CISM-Lecture 2007

The Γ-limit of a finite-strain Cosserat model for asymptotically thin domains versus a formal dimensional reduction

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Plan of the lecture

- Notation
- The finite-strain Cosserat model
- The rescaled Cosserat problem
- Γ-convergence and the Γ-limit
- Comparison: formal finite-strain Cosserat plate
- Linearization and the linear Reissner-Mindlin model
- Open problems and Conclusions

Notation: Bulk behaviour

- $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, the deformation, Ω reference configuration.
- $F = \nabla\varphi \in \mathrm{GL}^+(3,\mathbb{R})$, the deformation gradient, $1\!\!1$ the identity.
- $W(F, \overline{R})$ elastic free energy density, $D_F W(F, \overline{R})$ first P. K. stresses.
- $\|A\|,\ \langle A, B\rangle,\ \mathrm{tr}\,[A]$ norm, scalar product and trace.
- $F = R\,U = \mathrm{polar}(F)\,U, \quad R \in \mathrm{SO}(3)$, polar decomposition.
- $\overline{R} : \Omega \mapsto \mathrm{SO}(3,\mathbb{R})$ Cosserat-rotation.
- $\overline{U} = \overline{R}^T F$, non-symmetric micropolar-stretch tensor.

The Cosserat approach

- Idea: relax constraint on the rotations $R(\nabla\varphi)$ to coincide with continuum rotations $\mathrm{polar}(\nabla\varphi)$. Allow for independent (micro) rotations $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ (Cosserat 1906, motivated by "triedre caché" of **surface theory**).

- Attribute energy to spatial variations of rotations (curvature): $W_{\mathrm{curv}}(\mathrm{D_x}\overline{R}) = \mu\, L_c{}^p \|\mathrm{D_x}\overline{R}\|^p$.

- Look for independent fields $\varphi, \overline{R}$ in the minimization problem. $\overline{R} \in \mathrm{SO}(3,\mathbb{R})$ nonlinear manifold.

- Suited for material with granular substructure (individual rotations of particles), spin, magnetization, homogenisation of heterogeneous materials. Explore lower energy paths, avoid stability problems.

The finite Cosserat problem in variational form

Thin domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, $\omega \subset \mathbb{R}^2$.

$$I(\varphi, \overline{R}) = \int_{\Omega_h} W_{\mathrm{mp}}(\overline{U}) + \mu \, {L_c}^p \|\mathrm{D_x} \overline{R}\|^p \, \mathrm{dV} \mapsto \min . \text{ w.r.t.} (\varphi, \overline{R})$$

$$\varphi_{|\Gamma_h} = g_{\mathrm{d}}, \quad \overline{R}_{|\Gamma_h} \quad \text{free}$$

$$W_{\mathrm{mp}}(\overline{U}) = \mu \, \|\operatorname{sym}(\overline{U} - 1\!\!1)\|^2 + \frac{\lambda}{2} \operatorname{tr}\left[\operatorname{sym}(\overline{U} - 1\!\!1)\right]^2 + \mu_c \, \|\operatorname{skew}(\overline{U} - 1\!\!1)\|^2$$

$$\overline{U} = \overline{R}^T F, \quad \text{non-symmetric micropolar stretch tensor.}$$

Existence: $\mu_c \geq 0$ Neff (PRSE06), general micromorphic model: $\overline{R} \in \mathrm{SO}(3, \mathbb{R}) \to P \in \mathrm{GL}^+(3, \mathbb{R})$, Neff/Forest (JEL07). Infinitesimal elasto-plastic extension, $\mu_c > 0$: Neff/Chełmiński (PRSE05).

The Cosserat balance equations

E-L: linear momentum+angular momentum: nonlinear second order PDE, non-constant coefficients

$$0 = \mathrm{Div}\left(S_1(F,\overline{R}) + 2\,\mu_c\,\overline{R}\,\mathrm{skew}(\overline{R}^T F)\right)$$

$$0 = \mathrm{skew}(D_{\overline{U}} W_{\mathrm{mp}}(\overline{U})\,\overline{U}^T) + \mathrm{skew}\left(\overline{R}^T\,\mathrm{Div}\left[\overline{R}\,D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\right]\right)$$

$$+\,\mathrm{skew}\left(D_{\mathfrak{K}} W_{\mathrm{curv}}(\mathfrak{K})\mathfrak{K}^T\right)$$

$$S_1(F,\overline{R}) = \overline{R}\left[\mu(F^T\overline{R} + \overline{R}^T F - 2\mathbb{1}) + \lambda\,\mathrm{tr}\left[F^T\overline{R} - \mathbb{1}\right]\mathbb{1}\right]$$

$$W_{\mathrm{curv}}(\mathrm{D_x}\overline{R}) = \mu\,{L_c}^p\,\|\mathrm{D_x}\overline{R}\|^p$$

The Cosserat curvature energy

The curvature term has the form

$$W_{\text{curv}}(\mathrm{D_x}\overline{R}) = \mu\, {L_c}^p\, \|\mathrm{D_x}\overline{R}\|^p\,,$$
$$\mathrm{D_x}\overline{R} = \big(\nabla(\overline{R}.e_1), \nabla(\overline{R}.e_2), \nabla(\overline{R}.e_3)\big)\ .$$

Coercivity of curvature

$$\exists\, c^+ > 0\, \exists\, r > 1 :\ \forall\, \mathfrak{K} \in \mathfrak{T}(3) : \quad W_{\text{curv}}(\mathfrak{K}) \geq c^+\, \|\mathfrak{K}\|^r$$
$$\exists\, r > 1 : \quad \frac{W_{\text{curv}}(\mathfrak{K})}{\|\mathfrak{K}\|^r} \to \infty \quad \text{as } \|\mathfrak{K}\| \to \infty$$
$$\Rightarrow \quad p > 1$$

$L_c > 0$: internal length, characteristic for the material (grain size).

The Cosserat couple modulus μ_c

- $\mu_c > 0 : \quad D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(H,H) \geq 2\,\mu_c\,\|H\|^2$
- $\mu_c = 0 : \quad D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(H,H) \geq 2\,\mu\|\frac{1}{2}(\overline{R}^T H + H^T \overline{R})\|^2$
- $\mu_c = 0 : \quad D_F^2 W_{\mathrm{mp}}(\overline{R}^T F).(\xi \otimes \eta, \xi \otimes \eta) \geq \mu\,\|\xi\|^2 \|\eta\|^2$
- $\mu_c = 0$: uniformly Legendre-Hadamard independent of $\overline{R}(x)$.
- $\mu_c = 0$: linearization coincides with classical linear elasticity.
- thinking linear one may believe: $\mu_c > 0$ necessary

The infinitesimal, linear Cosserat problem

Expand $F = 1\!\!1 + \nabla u$, $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, $\overline{A} \in \mathfrak{so}(3, \mathbb{R})$

$$\int_{\Omega_h} W_{\mathrm{mp}}(\nabla u, \overline{A}) + \mu \, L_c{}^2 \, \|\mathrm{D_x} \overline{A}\|^2 \, \mathrm{dV} \mapsto \min . \text{ w.r.t. } (u, \overline{A})$$

$$W_{\mathrm{mp}}(\nabla u, \overline{A}) = \mu \, \|\operatorname{sym} \nabla u\|^2 + \frac{\lambda}{2} \operatorname{tr} [\operatorname{sym} \nabla u]^2 + \mu_c \, \|\operatorname{skew} \nabla u - \overline{A}\|^2$$

Cosserat couple modulus $\mu_c = 0 \Rightarrow$ no coupling, linear elasticity.

$L_c = 0 \Rightarrow$ linear elasticity.

Well posed linear infinitesimal Cosserat problem.

The rescaled Cosserat problem

Follow Le Dret/Raoult (Membrane-model):
domain with constant thickness $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}]$
$\varphi^\sharp(x,y,z) := \varphi(x,y,h\,z), \quad \overline{R}^\sharp(x,y,z) := \overline{R}(x,y,h\,z)$.

$$I^\sharp(\varphi^\sharp, \overline{R}^\sharp) = h\int_{\Omega_1} W_{\mathrm{mp}}(\overline{U}_h{}^\sharp) + \mu\, L_c{}^p \|\mathrm{D_x}{}^h \overline{R}^\sharp\|^p \,\mathrm{dV} \mapsto \min . \text{ w.r.t.}(\varphi^\sharp, \overline{R}^\sharp)$$

$$\overline{U}_h^\sharp := \overline{R}^{\sharp,T}\,\nabla^h \varphi^\sharp, \quad \varphi^\sharp_{|_{\Gamma^1}} = g_{\mathrm{d}}^\sharp, \quad \overline{R}^\sharp_{|_{\Gamma^1}} \quad \text{free}$$

$$\nabla^h \varphi^\sharp := (\partial_x \varphi^\sharp | \partial_y \varphi^\sharp | \frac{1}{h} \partial_z \varphi^\sharp)$$

$$\mathrm{D_x}{}^h \overline{R}^\sharp := (\nabla^h(\overline{R}^\sharp.e_1) | \nabla^h(\overline{R}^\sharp.e_2) | \nabla^h(\overline{R}^\sharp.e_3))$$

Consider sequence of variational problems $I_h{}^\sharp(\varphi^\sharp, \overline{R}^\sharp) := \frac{1}{h}\, I^\sharp(\varphi^\sharp, \overline{R}^\sharp)$.

Γ-convergence- variational convergence

X metric space. $I_{h_j} : X \mapsto \overline{\mathbb{R}}$ Γ-converges in X to $I_0 : X \mapsto \overline{\mathbb{R}}$, if for all $x \in X$ we have

$$\forall\, x_{h_j} \to x : \quad I_0(x) \le \liminf_{h_j \to 0} I_{h_j}(x_{h_j}), \qquad \textbf{(}\liminf\textbf{-inequality)}$$

$$\exists\, x_{h_i} \to x : \quad I_0(x) \ge \limsup_{h_i \to 0} I_{h_i}(x_{h_i}), \qquad \textbf{(recovery sequence)} \; .$$

Γ-convergence ensures that limit of minimizers to I_{h_j} is a minimizer of the limit functional I_0.

Here: $X = L^r(\Omega_1, \mathbb{R}^3) \times W^{1,p}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))$, $r = p'$, $p > 3$.
$H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^r(\Omega_1, \mathbb{R}^3)$ compact.

Notation: Thin structures

- $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, midsurface-deformation.
- ω planar reference configuration, γ_0 Dirichlet boundary.
- $\nabla m = (\partial_x m | \partial_y m) \in \mathbb{M}^{3\times 2}$ the midsurface-deformation gradient.
- $(\xi|\eta|\theta)$ matrix composed of columns $\xi, \eta, \theta \in \mathbb{R}^3$.
- $1\!\!1_2$ the identity on $\mathbb{M}^{2\times 2}$.
- $\overline{R}_i = \overline{R}.e_i$, i.th column of matrix $\overline{R}$.

The Γ-limit "membrane" plate

Find midsurface deformation $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ and microrotation of the plate $\overline{R} : \omega \subset \mathbb{R}^2 \mapsto \mathrm{SO}(3,\mathbb{R})$:

$$I_0^\sharp(m,\overline{R}) = \int_\omega W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) + \mu\, {L_c}^p \, \|\mathfrak{K}_s\|^p \, \mathrm{d}\omega \mapsto \min . \text{ w.r.t. } (m,\overline{R})$$

$$m_{|_{\gamma_0}} = g_{\mathrm{d}}(x,y,0)\,, \qquad \text{simply supported}$$

$$\overline{R}_{|_{\gamma_0}} \qquad \text{free: Neumann condition}$$

$$\mathfrak{K}_s = \Big(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\Big) \in \mathfrak{T}(3)$$

third order curvature tensor

The homogenized reduced energy density

$$
\begin{aligned}
&W_{\rm mp}^{\rm hom}(\nabla m, \overline{R}) \\
&= \mu \underbrace{\|\,\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\|^2}_{\text{"intrinsic" shear-stretch energy}} + \mu_c \underbrace{\|\,\mathrm{skew}((\overline{R}_1|\overline{R}_2)^T\nabla m)\|^2}_{\text{"intrinsic" first order drill energy}} \\
&\underbrace{+\,2\mu\,\frac{\mu_c}{\mu+\mu_c}\Big(\langle \overline{R}_3, m_x\rangle^2 + \langle \overline{R}_3, m_y\rangle^2\Big)}_{\text{homogenized transverse shear energy}} \\
&+ \frac{\mu\lambda}{2\mu+\lambda}\underbrace{\mathrm{tr}\left[\mathrm{sym}((\overline{R}_1|\overline{R}_2)^T\nabla m - 1\!\!1_2)\right]^2}_{\text{homogenized elongational stretch energy}} .
\end{aligned}
$$

Not coercive if $\mu_c = 0$, missing transverse shear energy. Loss of coercivity not related to missing drill energy.

Sketch of proof for $\mu_c > 0$: equi-coercivity

- $\mu_c > 0$: elastic energy estimates strain part: $\|\overline{R}^{k,T}\nabla^h\varphi_k - 1\!\!1\|^2 = \|\overline{R}^{k,T}\nabla^h\varphi_k\|^2 - 2\langle\overline{R}^k, \nabla^h\varphi_k\rangle + 3 \geq \|\nabla^h\varphi_k\|^2 - 2\sqrt{3}\|\nabla^h\varphi_k\| + 3$.

- Control of φ_k in $H^{1,2}(\Omega_1, \mathbb{R}^3)$ is independent of $\overline{R}^k$.

- Since $\|\mathrm{D_x}^h\overline{R}\|^p$ in the energy, control $\overline{R}^k$ in $W^{1,p}(\Omega_1)$.

- Restrict attention to strongly convergent subsequence of rotations $\overline{R}^k \to \overline{R}$ in $L^p(\omega, \mathrm{SO}(3,\mathbb{R}))$.

- Use compact embedding $H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^r(\Omega_1, \mathbb{R}^3)$.

- $I_h^\sharp$ is equi-coercive over $X = L^r(\Omega_1, \mathbb{R}^3) \times W^{1,p}(\Omega_1, \mathrm{SO}(3,\mathbb{R}))$.

Sketch of proof for $\mu_c > 0$: lower-bound

- Local minimization w.r.t. "transverse" derivatives $\partial_z \varphi$.
- $W_{\mathrm{mp}}^{\mathrm{hom}}(\nabla m, \overline{R}) := W_{\mathrm{mp}}(\overline{R}^T(\nabla m | b^*)) := \inf_{b \in \mathbb{R}^3} W_{\mathrm{mp}}(\overline{R}^T(\nabla m | b))$.
- Determine "director" b^* explicitely: $D_F W_{\mathrm{mp}}((\nabla m | b^*), \overline{R}).e_3 = 0$. "Plane stress", membrane state.
- Use convexity of energy w.r.t. $\nabla^h \varphi$ and weak convergence of $\nabla^h \varphi_k$.
- Use convexity of energy w.r.t. $\mathrm{D_x}^h \overline{R}$ and strong convergence of $\overline{R}^k$.
- Obtain $\liminf$-inequality.

Sketch of proof $\mu_c > 0$: recovery sequence

- Given $m \in L^r(\omega, \mathbb{R}^3), \overline{R} \in W^{1,p}(\omega, \mathrm{SO}(3,\mathbb{R}))$.
- Reconstruction: $\varphi_{h_j}(x,y,z) = m(x,y) + h_j\, z\, b^*(\nabla m(x,y), \overline{R}(x,y))$.
- Reconstruction: $\overline{R}_{h_j}(x,y,z) = \overline{R}(x,y)$, constant in thickness.
- Approximation and continuity yield upper bound: $\limsup$-inequality.

Ideas of proof for $\mu_c = 0$:

- Unusual difficulty: sequence I_h is not equi-coercive. (equi-coercivity: uniform coercivity independent of $h > 0$).

- Construct suitable comparison functionals. Establish $\Gamma - \limsup$ estimate.

- Circumvent lack of compactness in φ_k by looking at in-plane components $(\overline{R}_1|\overline{R}_2)^{k,T}\nabla_{(x,y)}\varphi_k$.

- Use Korn's second inequality for a suitable quantity.

- Establish weak convergence of $(\overline{R}_1|\overline{R}_2)^{k,T}\nabla_{(x,y)}\varphi_k$, which does not imply weak convergence of $\nabla_{(x,y)}\varphi_k$! Establish $\Gamma - \liminf$ estimate. Show $\Gamma - \liminf \geq \Gamma - \limsup$.

Comparison: formal Cosserat plate

Find $m : \omega \mapsto \mathbb{R}^3$, $\overline{R} \in \mathrm{SO}(3, \mathbb{R})$:

$$I(m, \overline{R}) = \int_\omega h W_{\mathrm{mp}}(\nabla m, \overline{R}) + h\,\mu {L_c}^p\,\|\mathfrak{K}_s\|^p + \frac{h^3}{12} W_{\mathrm{bend}}(\mathfrak{K}_b)\,\mathrm{d}\omega \mapsto \min\,.$$

$$\mathfrak{K}_s = \left(\overline{R}^T(\nabla(\overline{R}.e_1)|0), \overline{R}^T(\nabla(\overline{R}.e_2)|0), \overline{R}^T(\nabla(\overline{R}.e_3)|0)\right) \in \mathfrak{T}(3)\,,$$

$$m_{|\gamma_0} = g_{\mathrm{d}}(x, y, 0)\,, \qquad \text{simply supported}\,.$$

$$\overline{R}_{|\gamma_0} \quad \text{free: Neumann condition}$$

$$\varphi(x, y, z) = m(x, y) + \left(\varrho_m(x, y)\,z + \varrho_b(x, y)\frac{z^2}{2}\right)\overline{R}(x, y).e_3$$

$$\overline{R}(x, y, z) = \overline{R}(x, y).$$

Quadratic ansatz through thickness. ϱ_m thickness stretch, ϱ_b asymmetric shift of the midsurface due to bending.

The formal reduced energy densities: stretch

$$W_{\mathrm{mp}}(\nabla m, \overline{R})$$
$$= \mu \underbrace{\| \operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2)\|^2}_{\text{"intrinsic" shear-stretch energy}} + \mu_c \underbrace{\| \operatorname{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2}_{\text{"intrinsic" first order drill energy}}$$
$$+ \underbrace{\frac{\kappa(\mu+\mu_c)}{2} \left(\langle \overline{R}_3, m_x \rangle^2 + \langle \overline{R}_3, m_y \rangle^2 \right)}_{\text{classical transverse shear energy}}$$
$$+ \frac{\mu\lambda}{2\mu+\lambda} \underbrace{\operatorname{tr}\left[\operatorname{sym}((\overline{R}_1|\overline{R}_2)^T \nabla m - 1\!\!1_2)\right]^2}_{\text{"intrinsic" elongational stretch energy}},$$

$0 < \kappa \leq 1$, transverse shear correction coefficient

The formal reduced energy densities: bending

$$W_{\text{bend}}(\mathfrak{K}_b) = \mu \,\|\,\text{sym}(\mathfrak{K}_b)\|^2 + \mu_c \,\|\,\text{skew}(\mathfrak{K}_b)\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\text{tr}\left[\text{sym}(\mathfrak{K}_b)\right]^2,$$

$$\mathfrak{K}_b = \overline{R}^T (\nabla \overline{R}_3 | 0) = \mathfrak{K}_s^3$$

Neff (CMT04), Existence $\mu_c = 0$: Neff (M3AS07). Main idea: new Korn's inequality for shells/plates.

Linearization

$m(x,y) = (x,y,0)^T + v(x,y)$, $\overline{R} = 1\!\!1 + \overline{A} + \ldots$, $\overline{A} \in \mathfrak{so}(3,\mathbb{R})$. First order reduction of formal Cosserat plate model ($\kappa = 1$, $p > 2$)

$$\int_\omega h \Big(\mu \,\|\, \mathrm{sym}((\nabla v|\overline{A}_3))\|^2 + \mu_c \,\|\, \mathrm{skew}((\nabla v|\overline{A}_3) - \overline{A})\|^2$$

$$+ \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[(\nabla v|\overline{A}_3)\right]^2 \Big) + \frac{h^3}{12} \left(\mu \,\|\, \mathrm{sym}((\nabla \overline{A}_3|0))\|^2 \right.$$

$$\left. + \mu_c \,\|\, \mathrm{skew}((\nabla \overline{A}_3|0))\|^2 + \frac{\mu\lambda}{2\mu+\lambda} \mathrm{tr}\left[(\nabla \overline{A}_3|0)\right]^2 \right) \mathrm{d}\omega$$

$$\mapsto \min . \text{ w.r.t. } (v, \overline{A})$$

$\mu_c = 0$: infinitesimal in-plane rotations do not "survive" linearization: classical Reissner-Mindlin model, $-\overline{A}_3 = (\theta_1, \theta_2, 0)^T$. Infinitesimal "director" $\theta \in \mathbb{R}^2$.

The infinitesimal, linear Reissner-Mindlin plate

Displacement $v : \omega \mapsto \mathbb{R}^3$, infinitesimal "director" $\theta : \omega \mapsto \mathbb{R}^2$

$$\int_\omega h \left(\mu \,\| \operatorname{sym} \nabla (v_1, v_2) \|^2 + \underbrace{\kappa \frac{\mu}{2} \, \| \nabla v_3 - \theta \|^2}_{\text{transverse shear energy}} + \frac{\mu \lambda}{2\mu + \lambda} \operatorname{tr} \left[\operatorname{sym} \nabla (v_1, v_2) \right]^2 \right)$$

$$+ \frac{h^3}{12} \left(\mu \,\| \operatorname{sym} \nabla \theta \|^2 + \frac{\mu \lambda}{2\mu + \lambda} \operatorname{tr} \left[\operatorname{sym} \nabla \theta \right]^2 \right) \, d\omega \mapsto \min . \text{ w.r.t. } (v, \theta)$$

$$v_{|\gamma_0} = u^{\mathrm{d}}(x, y, 0) \qquad \text{simply supported}$$

$$-\theta_{|\gamma_0} = (u^{\mathrm{d}}_{1,z}, u^{\mathrm{d}}_{2,z})^T$$

Open problems

- Γ-limit for $\mu_c = \infty$ is perhaps not the existing limit $\mu_c \to \infty$ of the Γ-limit formula? (work in progress Neff/Chełmiński)
- Optimal value of shear correction term $0 < \kappa \leq 1$?
- Other boundary conditions for $\overline{R}$ on Dirichlet boundary yield other Γ-limit result? Not present in classical theories.
- What is the boundary condition for 3D-Cosserat problem on Dirichlet boundary for the microrotations?
- Extension to (more realistic) non-quadratic bulk energy in $\overline{U}$, e.g. $W_{\mathrm{mp}}(\overline{U}) \to \infty$ for $\det[\overline{U}] \to 0$?

Conclusion

- Γ-limit underestimates the transverse shear stresses (Reuss-bound), modulus $2\mu\,\frac{\mu_c}{\mu+\mu_c}$.
- Rigourous Γ-limit is not coercive for $\mu_c = 0$. But $\mu_c = 0$ is the consistent value.
- Formal limit (ansatz) perhaps overestimates the transverse shear stresses (Voigt-bound), modulus $\frac{(\mu+\mu_c)}{2}$.
- Transverse shear correction factor $0 < \kappa \leq 1$ interpolates between both extremes. Stable computations need $\kappa > 0$!
- For $\mu_c = 0$ and $p > 2$ classical Reissner-Mindlin is the linearization of the formal Cosserat plate model.

Extended Korn's first inequality

Let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL^+(3,\mathbb{R}))$ be given with $\det[F_p(x)] \geq c > 0$. Suppose dislocation density $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^1(\Omega),\ \varphi_\Gamma = 0 :$$
$$\|\nabla\phi\, F_p^{-1}(x) + F_p^{-T}(x)\, \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)} \,.$$

Proof: [Neff (PRSE02)], recent improvement: $F_p \in C^0(\overline{\Omega})$. But work in progress suggests $F_p \in L^\infty(\Omega), \operatorname{Curl} F_p \in L^{3+\delta}(\Omega)$ suffices. Counterexample for $F_p \in L^\infty(\Omega)$.

Reminder: Classical Korn's first inequality:

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^1(\Omega),\ \varphi_\Gamma = 0 : \qquad \|\nabla\phi + \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)} \,.$$

Sketch of proof for Korn's inequality

Gårding's inequality: F_p invertible and smooth, exists $\lambda > 0$:

$$\|\nabla\phi\, F_p^{-1} + F_p^{-T}\, \nabla\phi^T\|^2_{L^2(\Omega)} + \lambda\, \|\phi\|^2_{L^2(\Omega)} \geq c^+\, \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Show $\|\nabla\phi\, F_p^{-1} + F_p^{-T}\, \nabla\phi^T\|^2_{L^2(\Omega)} = 0 \Rightarrow \phi = 0$, then use compactness. Let $\phi \in C_0^\infty(\Omega, \Gamma) \cap H^{1,2}(\Omega)$.

$$\nabla\phi F_p^{-1} = A \in L^2(\Omega, \mathfrak{so}(3)) \;\Rightarrow\; \nabla\phi = A\, F_p \;\Rightarrow$$

$0 = \operatorname{Curl}\nabla\phi = L_{F_p}\cdot\nabla A + A\, \operatorname{Curl} F_p,\; L_{F_p}$ invertible, if $\det[F_p] \neq 0 \Rightarrow$

$$\nabla A = -L_{F_p}^{-1}\cdot[A\, \operatorname{Curl} F_p] \Rightarrow\; A \in W^{1,2}(\Omega)$$

$$\phi_{|_\Gamma} = 0 \;\Rightarrow\; 0 = \nabla\phi.\tau_i = A\,(F_p.\tau_i),\; i = 1, 2 \;\Rightarrow\; \operatorname{rank}(A)_{|_\Gamma} \leq 1 \;\Rightarrow\; A_{|_\Gamma} = 0$$

$\nabla A = -L_{F_p}^{-1}\cdot[A\, \operatorname{Curl} F_p],\; A_{|_\Gamma} = 0$, PDE 1. Ord., linear $\Rightarrow\; A \equiv 0$

$$A \equiv 0 \;\Rightarrow\; \nabla\phi = A\, F_p = 0 \;\Rightarrow\; \nabla\phi = 0 \;\Rightarrow\; \phi = 0\,.$$

Extended Korn's inequality for shells

Theorem [N. M3AS07]
Let $\omega \subset \mathbb{R}^2$ be a smooth domain and $\gamma_0 \subset \partial\omega$ has full one-dimensional Hausdorff-measure. Moreover $F_p, F_p^{-1} \in W^{1,2+\delta}(\overline{\omega}, GL(3))$. Then

$$\exists\, c^+ > 0 \quad \forall\, m \in H^1(\omega),\, m_{|_{\gamma_0}} = 0 :$$
$$\|(\nabla m|0)\, F_p^{-1}(x) + F_p^{-T}(x)\, (\nabla m|0)^T\|^2_{L^2(\omega)} \geq c^+\, \|m\|^2_{H^1(\omega)}\,.$$

Apply to energy

$$\frac{1}{4}\, \|R^T(\nabla m|R_3) + (\nabla m|R_3)^T R - 2\, 1\!\!1\|^2$$
$$\sim \|\, \mathrm{sym}((R_1|R_2)^T \nabla m - 1\!\!1_2)\|^2 + \left(\langle m_x, R_3\rangle^2 + \langle m_y, R_3\rangle^2\right)$$
$$\sim \|R^T(\nabla m|0) + (\nabla m|0)^T R\|^2 + \text{ l.o.t}$$

Rigidity estimates

Qualitative Version with relation to well known rigidity statements (John, Sverak, Müller, Friesecke)

Theorem [N./Münch ESAIM:COCV07]

$$\forall\, R \in C^1(\mathbb{R}^3, \mathrm{SO}(3)) : \quad \|\operatorname{Curl} R\|^2 \geq \frac{1}{2}\|\nabla R\|^2 \,.$$

No integration! Special case: $R = \nabla\varphi$ and $\operatorname{Curl}\nabla\varphi = 0$ yields $\varphi(x) = R.x + b$.

Known "linear" result: $R = 1\!\!1 + A + \ldots, \qquad A^T + A = 0$

$$\forall\, A \in C^1(\mathbb{R}^3, \mathfrak{so}(3)) : \quad \|\operatorname{Curl} A\|^2 \geq \frac{1}{2}\|\nabla A\|^2 \,.$$

Is implicitly used in the proof of Korn's first inequality:

$$\|\nabla u^T + \nabla u\|^2_\Omega = 0 \;\Rightarrow\; \nabla u = A \in \mathfrak{so}(3) \;\Rightarrow 0 = \operatorname{Curl}\nabla u = \operatorname{Curl} A \;\Rightarrow\; \nabla A = 0 \,.$$

Kill constant skew-symmetric matrices A with boundary conditions!