# Chapter 1

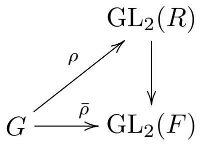
# Deformation theory of Galois representations

**The Big Picture.** Given a residual representation $\bar{\rho}$, we consider its *lifts*. We want to understand all lifts that enjoy certain properties of $\bar{\rho}$ that are important to us. If the properties of $\bar{\rho}$ are nice enough, there is a *universal lift* that "gives" all the other lifts of $\bar{\rho}$ satisfying these nice conditions. In this section, we prove that given an absolutely irreducible, semistable $\bar{\rho}$ and a finite set of primes $\Sigma$, there is a universal lift $\xi_\Sigma$ which *parametrizes* all lifts of $\bar{\rho}$ that are semistable at $\ell$ and that have no worse ramification at $p \notin \Sigma$ than $\bar{\rho}$ has.

A good description of this material can be found in [**?**]. The original source is [**?**].

Fix a profinite group $G$, a finite field $F$ with characteristic $\ell$ and a continuous representation $\bar{\rho} : G \to GL_n(F)$. Let $\mathbf{C}_F$ denote the category introduced in Chapter 2. Recall that the objects of $\mathbf{C}_F$ are complete, Noetherian local (commutative) rings $R$ with maximal ideal $\mathbf{m}_R$ and residue field $R/\mathbf{m}_R \cong F$. The morphisms of $\mathbf{C}_F$ are ring homomorphisms $\phi : R \to S$ such that $\phi(\mathbf{m}_R) \subseteq \mathbf{m}_S$ and such that the induced map $R/\mathbf{m}_R \to S/\mathbf{m}_S$ is the identity map on $F$. Cohen's theorem **??** says that every ring in $\mathbf{C}_F$ is a quotient ring of $W(F)[[T_1, ..., T_m]]$ for some $m$ and so is a $W(F)$-algebra where $W(F)$ is the ring of Witt vectors over $F$.

**Definition 1.1** *A lift of $\bar{\rho}$ to a ring $R$ in $\mathbf{C}_F$ is a continuous homomorphism $\rho : G \to GL_n(R)$ which modulo $\mathbf{m}_R$ produces $\bar{\rho}$. Two such lifts $\rho_1$ and $\rho_2$ are called strictly equivalent if there exists $M \in \Gamma_n(R) := ker(GL_n(R) \to GL_n(F))$ such that $\rho_2 = M^{-1}\rho_1 M$. A deformation of $\bar{\rho}$ is a strict equivalence class of lifts.*




Let $E(R)$ be the set of deformations of $\bar{\rho}$ to $R$. A morphism $R \to S$ induces a map of sets $E(R) \to E(S)$, making $E$ a functor $\mathbf{C}_F - - \to \mathit{Sets}$, which we call the *deformation functor*. We establish conditions under which $E$ is representable, in which case there is one representation parametrizing all lifts of $\bar{\rho}$ in a sense that will be made precise soon.

Say that $G$ satisfies the *finiteness condition* $\Phi_p$ if the maximal elementary $p$-abelian quotient (the $p$-Frattini quotient) of every open subgroup of $G$ is finite. Examples include:

(1) $G_{\mathbf{Q}_p}$,

(2) let $S$ be a finite set of primes of $\mathbf{Q}$ and $G_{\mathbf{Q},S}$ denote the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside $S$, i.e. the quotient of $G_{\mathbf{Q}}$ by the normal subgroup generated by the inertia subgroups $I_p$ for $p \notin S$.

Using class field theory, one can show that these groups satisfy $\Phi_p$ for any $p$ (see Exercise 1). Note also that by the Néron-Ogg-Shafarevich criterion, a Galois representation associated to an elliptic curve or a modular form factors through $G_{\mathbf{Q},S}$ for some $S$.

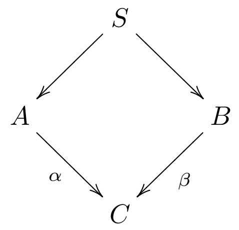
**Theorem 1.2** *(Mazur) Suppose that $\bar{\rho}$ is absolutely irreducible and that $G$ satisfies $\Phi_\ell$. Then the associated deformation functor $E$ is representable.*

This means that there is a ring $\Re(\bar{\rho})$ in $\mathbf{C}_F$ and a continuous homomorphism $\xi : G \to GL_n(\Re(\bar{\rho}))$ such that each lift of $\bar{\rho}$ up to $R$ in $\mathbf{C}_F$ arises, up to strict equivalence, via a unique morphism $\Re(\bar{\rho}) \to R$. The representing ring $\Re(\bar{\rho})$ is called the *universal deformation ring* of $\bar{\rho}$ and (the strict equivalence class of) $\xi$ the *universal deformation* of $\bar{\rho}$. For example, if $n = 2$, $\bar{\rho}$ is odd, $F = \mathbf{Z}/\ell$, and $G = G_{\mathbf{Q},S}$ with $\ell \in S$, then $\Re(\bar{\rho})$ is typically $\mathbf{Z}_\ell[[T_1, T_2, T_3]]$.

We will prove Mazur's theorem using Schlessinger's criteria [**?**]. In order to state Schlessinger's criteria, which we will not prove, let $\mathbf{C}_F^0$ denote the subcategory of the finite objects of $\mathbf{C}_F$. If $R$ is in $\mathbf{C}_F$, then every $R/\mathbf{m}_R^i$ is in $\mathbf{C}_F^0$ and $R$ is the inverse limit of $R/\mathbf{m}_R^i$. We say that $\mathbf{C}_F$ is a pro-category of $\mathbf{C}_F^0$. It can be shown that our functor $E$ is continuous, in the sense that it respects inverse limits, thus $E$ is determined by its values on $\mathbf{C}_F^0$.

A pivotal object in $\mathbf{C}_F^0$ is the *dual numbers* $F[\epsilon] = F[T]/(T^2)$ with $\epsilon$ the image of $T$. A morphism $R \to S$ is called *small* if it is surjective with the kernel a principal ideal whose product with $\mathbf{m}_R$ is 0, for example the map $\pi : F[\epsilon] \to F$ given by $a + b\epsilon \mapsto a$. Small maps are the building blocks in the sense that any surjective homomorphism in $\mathbf{C}_F^0$ can be written as the composition of small homomorphisms.

Let $A, B, C$ be objects in some category and $\alpha : A \to C$ and $\beta : B \to C$ be two morphisms in this category. If a terminal object exists in the subcategory of objects $S$ fitting into the following commutative diagram




then we call it the *fiber product* of $A$ and $B$ over $C$.

In the category of sets, the fiber product exists and is given by $A \times_C B = \{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$. In the category of rings, $A \times_C B$ has a natural ring structure and is the fiber product in this category. If $A, B, C$ are in $\mathbf{C}_F^0$, then $A \times_C B$ is in $\mathbf{C}_F^0$ as well and thus is the fiber product in $\mathbf{C}_F^0$.

We are ready to state Schlessinger's criteria.

**Definition 1.3** *Suppose $D : \mathbf{C}_F - - \to$ Sets is a functor satisfying $|D(F)| = 1$. Let the rings $R_0, R_1, R_2$ and the morphisms $R_1 \to R_0$ and $R_2 \to R_0$ be in $\mathbf{C}_F^0$. Consider the natural map*

$$(*)\ D(R_1 \times_{R_0} R_2) \to D(R_1) \times_{D(R_0)} D(R_2),$$

*which exists because $D$ is a functor and the right hand side is the fiber product in the category of sets. Schlessinger's criteria are as follows:*

**H1.** *$R_2 \to R_0$ small implies $(*)$ surjective.*

**H2.** *If $R_0 = F, R_2 = F[\epsilon]$, and $R_2 \to R_0$ is the map $\pi$ above, then $(*)$ is bijective.*

*Note: If* **H2** *holds then $t_E := E(F[\epsilon])$ has an $F$-vector space structure (see Exercise 2). It is called the tangent space of $E$.*

**H3.** *$t_D$ is a finite-dimensional $F$-vector space.*

**H4.** *If $R_1 = R_2$ and $R_i \to R_0$ $(i = 1, 2)$ are the same small map, then $(*)$ is bijective.*

**Theorem 1.4** *(Schlessinger)* **H1, H2, H3, H4** *hold if and only if $D$ is representable.*

Now we are ready to prove Mazur's result:

*Proof:* (proof of Theorem 1.2) We need to show that his functor $E$ satisfies **H1, H2, H3, H4**. Set $R_3 = R_1 \times_{R_0} R_2$. Let $E_i$ denote the set of continuous homomorphisms $G \to GL_n(R_i)$ lifting $\bar{\rho}$ and set $K_i = \Gamma_n(R_i)$. Since deformations are strict equivalence classes, we have $E(R_i) = E_i/K_i$. We are interested in the map

$$(*)\ E_3/K_3 \to E_1/K_1 \times_{E_0/K_0} E_2/K_2,$$

when $R_2 \to R_0$ is small.

To show $(*)$ is surjective, we take $\rho_1 \in E_1, \rho_2 \in E_2$ such that their images $\bar{\rho}_1$ and $\bar{\rho}_2$ in $E_0/K_0$ are the same, i.e. $\bar{\rho}_1 = M^{-1}\bar{\rho}_2 M$ for some $M \in K_0$. Since $R_2 \to R_0$ is surjective, so is $K_2 \to K_0$. If $N \in K_2$ maps to $M$, then $(\rho_1, N^{-1}\rho_2 N)$ gives the desired element of $E_3$. This proves **H1**.

Now assume that $R_0 = F, R_2 = F[\epsilon]$, and $R_2 \to R_0$ is the map $\pi$. Set $G_i(\rho_i) = C_{GL_n(R_i)}(\rho_i(G)) \cap K_i$ where $C$ is the centralizer subgroup. Let $\rho_2 \in E_2$ and let $\rho_0 \in E_0$ be its image. There is a natural map

$$(**) \quad G_2(\rho_2) \to G_0(\rho_0)$$

One can prove that $(*)$ is injective if $(**)$ is surjective for every $\rho_2$ (Exercise 3). Since $\overline{\rho}$ is absolutely irreducible, it follows by Schur's Lemma that $G_i(\rho_i)$ consists of scalar matrices. Then $(**)$ is surjective and thus $(*)$ is injective, ensuring **H4** holds. If $R_0 = F$, then $K_0 = \{1\}$ and so $(**)$ is surjective, whence **H2** follows.

For any lift $\rho : G \to GL_n(F[\epsilon])$ of $\bar{\rho}$, we have $\rho(ker(\bar{\rho})) = \Gamma_n(F[\epsilon]) \cap Im(\rho)$ and any two lifts agreeing on $ker(\bar{\rho})$ are identical. Note that $\Gamma_n(F[\epsilon])$ is isomorphic to the direct product of $n^2$ copies of $F^+$, so is an elementary abelian $\ell$-group. Thus by $\Phi_\ell$, there can only be finitely many maps from $ker(\bar{\rho})$ to $\Gamma_n(F[\epsilon])$ (see Exercise 1), proving **H3**. QED

We are interested in families of representations satisfying some further condition, such as semistability. To establish their representability, we need Ramakrishna's refinement of Mazur's result.

Let $X$ be a property of $W(F)[G]$-modules of finite cardinality which is closed under isomorphism, direct sums, taking submodules, and quotienting. Fix $\bar{\rho} : G \to GL_n(F)$ such that $F^n$ considered as a $W(F)[G]$-module via $\bar{\rho}$ satisfies $X$. For $R \in \mathbf{C}_F^0$ , let $E_X(R)$ denote the set of deformations in $E(R)$ satisfying $X$.

**Theorem 1.5** *(Ramakrishna) $E_X$ is a functor on $\mathbf{C}_F^0$. Moreover, if the deformation functor $E$ satisfies* **H1, H2, H3, H4***, then so does $E_X$ (in which case both functors are representable, where $E_X$ is extended to the category $\mathbf{C}_F$ by $E_X(R) = \varprojlim E_X(R/\mathbf{m}_R^i)$).*

*Proof:* Let $R, S$ be objects in $\mathbf{C}_F^0$ and $\phi : R \to S$ a morphism. To show $E_X$ is a functor, we need to show that if $\rho : G \to GL_n(R)$ satisfies $X$, then after composing with a $R \to S$ morphism , $GL_n(S)$ also satisfies $X$. If $B = R^n$ and $D = S^n$ are both with the given $G$-action, then $\phi$ induces $B \to D$ making $D$ a finitely generated (it is finite) $B$-module, say a quotient of $B^m$ - since satisfying $X$ is closed under direct product and quotient, $B$ and so $B^m$ and so $D$ all satisfy $X$.

The next thing to note is that **H1** for $E_X$ implies **H2,H3,H4** too. This follows since restrictions of injective maps to subsets are still injective. This gives injectivity in **H2** and **H4** with **H1** giving surjectivity. As for **H3**, since $E_X(F[\epsilon]) \subseteq E(F[\epsilon])$, the tangent space of $E_X$ is also finite-dimensional.

To prove **H1** for $E_X$, set $R_3 = R_1 \times_{R_0} R_2$. Let $\rho_1 \times_{\rho_0} \rho_2 \in E_X(R_1) \times_{E_X(R_0)} E_X(R_2)$. By **H1** for $E$, we get $\rho \in E(R_3)$ mapping to this element. We just need to show that $\rho$ has $X$. Well, $R_3 \hookrightarrow R_1 \times R_2$ induces $R_3^n \hookrightarrow R_1^n \times R_2^n$, making $R_3^n$ a submodule of a direct product of $W(F)[G]$-modules with $X$, whence $R_3^n$ with this $G$-action has $X$. QED

**Theorem 1.6** *Suppose that $E$ and $E_X$ satisfy the hypotheses of the previous theorem. Let $\Re$ and $\Re_X$ be the respective deformation rings. Then there is a natural surjection $\Re \to \Re_X$.*

*Proof:* Let $\xi : G \to GL_n(\Re)$ and $\xi_X : G \to GL_n(\Re_X)$ denote the universal deformations. Since $\xi_X$ is a lift of $\bar{\rho}$ and $\xi$ parametrizes all such, there is a (unique) morphism $\phi : \Re \to \Re_X$ which after composition with $\xi$ yields $\xi_X$. Let the image of $\phi$ be $S$. We thereby get a representation $\rho : G \to GL_n(S)$, which is of type $X$. By universality of $\Re_X$ we get a unique morphism $\Re_X \to S$ producing $\rho$. The composition $\Re_X \to S \hookrightarrow \Re_X$, by universality again, has to be the identity map, so $S = \Re_X$. QED

Let $\ell > 2$ and assume $\bar{\rho}$) is absolutely irreducible and semistable. Fix a finite set $\Sigma$ of rational primes. Let $R$ be in $\mathbf{C}_F^0$. If $\rho : G_{\mathbf{Q}} \to GL_2(R)$ is a lift of $\bar{\rho}$, we say that $\rho$ is of *type* $\Sigma$ if

1. det $\rho$ is the cyclotomic character;
2. $\rho$ is semistable at $\ell$;
3. (a) if $\ell \notin \Sigma$ and $\bar{\rho}$ is good at $\ell$, then $\rho$ is good at $\ell$;
   (b) if $p \notin \Sigma \cup \{\ell\}$ and $\bar{\rho}$ is unramified at $p$, then $\rho$ is unramified at $p$;
   (c) if $p \notin \Sigma \cup \{\ell\}$ and $\bar{\rho}$ is ramified (so ordinary) at $p$, then $\rho$ is ordinary at $p$.

The followings are immediate observations that follow from the above definition.

- If $E$ is a semistable elliptic curve over $\mathbf{Q}$ and $\rho_{E,\ell}$ is absolutely irreducible, then $\rho_{E,\ell^\infty}$ is a lift of type $\Sigma$ if $\Sigma$ contains all the primes of bad reduction for $E$.
- If $\rho$ is of type $\Sigma \subseteq \Sigma'$, then $\rho$ is of type $\Sigma'$.
- If $\rho$ is of type $\Sigma$, then $\rho$ is unramified outside $\{p : p | \ell N(\bar{\rho})\} \cup \Sigma$.
- A lift $\rho$ of $\bar{\rho}$, unramified outside $\Sigma$, with $\det(\rho)$ the cyclotomic character, semistable at $\ell$, is of type $\Sigma \cup \{\ell\}$.

- $\bar{\rho}$ is of type $\Sigma$ as semistability includes that det $\bar{\rho}$ be the cyclotomic character. (Note that this makes $\bar{\rho}$ odd.)

**Theorem 1.7** *Given a continuous absolutely irreducible semistable homomorphism $\bar{\rho} : G_{\mathbf{Q}} \to GL_2(F)$ and given $\Sigma$, there exists an object $\Re_\Sigma$ in $\mathbf{C}_F$ and a continuous homomorphism $\rho_\Sigma : G_{\mathbf{Q}} \to GL_2(\Re_\Sigma)$ such that every lift $\rho$ of $\bar{\rho}$ to $R$ in $\mathbf{C}_F$, of type $\Sigma$, is strictly equivalent to the composition of $\rho_\Sigma$ with some $\phi$, a unique morphism from $\Re_\Sigma$ to $R$.*

The proof proceeds by dealing with each of the conditions required for type $\Sigma$ in turn.

Let $G$ be a profinite group and $I$ a closed subgroup. Call $\rho : G \to GL_2(R)$ $I$-ordinary if the fixed points of $R^2$ under $I$ form a free direct summand of rank 1. Note that if the determinant of $\rho$ is the cyclotomic character and $\rho$ is $I_p$-ordinary, then $\rho$ is ordinary at $p$ in the sense of Definition REF. Let $E_I(R)$ denote the set of $I$-ordinary deformations of $\bar{\rho}$ to $R$. One can check that $E_I$ is a subfunctor of the deformation functor $E$.

**Theorem 1.8** *If $\bar{\rho}$ is $I$-ordinary and absolutely irreducible and $G$ satisfies $\Phi_\ell$, then $E_I$ is representable.*

*Proof:* As in the proof of Theorem 1.5, we only need to prove **H1**. Let $R_3 = R_1 \times_{R_0} R_2$ and consider $\rho_1 \times_{\rho_0} \rho_2 \in E_I(R_1) \times_{E_I(R_0)} E_I(R_2)$. Then since the deformation functor $E$ is representable, we get a $\rho_3 \in E(R_3)$ mapping to $\rho_1 \times_{\rho_0} \rho_2$. We leave it to the reader to show that $\rho_3$ is $I$-ordinary. QED

Let $p_1, ..., p_n$ be all the primes $p_i \notin \Sigma$ at which $\bar{\rho}$ is ramified at and let $I_{p_1}, ..., I_{p_n}$ be their inertia groups respectively. Let $E_I$ be the subfunctor of $E$ which gives all the lifts of $\bar{\rho}$ that are ordinary at every $I_{p_i}$. Then by the above theorem, then $E_I$ is representable.

Now we deal with the condition of being good at $\ell$ which involves group schemes. We consider the étale group scheme over $\mathbf{Q}_\ell$ corresponding to $G_{\mathbf{Q}_\ell} \to GL_n(R)$ with $R \in \mathbf{C}_F^0$ and say that $\rho$ has property $X$ if this group scheme extends to a finite flat group scheme over $\mathbf{Z}_\ell$. We just need to show that $X$ is preserved under direct sums, sub, and quotient. Then by Ramakrishna's result, the functor $E_{I,X}$ giving all lifts in $E_I$ that satisfy $X$ will be representable.

**Definition 1.9** *If $G$ is an affine group scheme over $\mathbf{Z}_\ell$, say represented by a Hopf algebra $A$, let $G_{gen}$ be its fibre (or base change) over $\mathbf{Q}_\ell$ represented by $A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Let $H_{gen}$ be a closed subgroup scheme of $G_{gen}$, say represented by $(A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell)/J$ where $J$ is a Hopf ideal (ideals that ensure the quotient is still a Hopf algebra). Let $I = \phi^{-1}(J)$ under $\phi : A \to A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Then the group scheme $H$ represented by $A/I$, a subgroup scheme of $G$, is called the schematic closure of $H_{gen}$. It is the smallest closed subscheme of $G$ with $H_{gen}$ as the generic fiber.*

The map $\phi$ induces an injection $A/I \hookrightarrow (A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell)/J$. As the latter is torsion-free ,since it is a $\mathbf{Q}_\ell$-algebra, the quotient $A/I$ s torsion-free as well.

Thus $H$ is a finite flat group scheme over $\mathbf{Z}_\ell$. This shows that goodness is preserved under sub.

As for quotient group schemes, let $H$ be a closed subgroup scheme of $G$ over $R$ with $H(A)$ a normal subgroup of $G(A)$ for all $R$-algebras $A$. Let $A$ be the Hopf algebra representing $G$ and $J$ be its Hopf ideal corresponding to $H$. Put $A' = \{a \in A | \mu(a) \equiv 1 \otimes a \pmod{I \otimes A})\}$ where $\mu$ is the comultiplication $\mu(a) = a \otimes 1$. Raynaud showed (see [?]) that the quotient $G/H$ is a finite flat group scheme over $R$ that is represented by $A'$.

Finally, as regards direct sums, if $G, H$ are finite flat group schemes over $\mathbf{Z}_\ell$ represented by $R, S$ respectively, let $F(A) = G(A) \times H(A)$ for every $\mathbf{Z}_\ell$-algebra $A$. Then we check that $F$ is an affine group scheme represented by $R \otimes_{\mathbf{Z}_\ell} S$, which is a finite flat $\mathbf{Z}_\ell$-algebra.

Lastly, we consider the condition that $\rho$ has determinant the cyclotomic character. Without imposing this condition, we so far have a universal deformation $\xi : G_{\mathbf{Q}} \to GL_2(\Re)$ satisfying all other conditions of being of type $\Sigma$. In particular, for $p \notin S = \{p : p \mid \ell N(\bar{\rho})\} \cup \Sigma$, $\xi$ is unramified at $p$. Thus, $\xi(Fr_p)$ (for $p \notin S$) are defined. Let $\det \xi(Fr_p) = r_p \in \Re$ and let $I$ be the ideal of $\Re$ generated by $r_p - p$ for $p \notin S$. The representation $\tilde{\xi} = \xi \pmod{I} : G_{\mathbf{Q}} \to GL_2(\Re/I)$ now has $\det \tilde{\xi}(Fr_p) = r_p = p = \chi(Fr_p)$ for $p \notin S$. By Chebotarev's density theorem, the $Fr_p(p \notin S)$ are dense in $G_{\mathbf{Q},S}$ and so $\det \tilde{\xi} = \chi$.

Imposing all these conditions has actually reduced us to a ring $\Re/I$ which is finite over $\mathbf{Z}_\ell$. This ring is the universal deformation ring for the functor $E_\Sigma$ giving all the deformations of type $\Sigma$. This finishes the proof of Theorem 1.7.

We will end our chapter with another description of the tangent space of our main deformation functor $E$. Suppose $E$ is represented by $\Re$. Recall that then $t_E = E(F[\epsilon])$, the *tangent space* of $E$, a finite-dimensional $F$-vector space. Its dimension will turn out to be a useful invariant of both $E$ and $\Re$, calculated via Galois cohomology.

As $E$ is representable, we have $t_E = \hom(\Re, F[\epsilon])$. By working out how the elements of the right hand side should look like, one shows (see Exercise 5) that $t_E$ is the dual space of $\mathbf{m}_\Re/(\mathbf{m}_\Re^2 + \ell\Re)$. For example, if $\Re = W(F)[[T_1, ..., T_r]]$, then $\mathbf{m}_\Re = (T_1, ..., T_r, \ell)$ and so $\dim_F t_E = \dim_F(\mathbf{m}_\Re/(\mathbf{m}_\Re^2 + \ell\Re)) = r$. Furthermore, if $I$ is an ideal of $\Re$ such that $I \subseteq \mathbf{m}_\Re^2 + \ell\Re$, then the tangent space of $\Re/I$ is also $r$-dimensional.

# Exercises

1. Fix a rational prime $p$. Let $S$ be a finite set of primes of $\mathbf{Q}$. We prove that $G_{\mathbf{Q},S}$ satisfies $\Phi_p$ as follows:

   (i) Prove that our claim amounts to saying that the set of continuous homomorphisms from $G_{\mathbf{Q},S}$ to $\mathbf{F}_p$ is finite.

(ii) Use the theorem of Hermite and Minkowski which says that there are only finitely many extensions $K/\mathbf{Q}$ of degree $p$ which are unramified outside $S$.

2. Prove that $\mathbf{C}_F^0$ is indeed the subcategory of Artinian rings in $\mathbf{C}_F$. When the residue field $F$ is not finite, this is how we define $\mathbf{C}_F^0$. Schlessinger's criteria work in this generality as well.

3. Verify that the tangent space $t_E$ of $E$ carries an $F$-vector space structure when Schlessinger's second criterion holds. *Hint*: Note that taking $R_1 = R_2 = F[\epsilon]$, the map $(*)$ gives a bijection between $E(F[\epsilon] \times_F F[\epsilon])$ and $E(F[\epsilon]) \times_{E(F)} E(F[\epsilon])$. The latter is actually just $E(F[\epsilon]) \times E(F[\epsilon])$.

4. Prove that in the proof of Theorem 7.2 the map $(*)$ is injective if $(**)$ is surjective.

5. For a representable functor $E$, verify that $t_E = \hom(\Re, F[\epsilon])$ is indeed isomorphic to $\mathrm{Hom}(\mathbf{m}_\Re/(\mathbf{m}_\Re^2 + \ell\Re), F)$ as $F$-vector spaces. *Hint*: Since morphisms in our category induce identity on residue fields, a homomorphism $\Re \to F[\epsilon]$ must have the form $r \mapsto \bar{r} + \phi(r)\epsilon$ where $\bar{r}$ is reduction modulo $\mathbf{m}_\Re$ and $\phi(r) \in F$. The map $\phi$ is completely determined by its values on $\mathbf{m}_\Re$. Now observe that $\mathbf{m}_\Re^2 + \ell\Re$ is the kernel of $\phi$.