Optimal BV Estimates for a Discontinuous Galerkin Method in Linear Elasticity

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Introductory Remarks

- *Discontinuous Galerkin Methods*:
  - **Uses discontinuous interpolation across elements**
  - **Traditionally used for 1st and 2nd order hyperbolic eqns.**
  - **Little or no advantages over conforming FE for elliptic problems (For "*traditional*" situations)**

- *Motivations:*
  - **Linear Fracture Mechanics**
    - Crack propagation path and convergence of solutions
  - **Adaptive Mesh Refinement**
    - Efficient hp-adaptivity
  - **Dynamical response of materials**
    - Shock-Capturing schemes

Scalar Case

- *DG for scalar diffusion equation*:
  - **Developed for convection dominated difussion problems**
    - Ej: Bassi and Rebay(1997), Arnold, Brezzi, Cockburn and Marini (2002), etc.
  - **Optimal energy, *mesh-dependent*, error estimates for a stabilized formulation. (Brezzi, Manzini, et. al (2000))**
    - Completely discontinuous approximation

$$\begin{aligned} |||v - v_h|||^2 &\le C\, h^{m-1}|v|_m \\ &= \sum_{E\in\mathcal{T}_h} \|\nabla v - \nabla v_h\|^2_{0,E} + \underbrace{\sum_{e\in\mathcal{E}_h} \|r_e([\![v - v_h]\!])\|^2_{0,B}}_{\text{mesh-dependent}} \end{aligned}$$

- $\|r_e([\![v]\!])\|^2_{0,B}$ measures the jump of v across face e
- Optimal L^2-estimates: $\|v - v_h\| \le C\, h^m\, |v|_m$

Vectorial Case

◇ *DG for Linear Elasticity:*

- **An analog stabilized method to the scalar case can be formulated and analyzed for the *vectorial* case**
- **Error estimates obtained in $(L^2)^d$ and $\|\!| \cdot |\!\|_s$, given by**

$$|\!|\!| v |\!|\!|_s^2 = \sum_{E \in \mathcal{T}_h} \|\nabla_s v\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|r_e([\![v]\!])\|_{0,B}^2$$

- **$|\!|\!| \cdot |\!|\!|_s$ is not a *norm*.**
- **Need a discrete Korn's-like inequality, i. e,**

$$|\!|\!| v |\!|\!| \leq C |\!|\!| v |\!|\!|_s \qquad \forall v \in V_h + (H_0^1(B))^d$$

with C independent of h (it does not necessarily hold).

- **Want a *mesh-independent* error estimate → BV norm.**

Problem Formulation

- *$\sigma - u$ formulation of Linear Elasticity:*
  - **Hellinger-Reissner Energy:**

$$I[u,\sigma] = \int_B \left(\frac{1}{2}\sigma : \mathbb{C}^{-1} : \sigma \quad -\sigma : \nabla_s u + f \cdot u\right) + \int_{\partial_d B} n \cdot \sigma \cdot (u - \bar{u}) + \int_{\partial_\tau B} \bar{T} \cdot u$$

- *Discrete formulation allowing jumps*:
  - **Let E be an element, consider the energy:**

$$\begin{aligned} I_E &= \int_E \left(\frac{1}{2}\sigma : \mathbb{C}^{-1} : \sigma - \sigma : \nabla_s u + f \cdot u\right) + \int_{\partial E \cap \partial_\tau B} \bar{T} \cdot u \\ &+ \int_{\partial E \backslash \partial B} \frac{1}{2} n^- \cdot \sigma^- \cdot (u^- - u^+) + \int_{\partial E \cap \partial_d B} n \cdot \sigma \cdot (u - \bar{u}) \end{aligned}$$

  - **The total energy is given by:** $I_h = \sum_E I_E$

Euler-Lagrange equations

◇ *Discrete Spaces:*

- $u_h\Big|_E \in V_h^E$, **where** $(\mathcal{P}_k^E)^d \subseteq V_h^E \subset (H^1(E))^d$
- $\sigma_h\Big|_E \in W_h^{E,s}$, **where** $(\mathcal{P}_{k-1}^E)^{d\times d,s} \subseteq W_h^E \subset (H^{1,div}(E))^d$

◇ *Euler-Lagrange equations:*

- **For all** $\gamma \in W_h^s$ **and** $v \in V_h$

$$0 = \sum_{E\in\mathcal{T}_h} \int_E \Big(\gamma : \mathbb{C}^{-1} : \sigma - \gamma : \nabla_s u\Big) + \int_\Gamma [\![u]\!] : \{\gamma\} - \int_{\partial_d B} n \cdot \gamma \cdot \bar{u}$$

$$0 = \sum_{E\in\mathcal{T}_h} \int_E \Big(- \sigma : \nabla_s v + f \cdot v\Big) + \int_\Gamma [\![v]\!] : \{\sigma\} + \int_{\partial_\tau B} \bar{T} \cdot v$$

where

$$[\![v]\!] = v^+ \otimes n^+ + v^- \otimes n^- \textbf{ and } \{\sigma\} = \frac{1}{2}(\sigma^+ + \sigma^-)$$

across any face e**.**

Lifting Operators

- ◇ *Face Lifting Operator :* For a face e, let $r_e : (L^2(e))^{d\times d} \to W_h^s$ be such that

$$\int_B r_e(\tau) : \gamma = -\int_e \tau : \{\gamma\}, \qquad \forall \gamma \in W_h^s$$

- ◇ *Lifting Operator :* Let $R : (L^2(\cup_{e\in\mathcal{T}_h} e))^{d\times d} \to W_h^s$ such that for any element $E \in \mathcal{T}_h$,

$$R(\tau) = \sum_{e\subset\partial E} r_e(\tau) \qquad \text{on E}$$

- ◇ *It holds :* There exist $C_1 > 0$,$C_2 > 0$ independent of the face e and h such that for all $v \in V_h + (H_0^1(B))^d$

$$C_2\, h^{1/2}\|r_e([\![v]\!])\|_{0,B} \le \|[\![v]\!]\|_{0,e} \le C_1\, h^{1/2}\|r_e([\![v]\!])\|_{0,B},$$

- ◇ For simplicity, we assume $u\big|_{\partial B} = 0$

One Field Equation

- *Equivalent one-field problem :*
  - Find $u \in V_h$ such that

$$a(u,v) = \int_B f \cdot v \qquad \forall v \in V_h$$

where the symmetric bilinear form $a : V_h \times V_h \to \mathbb{R}$ is given by

$$a(u,v) = \sum_{E \in \mathcal{T}_h} \int_E (\nabla_s v + R(\llbracket v \rrbracket_s)) : \mathbb{C} : (\nabla_s u + R(\llbracket u \rrbracket_s)) \\ \underbrace{+\beta \sum_{e \in \mathcal{E}_h} \int_B r_e(\llbracket u \rrbracket_s) : \mathbb{C} : r_e(\llbracket v \rrbracket_s)}_{\text{Stabilization term}}$$

  - The stress σ is given by

$$\sigma = \sigma(u) = \mathbb{C} : \nabla_s u + \mathbb{C} : R(\llbracket u \rrbracket_s)$$

Summary of Results

◇ *DG for Linear Elasticity. We proved:*

- **Optimal error estimates in $(L^2(B))^d$ and in $\|\!|\cdot|\!\|_s$.**
- **An elemental version of Korn's inequality for the discretization.**
- **Therefore, optimal error estimates in $\|\!|\cdot|\!\|$.**
- **Then we also proved that**

$$\|v\|_{BV} \leq C\|\!|v|\!\| \qquad \forall v \in V_h + (H_0^1(B))^d$$

with C independent of h.

- **Alltogether we have the chain of inequalities**

$$\|v\|_{BV} \leq C\|\!|v|\!\| \leq \bar{C}\left(\|\!|v|\!\|_s + \frac{1}{h}\|v\|_{0,B}\right) \quad \forall v \in V_h + (H_0^1(B))^d$$

from where the optimal and *mesh-independent* BV estimate follows.

Mesh-dependent estimate

- The proof follows the steps in Brezzi, Manzini et. al. (2001).
- *From here it follows that:*
  - $a(\cdot,\cdot)$ **is coercive and continuous in** $\|\!|\cdot|\!\|_s$ **for** $\beta > N_e$**. (respectively** $\beta > 0$**).**
  - **If** $u \in (H^m(B))^d$ **is the exact solution,** $1 \leq m \leq k+1$**, then**

$$\|\!|u - u_h|\!\|_s < C\ h^{m-1}|u|_m$$

and

$$\|u - u_h\|_{0,B} < C\ h^m|u|_m$$

and

$$\|\sigma - \sigma_h\|_{0,B} < C\ h^{m-1}|u|_m$$

Discrete Korn's Inequality

- *Discrete Elemental Korn's inequality:*
  - **For a regular family of triangulations $\{\mathcal{T}_h\}$ composed of affine equivalent elements**

$$\exists \quad C > 0 : \quad \forall\, E \in \mathcal{T}_h, \quad \forall\, u \in H^1(E),$$
$$\|\nabla u^T + \nabla u\|_{0,E}^2 + \frac{1}{h_E^2}\,\|u\|_{0,E}^2 \geq C\left(\|\nabla u\|_{0,E}^2 + \frac{1}{h_E^2}\,\|u\|_{0,E}^2\right)$$

  - **Obtained by considering the effect of affine transformations on the reference elements.**
    - Consider the distortion and volumetric deformations separately

Convergence in $\|\!|\cdot|\!\|$

- *It holds that:*

$$|\!|\!| u - u_h |\!|\!| < C h^{m-1} |u|_m$$

- **Proof:**

$$\sum_{E\in\mathcal{T}_h} \|\nabla u^T + \nabla u\|_{0,E}^2 \quad + \frac{1}{h^2}\|u\|_{0,E}^2 + \sum_{e\in\mathcal{E}_h} \|r_e([\![u]\!]_s)\|_{0,B}^2$$

$$\geq C \sum_{E\in\mathcal{T}_h} \left(\|\nabla u\|_{0,E}^2 + \frac{1}{h^2}\|u\|_{0,E}^2 + \sum_{e\in\mathcal{E}_h} \|r_e([\![u]\!]_s)\|_{0,B}^2 \right)$$

$$|\!|\!| u |\!|\!|_s^2 + \frac{1}{h^2}\|u\|_{0,B}^2 \geq C \left(|\!|\!| u |\!|\!|^2 + \frac{1}{h^2}\|u\|_{0,B}^2 \right)$$

$$C h^{2m-2} = C \left(h^{2m-2} + \frac{1}{h^2} h^{2m} \right) \geq |\!|\!| u |\!|\!|^2$$

BV Estimate

◇ *Natural BV estimate:*

$$\exists\, C > 0 : \qquad \|u\|_{BV} \leq C\, \|\!|u|\!\|,$$

$\forall\, u \in V_h + (H_0^1(B))^d$ with C independent of h.

- **BV norm**

$$\|u\|_{BV(B)} = \|u\|_{L^1(B)} + \|Du\|(B)$$

- **Poincaré for BV:**

$$\exists\, C > 0 : \quad \forall\, u \in BV(\mathbb{R}^d), \quad \|u\|_{L^{d/(d-1)}(\mathbb{R}^d)} \leq C\, \|Du\|(\mathbb{R}^d).$$

- **From the definition of $\|\cdot\|_{BV}$ and the fact that $u \in V_h + (H_0^1(B))^d$ it is possible to obtain**

$$\|Du\|(B) \leq C \|\!|u|\!\|$$

The End

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