Optimal BV-estimates for a discontinuous Galerkin method in linear elasticity.

Adrian Lew, Patrizio Neff,
Deborah Sulsky, Michael Ortiz

California Institute of Technology and
TU DARMSTADT

Plan of the talk

- General framework of linear elasticity.
- Discontinuous Galerkin methods.
- Mesh dependent error estimates.
- BD and BV spaces.
- Korn's first and second inequality.
- Natural mesh independent BD error estimate.
- Optimal mesh independent BV error estimate.

Notation

- $u : B \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, the displacement, $\nabla_s u = \frac{1}{2}(\nabla u^T + \nabla u)$ symmetrized gradient. B reference configuration.
- $H_0^1(B)$ Sobolev space with zero trace on the boundary of B.
- E typical element of a triangulation $\mathcal{T}_h$ with $h > 0$ typical diameter of the elements and $e \in \mathcal{E}_h$ typical face of these elements.
- V_h discrete space subordinate to the triangulation: $u \in V_h \Leftrightarrow u_{|_E} \in H^1(E), u_{|_{\partial B}} = 0$. u_h discrete solutions.
- $[\![u]\!]$ jump over a face.

Linear Elasticity

The problem of linear elasticity reads

$$\begin{aligned} \operatorname{Div} \mathbb{C}.\nabla_s u &= f \\ u_{|\partial_D B} &= \bar{u} \\ \mathbb{C}.\nabla_s u_{|\partial_N B} &= \bar{T} \end{aligned}$$

$\mathbb{C}$ constant coefficient (not essential) uniformly positive definite elasticity tensor, i.e. $\langle \mathbb{C}.\eta, \eta \rangle \geq c^+ \|\eta\|^2$. For simplicity: $\bar{u} = 0$, not essential and $\partial_N B = 0$ (essential). For smooth boundary and data the unique solution satisfies $u \in H^2$.

Stress-displacement formulation

The problem of linear elasticity can be cast into the two field stress displacement problem

$$\begin{aligned} \operatorname{Div} \sigma &= f \\ \sigma &= \mathbb{C}.\nabla_s u \\ u_{|\partial_D B} &= \bar{u} \end{aligned}$$

Global Hellinger-Reissner Variational principle

The stress displacement problem can be formally obtained by taking free variations with respect to both u, σ of

$$I[u,\sigma] = \int_B \left(\frac{1}{2}\langle \sigma, \mathbb{C}^{-1}.\sigma \rangle - \langle \sigma, \nabla_s u \rangle + f\, u \right)$$
$$\int_{\partial_D B} n\, \sigma (u - \bar{u})$$

Here n is the unit outward normal on $\partial_D B$.

Local Hellinger-Reissner Variational principle

Assuming a subdivision of B into a finite union of elements E, we specialize the variational principle to one such element

$$I_E = \int_E \left(\frac{1}{2}\langle \sigma, \mathbb{C}^{-1}.\sigma \rangle - \langle \sigma, \nabla_s u \rangle + f\, u \right)$$
$$\int_{\partial_E \setminus \partial_D B} \frac{1}{2} n\, \sigma (u - u^{ext}) + \int_{\partial_E \cap \partial_D B} n\, \sigma (u - \bar{u})$$

and obtain a global principle for the subdivision by summing up: $I_h = \sum_E I_E$. Now free variations are taken w.r.t. $u_h \in V_h \Leftrightarrow u_{h|_E} \in H^1(E)$ and σ_h correspondingly. This gives a set of discrete equations.

Discrete primal problem

Some algebraic manipulations lead finally to the weak discrete problem: find $u_h \in V_h$ with

$$a_h(u_h, v_h) = \int_B f \cdot v_h \quad \forall\, v_h \in V_h$$

where the bilinear form is

$$a_h(u,v) = \sum_E \int_E (\langle \nabla_s v_h, \mathbb{C}.\nabla_s u_h \rangle + \langle \nabla_s v_h, \mathbb{C}.R([\![u_h]\!]) \rangle$$

$$+\langle R([\![v_h]\!]), \mathbb{C}.\nabla_s u_h \rangle) + \beta \sum_{e \in \partial E} \int_B \langle r_e([\![u_h]\!]), \mathbb{C}.r_e([\![v_h]\!]) \rangle$$

Here, R and r_e represent contribution of the possible jump-discontinuities. Question: How good is this method?

Discrete primal problem in variational form

Switch to corresponding variational problem: minimize

$$a_h(u_h, u_h) - \int_B f \cdot u_h$$

in the discrete space V_h with jumps. Task: show coercivity and continuity of a_h over V_h.

Mesh dependent norms

For $h > 0$ we consider two relevant mesh-dependent semi-norms on $\hat{V} = H_0^1(B) + V_h$ given by

$$\|v\|_S^2 = \sum_{E \in \mathcal{T}_h} \|\nabla^s v\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|\mathbf{r}_e([\![v]\!])\|_{0,B}^2 \qquad v \in H_0^1(B) + V_h$$

$$\|v\|^2 = \sum_{E \in \mathcal{T}_h} \|\nabla v\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|\mathbf{r}_e([\![v]\!])\|_{0,B}^2 \qquad \mathbf{v} \in H_0^1(B) + V_h$$

That $\|u\|$ is a norm on $\hat{V}$ is clear. What about $\|u\|_S$? Problem is absent for Poisson-equation (Laplace Operator harmless)!

Discrete norm

Consider $\|u_h\|^2 \approx \sum \|\nabla u_h\|_E^2 + [\![u_h]\!]_e^2$. This suggests that $\|u\|$ is a norm on $\hat{V}$.
Let $v_h \in \hat{\mathbf{V}} = H_0^1(B) + \mathbf{V}_h$. Then $\|\cdot\|_s : \hat{\mathbf{V}} \to \mathbb{R}$ defined as

$$\|v\|_s^2 = \sum_{E \in \mathcal{T}_h} \|\nabla^s v\|_{0,E}^2 + \sum_{e \in \mathcal{E}_h} \|\mathbf{r}_e([\![v]\!])\|_{0,e}^2 \qquad v \in \hat{\mathbf{V}}$$

is a norm on $\hat{V}$. **Proof.** $\|v\|_s = 0$ implies that $[\![v]\!] = 0$. Then apply Korn's first inequality on $H_0^1(B)$ which shows $v = 0$. ∎

Continuity and coercivity of the bilinear form

Let $\hat{\mathbf{V}} = H^1_0(B) + \mathbf{V}_h$. Then

- $a_h(u_h, v_h) \leq C |||u_h|||_S \cdot |||v_h|||_S, \quad \forall\, u_h, v_h \in \hat{\mathbf{V}}$
- $a_h(u_h, u_h) \geq c^+ \, |||u_h|||_S^2, \quad \forall\, u_h \in \hat{\mathbf{V}}$

Proof. Estimates involving the jumps and trace inequalities. ■

Hence unique solutions u_h of the discrete primal problem exist!

Mesh dependent estimates

Let u be the exact solution of the linear elasticity problem with $u \in H^m(B)$ for $2 \leq m$ and let u_h be the solution of the dicretized primal problem. The coercivity of the bilinear form together with consistency and Galerkin orthogonality implies, along the standard arguments, that

- $\|u - u_h\|_S \leq C\, h^{m-1}\, \|u\|_{m,B}$, mesh-dependent estimate.
- $\|u - u_h\|_{0,B} \leq C\, h^m\, \|u\|_{m,B}$, mesh-independent L^2- estimate.

These estimates are the state of the art for discontinuous Galerkin methods.

What is the real value of these estimates?

Korn's first inequality

Theorem 0.1. [Korn's first inequality] *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and let $u : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}^d$. Then*

$$\exists\, c^+ > 0 \quad \forall\, u \in H^{1,2}_\circ(\Omega) \quad \|\nabla u^T + \nabla u\|^2_\Omega \geq c^+ \, \|u\|^2_{H^{1,2}(\Omega)}$$

where the constant $c^+ > 0$ depends only on the geometry of the given domain Ω and the boundary.

Proof. Standard inequality in linear elasticity, see e.g., Ciarlet. A generalization to nonconstant coefficients has been proved by the author with applications in finite plasticity. ∎

Korn's second inequality

Theorem 0.2. [Korn's second inequality] *Assume that* $\Omega \subset \mathbb{R}^d$ *is a bounded domain with Lipschitz boundary* $\partial\Omega$ *and let* $u : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}^d$. *Then*

$$\exists\, c^+ > 0 \quad \forall\, u \in H^{1,2}(\Omega) \quad \|\nabla u^T + \nabla u\|_\Omega^2 + \|u\|_{2,\Omega}^2 \geq c^+ \, \|u\|_{H^{1,2}(\Omega)}^2$$

where the constant $c^+ > 0$ *depends only on the geometry of the given domain* Ω *and the boundary.*

Proof. Standard inequality in linear elasticity. ■

Scaled Korn's second inequality

Theorem 0.3. [Homogeneous Korn's second inequality] *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $|\Omega| \geq 1$. Let $u : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}^d$. Let Ω_h be the scaled domain $h \cdot \Omega$. Then*

$$\exists\, c^+ > 0 \quad \forall\, u \in H^{1,2}(\Omega_h)$$

$$\|\nabla u^T + \nabla u\|^2_{\Omega_h} + \frac{1}{|\Omega_h|^{\frac{2}{d}}} \|u\|^2_{2,\Omega} \geq c^+(\Omega) \left(\|\nabla u\|^2_{\Omega_h} + \frac{1}{|\Omega_h|^{\frac{2}{d}}} \|u\|^2_{2,\Omega_h} \right)$$

where the constant $c^+(\Omega) > 0$ is independent of $h > 0$.

Proof. Consider the scaling $u \mapsto u(h \cdot x)$. ■

The space of bounded variations BV

The displacement-gradient is a bounded measure allowing for jumps in the displacements along internal surfaces:

$$\|u\|_{BV(B)} := \|u\|_{L^1(B)} + \|Du\|(B)$$

$$\|Du\|(B) = \sup \left\{ \int_B \langle u, \operatorname{Div} \Psi \rangle \, dx \mid \Psi \in C_0^1(B, \mathbb{M}^{d\times d}), \ \|\Psi\|_\infty \leq 1 \right\}.$$

Theorem 0.4. [Poincar$\acute{e}$ for BV] *For $u \in BV(\Omega)$ with generalized trace $u_{|\partial\Omega} = 0$*

$$\|u\|_{L^1(\Omega)} \leq C^+ \cdot \|Du\|(\Omega)$$

The space of bounded deformations BD

The symmetrized displacement gradient is a bounded measure:

$$\|u\|_{BD(B)} := \|u\|_{L^1(B)} + \|\mathcal{E}(Du)\|(B),\ \mathcal{E}(Du) = \frac{1}{2}\left(Du^T + Du\right)$$

$$\|\mathcal{E}(Du)\|(B) = \sup\left\{\int_B \langle u, \mathrm{Div}[\Psi^T + \Psi]\rangle\, dx \mid \Psi \in C_0^1(B, \mathbb{M}^{d\times d}),\ \|\Psi\|_{\infty}\right.$$

Theorem 0.5. [Poincaré for BD] *For* $u \in BD(\Omega)$ *with generalized trace* $u_{|\partial\Omega} = 0$

$$\|u\|_{L^1(\Omega)} \leq C^+ \cdot \|\mathcal{E}(Du)\|(\Omega)$$

Natural estimates

All constants independent of $h > 0$: it can be shown that

- $\|u\|_{BD} \leq C\|u\|_{BV}$, well known.
- $\|u_h\|_S \leq C\|\|u_h\|\|, \quad \forall u_h \in \hat{V}$, corresponds to the former for the discretization.
- $\|u_h\|_{BV} \leq C\|\|u_h\|\|, \quad \forall u_h \in \hat{V}$, use Hoelders inequality and properties of the subdivision.
- $\|u_h\|_{BD} \leq C\|u_h\|_S, \quad \forall u_h \in \hat{V}$, use Hoelders inequality and properties of the subdivision.

Natural mesh independent BD-estimate

As a consequence of the already established convergence of the method we get with the last inequality

$$\|u - u_h\|_{BD} \leq C|||u - u_h|||_S \leq Ch^{m-1}\,\|u\|_{m,B}$$

a mesh independent estimate of the error in BD. Can we hope for more? In any case, BD is strictly an artefact of the linearization procedure!

Loss of rigidity

By introducing the possibility of internal jumps, rigidity is lost (internal elements are free to rotate against each other) and as consequence a replacement of Korn's first inequality on the BV/BD-level does not exist:

- $\|u\|_{BV} \leq C\|u\|_{BD}$, is false.
- $\|u_h\| \leq C\|u_h\|_S$ is very probably false!
- $\|u_h\| \leq C(h)\|u_h\|_S$ true at a constant $h > 0$, but $h \to 0$?

The same is true for the respective discrete realizations of BV/BD, i.e V_h

Convergence of the discretization

However, convergence in the symmetric triple norm and L^2 convergence at specific rates already imply convergence in the simple triple norm.

Theorem 0.6. [Convergence in simple triple norm] *Assume there is sequence $v_h \in V_h$ given such that $\|v_h\|_S \leq Ch^{m-1}$ and $\|v_h\|_{0,B} \leq C\ h^m$ for $h \to 0$. Then*

$$|||v_h||| \leq C^+\ h^{m-1}$$

Proof. Use the scaled version of Korn's second inequality on each element and shrink the elements uniformly (minimal aspect ratio for a subdivision- assume regular subdivision). ■

Observe that this is no general estimate!

Optimal mesh independent BV-estimate

Let u be the exact solution of the linear elasticity problem with $u \in H^m(B)$ for $2 \leq m$ and let u_h be the solution of the dicretized problem. Then we already had

- $|||u - u_h|||_S \leq C\, h^{m-1}\, \|u\|_{m,B}$
- $\|u - u_h\|_{0,B} \leq C\, h^m\, \|u\|_{m,B}$,

which implies by the former general result

$$\|u - u_h\|_{BV} \leq C\, |||u - u_h||| \leq C^+\, h^{m-1} \|u\|_{m,B}$$

Optimal in the sence that discrete solution might have jumps but classical solution is smooth.

Summary

- Derivation of a discontinuous Galerkin method in linear elasticity.
- Based on local Hellinger-Reissner principle.
- Derivation of standard mesh-dependent error estimates.
- Improvement to mesh independent error estimates.
- Optimal BV-estimates.
- Decisive: Korn's second inequality on the element level.
- Results true for general isoparametric families of triangulations.

Outlook

- Implementation
- Application to infinitesimal plasticity: prominent role of BD-spaces, slip lines and shear bands etc.
- Model fracture
- Transfer the methods to finite elasticity.