A Novel Dimensionally Reduced Finite Viscoelastic-Plastic Shell Theory.

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Plan of the talk

- General Framework of Finite Plasticity
- A Model with Grain Boundary Relaxation
- Dimensional Reduction for Thin Elastic Plates
- Thin Plastic Plates and Thin Elastic Shells
- Further Reduction: Thin Elastic Beam
- The Completely Relaxed Thin Elastic Beam

Notation 3D

- $\varphi : \Omega \subseteq \mathbb{R}^3 \mapsto \mathbb{R}^3$, the deformation, Ω reference configuration.
- $F = \nabla\varphi$, the deformation gradient, $1\!\!1$ the identity.
- $F = F_e \cdot F_p$, multiplicative decomposition.
- W elastic free energy density, $D_F W$ first P. K. stresses.
- Σ_E elastic Eshelby tensor.
- $\|A\|$, $\langle A, B\rangle$, $\operatorname{tr}[A]$ norm, scalar product and trace.
- $F = R\,U = \operatorname{polar}(F)\,U, \quad R \in \mathrm{SO}(3)$

A general framework in finite plasticity

A large class of materials, not only metals, can be modelled in the framework of

$$\int_\Omega W(F_e)\,\det[F_p]\,dx \mapsto \min. \quad \text{w.r.t. } \varphi \text{ at given } F_p.$$

$$F_e = \nabla\varphi\, F_p^{-1}$$

$$\Sigma_E = F_e^T D_{F_e} W(F_e) \det[F_p] - W(F_e)\,\det[F_p]\,1\!\!1$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\,[F_p^{-1}] \in -F_p^{-1}\cdot f\,(\Sigma_E)$$

Properties of the system

- thermodynamically consistent for proper choice of f.
- for $f = \partial X$ associated plasticity.
- elastic energy density $W(F_e)$ defined per unit volume of intermediate configuration set by F_p.
- Eshelby tensor Σ_E driving force for inhomogeneities in plasticity.
- intermediate configuration unambiguously given up to a global constant rotation.
- allows for $\det[F_p] \neq 1$ in general (e.g. soil mechanics).

A model for small elastic strains with grain boundary relaxation

$$\int_\Omega W(F_e, R_e)\,\det[F_p]\,dx \mapsto \text{min.w.r.t. } \varphi \text{ at given } R_e, F_p.$$

$$F_e = \nabla\varphi\, F_p^{-1}$$

$$\Sigma_E = F_e^T D_{F_e} W(F_e, R_e) \det[F_p] - W(F_e, R_e)\det[F_p]\, 1\!\!1$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] \in -F_p^{-1}\cdot f(\Sigma_E)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R_e = \frac{1}{\eta_e}\cdot \mathrm{skew}(F_e R_e^T)\cdot R_e, \quad R_e \in \mathrm{SO}(3),\ [\eta_e] = \mathrm{sec.}$$

$$W(F_e, R_e) = \tfrac{\mu}{4}\,\|F_e^T R_e + R_e^T F_e - 2 1\!\!1\|^2 + \tfrac{\lambda}{8}\,\mathrm{tr}\left[F_e^T R_e + R_e^T F_e - 2 1\!\!1\right]^2.$$

Physical motivation

...As long as the inelastic deformation is concentrated within the crystal boundaries which restrain the crystalline domains from deforming by slip, the metal responds essentially like a relatively simple viscoelastic material, in which the response of the crystals is elastic. ... The material then responds essentially as a **viscoelastic body** which deforms by **creep** of the **boundary material** and by relative motion and **rotation** of the **grains** along the boundaries.

Viscous creep of the polycrystalline metal aggregate associated with rotation of the grains along the grain boundaries is thus characteristic for the initial stage of creep tests

Classical Textbook: Freudenthal: The inelastic behaviour of engineering materials and structures, 1950)

Properties of the model with grain boundary relaxation

- consistent Eshelby structure, frame-indifferent, finite rotations.
- viscoelastic effect in the elastic domain, i.e., for arbitrary small loads. Internal friction with source at the grain boundary.
- linear balance equations at given inelastic configuration R_e, F_p.
- R_e describes rotation of a substructure: extended balance of angular momentum!
- intrinsically dissipative: keeping R_e constant does not yield a frame indifferent model in contrast to keeping F_p constant.

Notation 2D

- $M \subset \mathbb{R}^2$ midsurface of the plate, $h > 0$ thickness of the plate.
- $\Omega_h := M \times [-\frac{h}{2}, \frac{h}{2}]$ thin planar reference configuration of the plate.
- $m : M \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, deformation of the midsurface.
- $a, b, c \in \mathbb{R}^3,\ (a|b|c)$ matrix with columns a, b, c.
- $\varphi_s(x, y, z)$ reconstructed deformation.
- $F_s = \nabla\varphi_s(x, y, z)$ reconstructed deformation gradient.

Small elastic strains for thin structures

- Smooth deformation of thin structure $\varphi : \Omega_h \mapsto \mathbb{R}^3$, s. that $\nabla\varphi \in \mathrm{GL}(3,\mathbb{R})$.
- Small elastic strains: $\|\nabla\varphi^T\nabla\varphi - 1\!\!1\|_\infty \approx$ small.
- $\nabla\varphi \approx \mathrm{polar}(\nabla\varphi) = R(x,y,z) \in \mathrm{SO}(3)$. Taylor expansion $\Rightarrow$
- $\varphi((x,y,z) = \varphi(x,y,0) + z\,\nabla\varphi(x,y,0).e_3 + o(z), \quad z \in [-\frac{h}{2},\frac{h}{2}]$
- $\varphi((x,y,z) \approx \varphi(x,y,0) + z\,R(x,y,0).e_3 \quad z \in [-\frac{h}{2},\frac{h}{2}]$

Kinematical ansatz for the plate

$$\varphi_s(x,y,z) = m(x,y) + z\,R(x,y).e_3, \qquad R : M \mapsto \mathrm{SO}(3)$$

$$F_s := \nabla\varphi_s(x,y,z) = \begin{pmatrix} m_{1,x} & m_{1,y} & R_{13} \\ m_{2,x} & m_{2,y} & R_{23} \\ m_{3,x} & m_{3,y} & R_{33} \end{pmatrix} + z \begin{pmatrix} R_{13,x} & R_{13,y} & 0 \\ R_{23,x} & R_{23,y} & 0 \\ R_{33,x} & R_{33,y} & 0 \end{pmatrix}$$

$$= (\nabla m | R_3) + z \begin{pmatrix} R_{13,x} & R_{13,y} & 0 \\ R_{23,x} & R_{23,y} & 0 \\ R_{33,x} & R_{33,y} & 0 \end{pmatrix}$$

$$= A_m + z\,A_r$$

Reduction of the energy

- 3D: $W(F,R) = \frac{1}{4}\|F^T R + R^T F - 2\mathbb{1}\|^2$.
- Insert F_s instead of F and $R(x,y)$ instead of R
- $W(F_s,R) = \frac{1}{4}\|A_m^T R + R^T A_m - 2\mathbb{1}\|^2 + z\,\langle A_m^T R + R^T A_m - 2\mathbb{1}, A_r^T R\rangle$
  $+ \frac{z^2}{4}\|A_r^T R + R^T A_r\|^2.$
- Integrate over referential domain $\Omega_h = M \times [-\frac{h}{2}, \frac{h}{2}]$.
- $\int_M \int_{-\frac{h}{2}}^{\frac{h}{2}} W(F_s,R)\, d(x,y,z) = \int_M h\, \frac{1}{4}\|A_m^T R + R^T A_m - 2\mathbb{1}\|^2\, dM +$
  $0 + \int_M \frac{h^3}{12}\, \frac{1}{4}\|A_r^T R + R^T A_r\|^2\, dM$

Elastic equilibrium

- 3D: $\int_{\Omega_h} W(F,R)\,dx \mapsto \min$ w.r.t. φ at given R.
- 2D: minimize w.r.t. deformation of the midsurface m. Only contribution is membrane term

$$\frac{h}{4}\int_M \|(\nabla m|R_3)^T R + R^T(\nabla m|R_3) - 2\mathbb{1}\|^2\, dM$$

- Quadratic in ∇m, uniformly Legendre-Hadamard elliptic at given $R(x,y) \in \mathrm{SO}(3)$. Linear balance equation in m.
- Kirchhoff-Love and Reissner-Mindlin: membrane limit degenerated, non-elliptic.

Reduction of the evolution equation: Resultant rotations

- 3D: $\frac{\mathrm{d}}{\mathrm{dt}}R(x,y,z) = \frac{1}{\eta_e}\,\mathrm{skew}(FR^T)\,R$.
- Complete reconstruction: $\frac{\mathrm{d}}{\mathrm{dt}}R(x,y,z) = \frac{1}{\eta_e}\,\mathrm{skew}(F_sR^T)\,R$.
- $\frac{\mathrm{d}}{\mathrm{dt}}R(x,y,z) = \frac{1}{\eta_e}\,\mathrm{skew}\left(\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}(A_m + z\,A_r)\,R^T(x,y,z)\,dz\right)\,R(x,y,z)$.
- Assume $R(x,y,z) = R(x,y,0)$ uniformly over thickness.
- $\frac{\mathrm{d}}{\mathrm{dt}}R(x,y,0) = \frac{1}{\eta_e}\,\mathrm{skew}\left((\nabla m|R_3(x,y))\,R^T(x,y,0)\right)\,R(x,y,0)$.
- $\frac{\mathrm{d}}{\mathrm{dt}}R(x,y) = \frac{1}{\eta_e}\,\mathrm{skew}\left((\nabla m|R_3(x,y))\,R^T(x,y)\right)\,R(x,y)$.

The thin elastic plate

Find $m : [0,T] \times \overline{M} \mapsto \mathbb{R}^3$ and rotation $R : [0,T] \times \overline{M} \mapsto \mathrm{SO}(3)$

$$\int_M hW(F,R) - \langle h\,\hat{f} + \bar{\mathrm{N}}, m\rangle\, dM - \int_{\partial M} h\langle \hat{\mathrm{N}}, m\rangle\, dS \mapsto \min . \text{ w.r.t. } m \text{ at } R$$

$$F = (\nabla m | R_3) = A_m$$

$$W(F,R) = \frac{\mu}{4}\,\|F^T R + R^T F - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\,\mathrm{tr}\left[F^T R + R^T F - 2\mathbb{1}\right]^2$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R(t) = \frac{1}{\eta_e} \cdot \mathrm{skew}\left(F R^T\right) \cdot R(t)$$

The thin elastic-plastic plate

Find $m : [0,T] \times \overline{M} \mapsto \mathbb{R}^3$ and elastic rotation $R_e : [0,T] \times \overline{M} \mapsto \mathrm{SO}(3)$ and $F_p : [0,T] \times \overline{M} \mapsto \mathrm{GL}(3)$

$$\int_M h\, W(F_e, R_e)\, \det[F_p] -$$

$$\langle h\, \hat{f} + \bar{N}, m\rangle \det[F_p]\, dM \mapsto \min . \text{ w.r.t. } m \text{ at } R_e, F_p$$

$$F_e = (\nabla m | (R_e P)_3) \cdot F_p^{-1}, \quad P = F_p$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \left[F_p^{-1}\right](t) \in -F_p^{-1}(t) \cdot \partial\chi(\Sigma_E)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R_e(t) = \frac{1}{\eta_e} \cdot \mathrm{skew}\left(F_e R_e^T\right) \cdot R_e(t)$$

Reduction for the shell

- $\Theta : \Omega_h \mapsto \Omega_\xi$ smooth diffeomorphism.
- Ω_ξ curvilinear, stress free reference configuration of the shell.
- $\varphi_\xi : \Omega_\xi \mapsto \mathbb{R}^3$ finite deformation of the shell.
- $m(x, y) = m_\xi(\Theta(x, y, 0))$ deformation of the transformed midsurface. Strain $E_\xi = \frac{1}{2}\Big((\nabla_\xi \varphi_\xi)^T R_\xi + R_\xi^T (\nabla_\xi \varphi_\xi) - 2 1\!\!1\Big)$.
- $\varphi_\xi(\Theta(x, y, z)) = m_\xi(\Theta(x, y, 0)) + z\ \nabla_\xi \varphi_\xi(\Theta(x, y, 0)) \nabla_x \Theta(x, y, 0).e_3 + o(z)$
- $\varphi_\xi(\Theta(x, y, z)) \approx m_\xi(\Theta(x, y, 0)) + z\ \nabla_\xi R_\xi \cdot \nabla_x \Theta(x, y, 0).e_3$

The thin elastic shell

Find $m : [0,T] \times \overline{M} \mapsto \mathbb{R}^3$ and $R_e : [0,T] \times \overline{M} \mapsto \mathrm{SO}(3)$

$$\int_M h\, W(F_e, R_e) \det[P] - \langle h\, \hat{f} + \bar{N}, m \rangle \det[P]\, dM \mapsto \min . \text{ w.r.t. } m \text{ at fixed } R_e$$

$$F_e = (\nabla m | (R_e P)_3) \cdot P^{-1}, \quad P = \nabla \Theta$$

$$W(F_e, R_e) = \frac{\mu}{4} \| F_e^T R_e + R_e^T F_e - 2 1\!\!1 \|^2 + \frac{\lambda}{8} \operatorname{tr} \left[F_e^T R_e + R_e^T F_e - 2 1\!\!1 \right]^2$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R_e(t) = \frac{1}{\eta_e} \cdot \operatorname{skew} \left(F_e R_e^T \right) \cdot R_e(t)$$

Frame indifference and rotation of the intermediate configuration.

- 3D: $F \mapsto QF$ implies $R \mapsto QR$. Check invariance of stresses under $(F,R) \mapsto (QF, QR)$, i.e., $QS_1(F,R) = S_1(QF,QR)$.
- Corotated rate of rotation: $\frac{\mathrm{d}}{\mathrm{dt}_\omega} R := \frac{\mathrm{d}}{\mathrm{dt}} R - \omega R,\ \omega = \dot{Q}Q^T$.
- 2D: Check $(m, R) \mapsto (Q.m, QR)$.
- 3D: $F = F_e F_p = F_e Q^T Q F_p$. Shell: $F_p = \nabla\Theta$, rotation of Ω_ξ
  Check $(F_e, R_e, F_p) \mapsto (F_e Q^T, R_e Q^T, QF_p)$.
- 2D: Check $(m, R_e, F_p) \mapsto (m, R_e Q^T, QF_p)$.

Further reduction: The thin planar elastic beam

- Constrain motion to $\{e_1, e_3\}$-plane.
- $m : [0, b] \mapsto \mathbb{R}^2$, deformation of the beam, Rotation $R(\alpha)$.
- Evolution equation for $R(\alpha) \Rightarrow$ Evolution equation for α.

$$\int_0^b \frac{2\mu+\lambda}{2}\left[m_1' \sin\alpha + m_3' \cos\alpha - 1\right]^2 + \frac{\mu}{2}\left[m_1' \cos\alpha - m_3' \sin\alpha\right]^2 dx \mapsto \min . \text{ at constant } \alpha$$

$$\frac{\mathrm{d}}{\mathrm{dt}}[\alpha(x,t)] = \frac{\nu_e^+}{2}\left[m_1'(x,t)\cdot\cos\alpha - m_3'(x,t)\cdot\sin\alpha\right], \quad \alpha(x,0) = \frac{\pi}{2},$$

The completely relaxed thin planar elastic beam

- Complete relaxation: $\frac{\mathrm{d}}{\mathrm{dt}}\alpha = 0.$
- Reconstructed rotation $\Rightarrow$ Normality $R(\alpha).e_3 = \vec{n}_m.$
- Actually solved

$$\int_0^b \frac{(2\mu+\lambda)}{2} \left[\|m'\| - 1\right]^2 \, dx \mapsto \mathrm{stat.} \quad \text{no claim of minimality!}$$

- Nonconvex double well problem.
- Dynamic relaxation of microstructure through rotations.
- Ljapunov analysis of the equilibrium limit.

- Complete elasto-plastic framework.
- elastic Shell = plastic Plate, with $F_p = \nabla\Theta$.
- Avoid: metric coefficients, Christoffel symbols, topology of the shell.
- Normality retained in the relaxation limit.
- Linear balance equations, nondegenerate membrane part!
- Locally wellposed by elliptic regularity and Korn for shells.
- Study nonconvex problems as relaxation limit via Ljapunov.
- Similar structure in rate-independent plasticity: nonconvex update.

The flow based approach

- Balance of linear momentum is side condition
- Solve elastic trial step at given R_e, F_p with extended Korn's inequality $\Rightarrow \varphi = \varphi(R_e, F_p)$ or $m = m(R_e, F_p)$
- Solve abstract ODE

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[F_p^{-1}\right] = -F_p^{-1} \cdot f\left(\Sigma_E(\varphi(R_e, F_p), R_e, F_p\right)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R_e = \frac{1}{\eta_e} \cdot \mathrm{skew}(F_e R_e^T) \cdot R_e$$

- Decisive: $\mathrm{Curl}(R_e F_p)$

3D-Local existence and uniqueness

- Displacement data $g \in C^1(\mathbb{R}^+, H^{5,2}(\Omega, \mathbb{R}^3))$.
- viscoelastic, viscoplastic.
- Time $T > 0$, unique solution

$$\varphi \in C([0,T], H^{5,2}(\Omega, \mathbb{R}^3)) \, ,$$

$$(F_p, R_e) \in C^1([0,T], H^{4,2}(\Omega, GL(3,\mathbb{R})), H^{4,2}(\Omega, SO(3))) \, .$$

2D-Local existence and uniqueness

- Displacement data $g \in C^1(\mathbb{R}^+, H^{3,2}(M, \mathbb{R}^3))$.
- viscoelastic, viscoplastic.
- Time $T > 0$, unique solution

$$m \in C([0,T], H^{3,2}(M, \mathbb{R}^3))\,,$$

$$(F_p, R_e) \in C^1([0,T], H^{2,2}(M, GL(3,\mathbb{R})), H^{2,2}(M, SO(3)))\;.$$

3D: Extended Korn's first inequality

Let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL^+(3,\mathbb{R}))$ be given with $\det[F_p(x)] \geq \mu^+ > 0$. Moreover suppose that for the **dislocation density** $\operatorname{Curl} F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H_0^1(\Omega)$$

$$\|\nabla\phi\, F_p^{-1}(x) + F_p^{-T}(x)\, \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Proof: [Neff'02], recent improvement: $F_p \in L^\infty, \quad \operatorname{Curl} F_p \in L^2(\Omega)$ and geometric condition on jumps of F_p suffice.

2D: Extended Korn's first inequality

Let $F_p, F_p^{-1} \in C^1(\overline{M}, GL^+(3,\mathbb{R}))$ be given with $\det[F_p(x)] \geq \mu^+ > 0$. Moreover suppose that for the **dislocation density** $\operatorname{Curl} F_p \in C^1(\overline{M}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H_0^1(M)$$

$$\|(\nabla\phi|0)\, F_p^{-1}(x) + F_p^{-T}(x)\, (\nabla\phi|0)^T\|^2_{L^2(M)} \geq c^+ \, \|\phi\|^2_{H^{1,2}(M)}\,.$$

Proof: Consequence of 3D, [Neff'02], recent improvement: $F_p \in L^\infty(M), \quad \operatorname{Curl} F_p \in L^2(M)$ suffice.

Examples for flow rules: Norton-Hoff

Associated, isochoric plasticity: $f = \partial \mathcal{X}$.

$$\partial \mathcal{X}(\Sigma_E) = \frac{1}{\eta}\left(1 + [\| \operatorname{dev}(\operatorname{sym} \Sigma_E)\| - \sigma_{\mathrm{y}}]_+^{r+1}\right)^m \times$$
$$[\| \operatorname{dev}(\operatorname{sym} \Sigma_E)\| - \sigma_{\mathrm{y}}]_+^r \frac{\operatorname{dev}(\operatorname{sym} \Sigma_E)}{\| \operatorname{dev}(\operatorname{sym} \Sigma_E)\|} .$$

$\langle f(\Sigma_E), \Sigma_e\rangle \geq 0$ premonotone type

The thermodynamically consistent 3D-model for small elastic strains with grain boundary relaxation

$$\int_\Omega W(F_e, R_e)\,\det[F_p]\,dx \mapsto \text{min.w.r.t. } \varphi \text{ at given } R_e, F_p.$$

$$F_e = \nabla\varphi\, F_p^{-1}$$

$$\Sigma_E = F_e^T D_{F_e} W(F_e, R_e) \det[F_p] - W(F_e, R_e)\,\det[F_p]\,1\!\!1$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\,[F_p^{-1}] \in -F_p^{-1} \cdot f\,(\Sigma_E)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R_e = \frac{1}{\eta_e} \cdot \mathrm{skew}(B)\, R_e$$

$$B = \left[\mu(2 \cdot 1\!\!1 - F_e R_e^T) + \lambda\,[3 - \langle F_e R_e^T, 1\!\!1\rangle]\right] \cdot F_e R_e^T$$

Reduction of the consistent evolution equation: Resultant rotations

- 3D: $\frac{\mathrm{d}}{\mathrm{dt}} R(x,y,z) = \frac{1}{\eta_e} \operatorname{skew}(B)\, R.$
- Complete reconstruction: $\frac{\mathrm{d}}{\mathrm{dt}} R(x,y,z) = \frac{1}{\eta_e} \operatorname{skew}(B_s)\, R.$
- $\frac{\mathrm{d}}{\mathrm{dt}} R(x,y,z) = \frac{1}{\eta_e} \operatorname{skew}\left(\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} B_s\, dz\right) R(x,y,z).$
- Assume $R(x,y,z) = R(x,y,0)$ uniformly over thickness.
- $\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} B_s\, dz = \mu(2\, A_m - A_m R^T A_m - \frac{h^2}{12} A_r R^T A_r) R^T$
  $+\, \lambda(3A_m - \langle A_m, R\rangle A_m - \frac{h^2}{12}\langle A_r, R\rangle A_r) R^T = B^{res}_{thermo}$

The moderately thin elastic plate

Find $m : [0, T] \times \overline{M} \mapsto \mathbb{R}^3$ and rotation $R : [0, T] \times \overline{M} \mapsto \mathrm{SO}(3)$

$$\int_M hW(F, R) - \langle h\,\hat{f} + \bar{\mathrm{N}}, m\rangle \, dM - \int_{\partial M} h\langle \hat{\mathrm{N}}, m\rangle \, dS \mapsto \min . \text{ w.r.t. } m \text{ at } R$$

$$F = (\nabla m | R_3) = A_m$$

$$W(F, R) = \frac{\mu}{4} \, \| F^T R + R^T F - 2 1\!\!1 \|^2 + \frac{\lambda}{8} \, \mathrm{tr} \left[F^T R + R^T F - 2 1\!\!1 \right]^2$$

$$\frac{\mathrm{d}}{\mathrm{dt}} R(t) = \frac{1}{\eta_e} \cdot \mathrm{skew}(B^{res}_{thermo}) \cdot R(t) \quad \text{first order PDE-system!}$$

Significance of $\mathrm{Curl}(R_eF_p)$?

- $\mathrm{Curl}(R_eF_p)$ not a priori bounded.
- (R_eF_p) determines the smoothness of the elastic moduli.
- Elasticity tensor deteriorates with $\mathrm{Curl}(R_eF_p) \mapsto \infty$.
- $\mathrm{Curl}(R_eF_p)$ variable of crack initiation?
- large viscosity η_e for small grain size, no internal friction for single crystals, [Ke'47], single crystals $\mapsto$ zero elastic viscosity limit!

Reduced, partially linearized structure

Assume $F = 1\!\!1 + \nabla u$ and $F_p = 1\!\!1 + p$ with ∇u and p small. The system may then be reduced to

$$\int_\Omega \tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle \left[1 + \mathrm{tr}\left[\varepsilon_p\right]\right] dx \mapsto \min . \quad \text{w.r.t. } u \text{ at given } \varepsilon_p\,,$$

$$\Psi(\varepsilon_e) = \tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle = \mu\,\|\varepsilon_e\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\varepsilon_e\right]^2, \quad T = \mathcal{D}.\varepsilon_e = \frac{\partial\Psi(\varepsilon_e)}{\partial\varepsilon}\,,$$

$$\varepsilon_e = \varepsilon - \varepsilon_p, \quad \varepsilon(\nabla u(x)) = \tfrac{1}{2}(\nabla u^T + \nabla u), \quad \varepsilon_p = \tfrac{1}{2}(p^T + p)\,,$$

$$\Psi^{thermo}(\varepsilon, \varepsilon_p) = \Psi(\varepsilon_e)\left[1 + \mathrm{tr}\left[\varepsilon_p\right]\right],$$

$$\dot{\varepsilon}_p(t) \in f(T_E)\,,$$

$$T_E = -\partial_{\varepsilon_p}\left[\Psi^{thermo}(\varepsilon, \varepsilon_p)\right] = T\left[1 + \mathrm{tr}\left[\varepsilon_p\right]\right] - \tfrac{1}{2}\langle \mathcal{D}.\varepsilon_e, \varepsilon_e\rangle\, 1\!\!1\,,$$

The partially linearized system

- T_E reduced Eshelby tensor.
- intrinsically thermodynamically admissible.
- correct of second order in ε, ε_p.
- linear balance equations at given ε_p.
- $\mathrm{tr}\,[\varepsilon_p] > 0$, plastic dilation, softening. $\mathrm{tr}\,[\varepsilon_p] < 0$, hardening.
- well-developed theory only for $\mathrm{tr}\,[\varepsilon_p] = 0$.
- Sometimes $[1 + \mathrm{tr}\,[\varepsilon_p]]$ not accounted for!