\dot p\end{pmatrix}\right)+\int_\Omega R(q)-R(\dot p)\,\mathrm{dV}\geq\int_\Omega\langle f(x,t),v-\dot u\rangle\,\mathrm{dV}\,.$$

Remaining local boundary condition for plastic distortion: $\mathrm{skew}\,p.\tau=0$ on Γ_D. $\mathcal{H}_\sharp$ linear space for plastic distortion p.

Abbreviate $j(\dot p):=\int_\Omega R(\dot p)\,\mathrm{dV}=\int_\Omega \mathcal{X}^*(\dot p)\,\mathrm{dV}$ (integrated dissipation) and $\ell(w):=\int_\Omega\langle f(x,t),w\rangle\,\mathrm{dV}$ (external body loads)

The associated bilinear form

$a : H(\Omega) \times H(\Omega) \mapsto \mathbb{R}^+$ for $w = (u_1, p_1)$, $z = (u_2, p_2)$

$$a(w,z) = \int_\Omega 2\mu \, \langle \operatorname{sym}(\nabla u_1 - p_1), \operatorname{sym}(\nabla u_2 - p_2) \rangle + \lambda \operatorname{tr}[\nabla u_1 - p_1] \operatorname{tr}[\nabla u_2 - p_2]$$
$$+ 2\mu \, H_0 \, \langle \operatorname{dev}\operatorname{sym} p_1, \operatorname{dev}\operatorname{sym} p_2 \rangle$$
$$+ \mu \, L_c^2 \, (\langle \operatorname{Curl} p_1, \operatorname{Curl} p_2 \rangle + sp_0 \, \langle \operatorname{Curl}\operatorname{skew} p_1, \operatorname{Curl}\operatorname{skew} p_2 \rangle) \, \mathrm{dV} \, .$$

Simple local observation: $p \in \mathfrak{sl}(3)$ and local kinematical backstress yield

$$a(w,w) \geq c^+ \int_\Omega \underbrace{\|\operatorname{sym} \nabla u\|^2}_{\text{Korn}} + \|\operatorname{sym} p\|^2 + \|\operatorname{Curl} p\|^2 + sp_0 \, \|\operatorname{Curl}\operatorname{skew} p\|^2 \, \mathrm{dV} \, .$$

No local control of $\operatorname{skew} p$! Is a positive definite/coercive on a suitable Hilbert-space $\mathcal{H}_\sharp$ with Dirichlet-boundary conditions $\operatorname{skew} p.\tau = 0$ on Γ_D?

Global Existence and Uniqueness

Theorem [Han/Reddy99]
W closed convex subset of Hilbert space $H(\Omega))$. Find $w \in W$ s.t. t a.e.

$$\forall\, z \in W : \quad a(w, z - \dot{w}) + j(z) - j(\dot{w}) \geq \ell(z - \dot{w})\,. \tag{1}$$

Assume $a(\cdot,\cdot)$ is symmetric, continuous on $H(\Omega)$, and coercive on W.

Functional j non-negative, convex, positively homogeneous of grade one and Lipschitz-continuous, i.e.,

$$\forall\, s \in \mathbb{R} \quad j(sw) = |s|\, j(w)\,, \qquad |j(z) - j(w)| \leq L\, \|z - w\|_W\,.$$

Then (1) has unique global solution $w(t) \in W$. Method: time-discretization and passage to the limit. Rate-independency of j essential.

Solution of V-inequality satisfies global a priori energy inequality automatically!

Important finite strain generalization: Time-incremental energetic formulation based on dissipation distances by Mielke.

Discussion of boundary conditions only essential for linearized case!

$\mathcal{H}_\sharp$ - A new Hilbertspace for plastic spin

Set $\mathcal{X}_\sharp := \{p \in C^\infty(\overline{\Omega}, \mathbb{M}^{3\times 3}) \mid \operatorname{skew} p(x).\tau = 0\,, x \in \Gamma_D\,\}$. Define quadratic form for non-symmetric p

$$\|p\|_\sharp^2 := \|\operatorname{sym} p\|^2_{L^2(\Omega)} + \|\operatorname{Curl} p\|^2_{L^2(\Omega)} + \|\operatorname{Curl}\operatorname{skew} p\|^2_{L^2(\Omega)}$$

Norm on $\mathcal{X}_\sharp$ (control of $\operatorname{skew} p$ with $\operatorname{Curl} p$ only after integration!).

Completion $\mathcal{H}_\sharp := \overline{\mathcal{X}_\sharp}^{\|\cdot\|_\sharp}$ is a Hilbert-space with scalar product

$$\langle p_1, p_2\rangle_\sharp = \int_\Omega \langle \operatorname{sym} p_1, \operatorname{sym} p_2\rangle + \langle \operatorname{Curl} p_1, \operatorname{Curl} p_2\rangle \\ + \langle \operatorname{Curl}\operatorname{skew} p_1, \operatorname{Curl}\operatorname{skew} p_2\rangle \,\mathrm{dV}\,.$$

$\mathcal{H}_\sharp$ continuously embedded in $L^2(\Omega, \mathbb{M}^{3\times 3})$.

This shows that a is positive (strain part is controlled locally with symmetric kinematic hardening) on $H(\Omega) := H_0^1(\Omega, \Gamma_D, \mathbb{R}^3)) \times \mathcal{H}_\sharp$.

Open problems

- Repeat analysis for multiplicative decomposition $F = F_e\,F_p$ with quadratic energy $W(F_e) = \|\,\mathrm{sym}\,F_e - 1\!\!1\,\|^2 = \frac{1}{4}\|\nabla\varphi F_p^{-1} + F_p^{-T}\nabla\varphi^T - 2\,1\!\!1\,\|^2$ and dislocation density $\|\,\mathrm{Curl}\,F_p\|^2$ ⇒ extended Korn's first inequality

- Local isotropic hardening (SSD - statistically stored dislocations) instead of local kinematical hardening? ⇒ (Neff/Ebobisse/Reddy)

- Physically motivated choice of boundary conditions for infinitesimal plastic distortion p? Status of 2.nd law of thermodynamics (insulation condition, strong formulation) versus a priori energy inequality satisfied by weak solution?

- Is the unique weak solution a strong solution if smooth? Relation to $\|\,\mathrm{Curl}\,F_p\|$-augmented time-incremental formulation (Ortiz, Mielke/Müller)?

- FEM for irrotational small strain model ⇒ (Neff/Sydow/Wieners, submitted to Int. J. Num. Meth. Engrg.)

Extended Korn's first inequality

Theorem [N. PRSE02]
Let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL^+(3,\mathbb{R}))$ be given with $\det[F_p(x)] \geq c > 0$. Suppose dislocation density $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^1(\Omega),\ \varphi_{|_\Gamma} = 0 :$$

$$\|\nabla\phi\, F_p^{-1}(x) + F_p^{-T}(x)\, \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Recent improvement: $F_p \in C^0(\overline{\Omega})$. Work in progress suggests $F_p \in L^\infty(\Omega), \operatorname{Curl} F_p \in L^2(\Omega)$ suffices. Counterexample for $F_p \in L^\infty(\Omega)$.

Classical Korn's first inequality:

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^1(\Omega),\ \varphi_{|_\Gamma} = 0 : \qquad \|\nabla\phi + \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Definition of the Norm

Define $\mathcal{X} := \{p \in C^\infty(\overline{\Omega}, \mathbb{M}^{3\times 3}) \,|\, p(x).\tau = 0\,, x \in \Gamma_D\}$. Why defines

$$\|p\|_\sharp^2 := \int_\Omega \|\operatorname{sym} p\|^2 + \|\operatorname{Curl} p\|^2 \,\mathrm{dV}$$

a norm on $\mathcal{X}$? Need to show $\|p\|_\sharp = 0 \;\Rightarrow p = 0$. First: $\|p\|_\sharp = 0 \;\Rightarrow\; p \in \mathfrak{so}(3)$. Two linear independent tangential vectors τ: $p(x).\tau = 0\,, x \in \Gamma_D$ and $p \in \mathfrak{so}(3) \quad \Rightarrow \quad p(x) = 0\,, x \in \Gamma_D$.

$$0 = \int_\Omega \|\operatorname{Curl} p\|^2 \,\mathrm{dV} = \int_\Omega \|\operatorname{Curl}\operatorname{skew} p\|^2 \,\mathrm{dV} \geq \frac{1}{2}\int_\Omega \|\nabla \operatorname{skew} p\|^2 \,\mathrm{dV}\,,$$

$$\int_\Omega \|\nabla[\operatorname{skew} p]\|^2 \,\mathrm{dV} \geq c^+ \int_\Omega \|\operatorname{skew} p\|^2 \,\mathrm{dV}$$

Poincaré-inequality $\;\Rightarrow \operatorname{skew} p = 0 \;\Rightarrow p = 0\,.$

Problem: with this norm no continuous embedding in L^2? (skew-symmetric part is not properly controlled). No possibility to define weak traces $p.\tau$? Remedy: add plastic spin control: $sp_0\,\|\operatorname{Curl}\operatorname{skew} p\|^2$.

Rigidity estimates

Qualitative Version with relation to well known rigidity statements (John, Sverak, Müller, Friesecke)

Theorem [N./Münch ESAIM:COCV07]

$$\forall\, R \in C^1(\mathbb{R}^3, \mathrm{SO}(3)) : \quad \|\,\mathrm{Curl}\, R\|^2 \geq \frac{1}{2}\|\nabla R\|^2 \,.$$

No integration! Special case: $R = \nabla\varphi$ and $\mathrm{Curl}\,\nabla\varphi = 0$ yields $\varphi(x) = R.x + b$.

Known "linear" result: $R = 1\!\!1 + A + \ldots, \qquad A^T + A = 0$

$$\forall\, A \in C^1(\mathbb{R}^3, \mathfrak{so}(3)) : \quad \|\,\mathrm{Curl}\, A\|^2 \geq \frac{1}{2}\|\nabla A\|^2 \,.$$

Is implicitly used in the proof of Korn's first inequality:

$$\|\nabla u^T + \nabla u\|^2_\Omega = 0 \;\Rightarrow\; \nabla u = A \in \mathfrak{so}(3) \;\Rightarrow 0 = \mathrm{Curl}\,\nabla u = \mathrm{Curl}\, A \;\Rightarrow\; \nabla A = 0\,.$$

Kill constant skew-symmetric matrices A with boundary conditions!

Some useful relations

The convex, one-homogeneous dissipation functional R (Mielke/Hackl)

No-spin: $R(\dot{\xi}) = \begin{cases} \sigma_{\mathrm{y}}\ \|\dot{\xi}\| & \dot{\xi} \in \mathrm{Sym}(3) \cap \mathfrak{sl}(3) \\ \infty & \mathrm{skew}(\dot{\xi}) \neq 0 \text{ or } \mathrm{tr}\left[\dot{\xi}\right] \neq 0 \end{cases}$

Determines the plastic distortion rate $\dot{p}$ to remain symmetric and trace-free, leads to irrotational model.

Free-spin : $R(\dot{\xi}) = \begin{cases} \sigma_{\mathrm{y}}\ \|\,\mathrm{sym}\,\dot{\xi}\| & \dot{\xi} \in \mathfrak{sl}(3) \\ \infty & \mathrm{tr}\left[\dot{\xi}\right] \neq 0 \end{cases}$

Determines the plastic distortion rate to remain trace-free, but local plastic rotational spin is not controlled.

For smooth fields $A, B \in \mathbb{M}^{3\times 3}$ Stokes Theorem reads:

$$\int_\Omega \langle \mathrm{Curl}\, A, \mathrm{Curl}\, B\rangle\, \mathrm{dV} = \int_\Omega \langle \mathrm{Curl}\,\mathrm{Curl}\, A, B\rangle\, \mathrm{dV} + \int_{\partial\Omega} \langle \mathrm{Curl}\, A \times B, N\rangle\, \mathrm{dS}\,.$$

Selected References

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The irrotational geometrically linear model: no spin

Elastic domain $K := \{\|\operatorname{dev}\operatorname{sym}\Sigma\| \leq \sigma_{\mathrm{y}}\}$ then $\partial\chi_K(\Sigma) = \mathbb{R}^+ \operatorname{dev}\operatorname{sym}\Sigma$ and $p \in \operatorname{Sym}(3)$ (irrotational). Define plastic strain $\varepsilon_p := \operatorname{sym} p$.

$$\operatorname{Div}\sigma = -f\,, \quad \sigma = 2\mu\,\operatorname{sym}(\nabla u - \varepsilon_p) + \lambda\,\operatorname{tr}\left[\nabla u - \varepsilon_p\right]\mathbb{1}\,,$$

$$\dot{\varepsilon}_p \in \partial\chi(\Sigma^{\mathrm{lin}})\,, \quad \Sigma^{\mathrm{lin}} = \Sigma^{\mathrm{lin}}_{\mathrm{e}} + \Sigma^{\mathrm{lin}}_{\mathrm{sh}} + \Sigma^{\mathrm{lin}}_{\mathrm{curl}}\,,$$

$$\Sigma^{\mathrm{lin}}_{\mathrm{e}} = 2\mu\,\operatorname{sym}(\nabla u - \varepsilon_p) + \lambda\,\operatorname{tr}\left[\nabla u - \varepsilon_p\right]\mathbb{1} = \sigma\,,$$

$$\Sigma^{\mathrm{lin}}_{\mathrm{sh}} = -2\mu\,H_0\operatorname{dev}\varepsilon_p \quad \text{(linear Prager hardening)}$$

$$\Sigma^{\mathrm{lin}}_{\mathrm{curl}} = -\mu\,L_c^2\,\operatorname{sym}\left(\operatorname{Curl}\operatorname{Curl}\varepsilon_p\right)\,,$$

$$u(x,t) = u_{\mathrm{d}}(x)\,, \qquad \varepsilon_p(x,0) = \varepsilon_p^0(x)\,, \qquad [\operatorname{Curl}\varepsilon_p].\tau = 0\,.$$

Suitable space for symmetric ε_p is well-known
$H_{\mathrm{curl}}(\Omega) := \{\varepsilon_p \in L^2(\Omega)\,,\ \operatorname{Curl}\varepsilon_p \in L^2(\Omega)\}$ instead of $\mathcal{H}_\sharp$.