The main focus of my research is the **arithmetic of Bianchi modular forms**. These are automorphic forms of cohomological type for GL_2 over imaginary quadratic fields. As a generalization of classical modular forms, they provide an important testing ground for different aspects of the Langlands Program which are well established over $\mathbb{Q}$.

Let us fix an imaginary quadratic field K with ring of integers O_K. Below I discuss some of the problems on which I am working as well as some of my contributions.

1. **Computing Bianchi modular forms**

   While there are well established methods and comprehensive databases for classical modular forms (in both characteristic 0 and mod p), our knowledge is limited in the case of Bianchi modular forms. Computations of eigenvalue systems for weight 2 have been done in the late seventies by Grunewald et al. [EGM] and Cremona [C]. Up to date, there has been no systematic work that treats higher weights and mod p forms. It is a very important feature of the theory that, unlike in the case of classical modular forms, reductions of (characteristic 0) Bianchi modular forms do not give all the mod p Bianchi modular forms. This is due to the torsion in the cohomology of Bianchi groups which I will talk about in the next section.

   Contribution I wrote MAGMA programs that compute the spaces of Bianchi modular forms over Euclidean imaginary quadratic fields together with the Hecke action on them in *all characteristics and weights* ≥ 2. I made these programs available on my website for the use of other mathematicians and provided a detailed manuscript [Sen4] explaining the basics of Bianchi modular forms and the techniques I use to compute them. The idea of the algorithm is to study the first cohomology of the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$ with certain irreducible coefficient modules over K via finite presentations of $\mathrm{PSL}_2(\mathcal{O}_K)$.

2. **Connections with Galois Representations**

   For the classical modular forms, there are geometric constructions that attach to each cuspidal eigenform a strictly compatible system of p-adic Galois representations. Unfortunately in the case of Bianchi modular forms, the relevant global symmetric space (the hyperbolic three-space) has no complex structure and thus there seems to be no direct link to algebraic geometry to exploit. Using theta lifts to Siegel modular forms, Taylor et al. [T, HST, BH], under mild conditions, were able to attach compatible families of p-adic Galois representations to cuspidal Bianchi eigenforms, in accordance with the Langlands philosophy. It is important to note that their result is not constructive even in weight 2.

   Project: The Torsion Phenomena As we mentioned above, Bianchi modular forms can be captured in the cohomology of the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$ with certain irreducible coefficient modules over K. Unlike the classical case, when one considers the same coefficient modules over $\mathcal{O}_K$, the torsion in the cohomology can get very large. The nature and the role of the torsion part of cohomology is not known.

Here are some of the main questions.

(a) What is the nature of the torsion in the cohomology ? What is the growth rate with respect to the level or weight ?

(b) Do reductions of the eigenvalue systems belonging to the p-torsion eigenclasses have attached mod p Galois representations?

(c) For a given torsion eigenclass, does there exist a non-torsion eigenclass (possibly with different weight and level) such that their respective eigenvalue systems agree when reduced to mod p ?

I believe systematic computations can bring insight to these questions. I am extending my computer program to compute the cohomology with coefficient modules over $\mathcal{O}_K$. This involves implementing a Smith algorithm over $\mathcal{O}_K$. Once the program is finished, I will start systematically computing the torsion that occurs in different levels and weights. I describe in the next section a theoretical approach to understand the torsion. After calculating the eigenvalue systems attached to the torsion classes, I will compare them with the eigenvalues attached to non-torsion parts in higher levels and weights. The torsion phenomena occurs also in the context of $\mathrm{GL}_3/\mathbb{Q}$ as seen in the work of Ash [A]. The *Bridge Conjecture* (the name coined by Barry Mazur) says that the answer to question (c) is "yes" in the setting of $\mathrm{GL}_3/\mathbb{Q}$ and there is supporting evidence collected by Avner Ash and his collaborators. I am expecting an affirmative answer to question (c) in the setting of Bianchi modular forms as well. Note that an affirmative answer would imply an affirmative one for question (b) as well via Taylor et al.'s result. It will be much more challenging to conjecture where (level and weight) to find a mathching non-torsion eigenvalue system given a torsion eigenvalue system.

Project: Serre Type Correspondence It is natural to expect a correspondence (in the sense of Serre [S]) between cuspidal mod p Bianchi eigenforms and irreducible mod p Galois representations of imaginary quadratic fields. But the situation is already more complicated than the classical situation as it is not known whether every mod p eigenvalue system has an attached mod p Galois representation.

If there is a "weak" two-way correspondence between these two classes of objects, then the non-existence of certain type of objects in one class implies the non-existence of the corresponding type of objects in the other class. I have established such a result which generalizes classical results of Tate [Ta] and Serre. It provides supporting evidence to a weak correspondence.

Contribution See [Sen2] and [Sen1].

(a) For the fields $K = \mathbb{Q}(\sqrt{d})$ with $d = 6, 5, 3, 2, -1, -2, -3, -5, -6$, there is no irreducible continuous representation $G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$ that is unramified away from $\{2, \infty\}$.

(b) There are no level 1 mod 2 cuspidal Bianchi modular forms for the fields with $d = 1, -2, -3$.

The first result says in other words that there is no irreducible level 1 mod 2 Galois representations for the above mentioned fields. I have proved this result using algebraic number theory and class field theory following ideas of Tate. The second result was established in my thesis by my MAGMA programs.

For a "strong" correspondence, given an irreducible mod p Galois representation one needs to predict the set of weights where one can find matching mod p cuspidal Bianchi modular eigenforms. There are such formulations in the case of totally real fields by Buzzard-Jarvis-Diamond [BDJ] and by Ash et al. [ADP] in the $\mathrm{GL}_n/\mathbb{Q}$ case. Recently, Herzig [H]has brought a conceptual approach that treats both of these cases. I am planning as a long term project to adapt these approaches to the Bianchi modular forms case. As a short term project, I have started collecting numerical data to test the future conjecture with. I would like to note that Rebecca Torrey, a student of Diamond, is also working on this problem.

3. **Dimensions of Spaces of Bianchi Modular Forms**

There is no known closed formula that gives the dimension of Bianchi modular forms with fixed level and weight. Such a formula would pave the way for important progress in the theory. For example, we would be able to learn the truth of the conjecture of Grunewald et al. [FGT] that level 1 cuspidal Bianchi modular forms all come from classical modular forms via base change. See also the work of Calegari-Mazur [CM] for further implications.

Project In a joint project with Nansen Petrosyan and Seyfi Turkelli, I am working on deriving an explicit dimension formula from the spectral sequence of the equivarient cohomology of the action of $\mathrm{PSL}_2(\mathcal{O}_K)$ on a certain complex. This approach, which looks very promising so far, can be adapted to study the torsion in the cohomology of $\mathrm{PSL}_2(\mathcal{O}_K)$.

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