

Bianchi Modular Forms

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Abstract

Bianchi modular forms are modular forms for GL_2 over imaginary quadratic fields. We discuss their theoretical background and present an algorithm to compute them. Finally, we explain an implementation in MAGMA.

1. Introduction

Modular forms (for $GL(2)$) over an imaginary quadratic field K with ring of integers \mathcal{O} can be seen as

- certain real analytic functions on the hyperbolic 3-space \mathcal{H} , or
- certain regular cuspidal automorphic representations of $GL_2(\mathbb{A}_K)$, or
- classes in the cohomology of quotients of \mathcal{H} by congruence subgroups of $SL_2(\mathcal{O})$

We follow the third approach in this note. As the groups $SL_2(\mathcal{O})$ are called Bianchi groups, we call these forms Bianchi modular forms.

The followings are mostly taken from my PhD thesis titled “Serre’s Conjecture Over Imaginary Quadratic Fields”. They provide details to the talk I gave at the workshop “Computing With Modular Forms”. Most of this work has been done with the guidance and help of Fritz Grunewald to whom I am grateful.

2. Theoretical Background

2.1. The Hyperbolic 3-Space

In this section, we will discuss the the hyperbolic 3-space and the action of $SL_2(\mathbb{C})$ on it. We refer the readers to the books by Elstrodt-Grunewald-Mennicke [6], Bearden [2], and Maclahlan-Reid [11] for proofs.

Three-dimensional hyperbolic space is the unique 3-dimensional connected and simply connected Riemannian manifold with constant sectional curvature equal to -1 . Let \mathbb{H} be the upper half-space of the three dimensional Euclidean space.

$$\begin{aligned}\mathbb{H} &:= \mathbb{C} \times \mathbb{R}^+ \\ &= \{(z, r) \mid z \in \mathbb{C}, r \in \mathbb{R}, r > 0\} \\ &= \{(x, y, r) \mid x, y, r \in \mathbb{R}, r > 0\}\end{aligned}$$

We endow \mathbb{H} with the hyperbolic metric d coming from the line element ds defined by

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$$

and \mathbb{H} becomes a model for the hyperbolic 3-space. With the metric d , the geodesics in \mathbb{H} are half circles or half lines which are orthogonal to the boundary plane \mathbb{C} in the Euclidean sense.

The group $\mathbf{SL}_2(\mathbb{C})$ of 2×2 complex matrices with determinant one act on \mathbb{H} . For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{C})$ and $(z, r) \in \mathbb{H}$, the action is given by $M \cdot (z, r) := (z^*, r^*)$ where

$$z^* = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2r^2},$$

$$r^* = \frac{r}{|cz + d|^2 + |c|^2r^2}.$$

This action can be given in a more compact way once quaternions are used to represent points in \mathbb{H} . Let $1, i, j, k$ be the standard \mathbb{R} -basis for Hamilton's quaternions \mathcal{H} . We regard \mathbb{H} as a subset of \mathcal{H} via $(z, r) \mapsto z + rj$. Then the action of $\mathbf{SL}_2(\mathbb{C})$ on a point $p = z + rj$ can be described as

$$M \cdot p = \frac{ap + b}{cp + d}.$$

The action can be extended to the boundary $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. If $(x : y)$ is an element of $\mathbb{P}^1(\mathbb{C})$, then

$$M \cdot (x : y) = (ax + by : cx + dy).$$

The action of $\mathbf{SL}_2(\mathbb{C})$ on \mathbb{H} is doubly transitive (any pair of elements can be sent to any pair of elements) and the stabilizer of the point $j = (0, 0, 1) \in \mathbb{H}$ inside $\mathbf{SL}_2(\mathbb{C})$ is $\mathbf{SU}(2)$. This implies that, with the fact that $-I_2$ acts trivially on \mathbb{H} ,

$$\mathbf{SL}_2(\mathbb{C})/\mathbf{SU}(2) \simeq \mathbf{PSL}_2(\mathbb{C})/\mathbf{PSU}(2) \simeq \mathbb{H}$$

In the language of Lie groups, this means that the global symmetric space of the groups \mathbf{SL}_2 and \mathbf{PSL}_2 is \mathbb{H} . We will discuss the situation from the perspective of automorphic forms in a later section.

We end this section by noting that $\mathbf{PSL}_2(\mathbb{C})$ is isomorphic to the group $\mathbf{Iso}^+(\mathbb{H})$ of orientation preserving isometries of \mathbb{H} .

2.2. Discrete Subgroups of $\mathbf{PSL}_2(\mathbb{C})$

We consider the topology on $\mathbf{SL}_2(\mathbb{C})$ that is given by the following norm

$$\|M\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A subgroup Γ of $\mathbf{PSL}_2(\mathbb{C})$ is *discrete* if its inverse image in $\mathbf{SL}_2(\mathbb{C}) \subset \mathbb{C}^{4 \times 1}$ is discrete. Discrete subgroups of $\mathbf{PSL}_2(\mathbb{C})$ are also known as *Kleinian groups*.

A subgroup $\Gamma < \mathbf{Iso}^+(\mathbb{H})$ is called *discontinuous* if for every compact subset $K \subset \mathbb{H}$ we have $g \cdot K \cap K = \emptyset$ for all but finitely many $g \in \Gamma$. The following is a result of Poincare.

Theorem 2.1. *A subgroup $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is discrete if and only if it is discontinuous.*

It follows from that for a point $P \in \mathbb{H}$, its stabilizer subgroup $\Gamma_P < \Gamma$ is finite if Γ is discrete. The classification of finite subgroups of $\mathbf{PSL}_2(\mathbb{C})$ tells us that Γ_P will be cyclic, dihedral, isomorphic to the alternating groups $\mathbf{A}_4, \mathbf{A}_5$ or the symmetric group \mathbf{S}_4 .

For the point ∞ of $\mathbb{P}^1(\mathbb{C})$, its stabilizer $\mathbf{PSL}_2(\mathbb{C})_\infty$ in $\mathbf{PSL}_2(\mathbb{C})$ is equal to the group

$$B(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid 0 \neq a, b \in \mathbb{C} \right\} / \{\pm I\}$$

which is a Borel subgroup. Given an element $\zeta \in \mathbb{P}^1(\mathbb{C})$ and a subgroup $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ then we have

$$\Gamma_\zeta = \Gamma \cap M^{-1}B(\mathbb{C})M$$

where $M \in \mathbf{PSL}_2(\mathbb{C})$ is such that $M\zeta = \infty$.

A closed subset $\mathcal{F} \subset \mathbb{H}$ is called a *fundamental domain* for $\Gamma < \mathbf{Iso}(\mathbb{H})$ if the followings are satisfied:

- (1) \mathcal{F} meets each Γ -orbit at least once,
- (2) the interior of \mathcal{F} meets each Γ -orbit at most once,
- (3) the boundary of \mathcal{F} has Lebesgue measure zero.

If Γ is discontinuous, then a fundamental domain \mathcal{F} exists. We say that Γ is of *finite covolume* or *cofinite* if

$$\text{vol}(\Gamma) = \int_{\mathcal{F}} dv < \infty$$

where $dv = \frac{dx dy dr}{r^3}$ is the hyperbolic volume element. The covolume $\text{vol}(\Gamma)$ is independent of the fundamental domain chosen. If Γ has a compact fundamental domain, then Γ is called *cocompact*.

If $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ is a discrete group which acts (fixed point) freely on \mathbb{H} then the quotient space $\Gamma \backslash \mathbb{H}$ inherits the structure of an orientable hyperbolic 3-manifold. Conversely, the universal cover of an orientable hyperbolic 3-manifold M will be isometric to \mathbb{H} and thus the fundamental group of M will be a covering group $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ that acts freely and is discrete.

Theorem 2.2. *If M is an orientable hyperbolic 3-manifold, then M is isometric to $\Gamma \backslash \mathbb{H}$ for some torsion-free discrete subgroup of $\mathbf{PSL}_2(\mathbb{C})$.*

It can be seen that Γ is cocompact if and only if $\Gamma \backslash \mathbb{H}$ is compact. Moreover, Γ is cofinite if and only if $\Gamma \backslash \mathbb{H}$ is of finite volume. This explains the prefix "co" in the above definitions.

The notion of a cusp is important for the sequel. An element $\zeta \in \mathbb{P}^1(\mathbb{C})$ is called a *cuspid* of a discrete group $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ if its stabilizer Γ_ζ contains a free abelian group of rank 2. We denote the set of cusps of Γ with C_Γ .

The following result provides a connection with algebraic number theory.

Theorem 2.3. *Let Γ be a discrete subgroup of $\mathbf{PSL}_2(\mathbb{C})$ which is cofinite. Then the field $\mathbb{Q}(\text{tr}\Gamma)$ is a finite extension of \mathbb{Q} .*

Here $\mathbb{Q}(\text{tr}\Gamma)$ is the field created by adjoining the traces of all preimages in $\mathbf{SL}_2(\mathbb{C})$ of elements of Γ .

2.3. Bianchi Groups

Let $d > 0$ be a square-free integer and let \mathcal{O}_d denote the ring of integers of the imaginary quadratic number field $K_d = \mathbb{Q}(\sqrt{-d})$. We will drop the d when we discuss situations that cover all d 's. The groups $\mathbf{PSL}_2(\mathcal{O}_d)$ are called *Bianchi groups*. Since \mathcal{O}_d is discrete in \mathbb{C} , Bianchi groups are discrete subgroups of $\mathbf{PSL}_2(\mathbb{C})$. Using a specific fundamental domain constructed by Bianchi, one sees that Bianchi groups are cofinite but are not cocompact.

Theorem 2.4. *Every discrete subgroup of $\mathbf{PSL}_2(\mathbb{C})$ that is not cocompact is commensurable with a conjugate of some Bianchi group.*

We first discuss the properties of cusps which is a first indicator of the strong connections with arithmetic.

Proposition 2.5. *Let $\Gamma < \mathbf{PSL}_2(\mathbb{C})$ be commensurable with a Bianchi group $\mathbf{PSL}_2(\mathcal{O}_d)$, then the set of cusps $C_\Gamma = \mathbb{P}^1(K_d) \subset \mathbb{P}^1(\mathbb{C})$.*

Theorem 2.6. *The “number of cusps” of $\mathbf{PSL}_2(\mathcal{O}_d)$ (that is, the number of $\mathbf{PSL}_2(\mathcal{O}_d)$ -orbits in $\mathbb{P}^1(K_d)$) is equal to the class number of K_d .*

It is effectively possible to write down a presentation for $\mathbf{PSL}_2(\mathcal{O}_d)$ from a fundamental domain. This was carried out by Bianchi, Humbert and Swan [16]. In this thesis, we will focus on the Euclidean imaginary quadratic fields K_1 and K_2 . We note the following presentations that Flöge [7] produced using a 2-dimensional $\mathbf{PSL}_2(\mathcal{O})$ -equivariant deformation retract of \mathbb{H} that is due to Mendoza [12].

$$\mathbf{PSL}_2(\mathcal{O}_1) = \left\langle A, B, U \mid \begin{array}{l} (AB)^3 = B^2 = AUA^{-1}U^{-1} = (BUBU^{-1})^3 = \\ (BU^2BU^{-1})^2 = (AUBAU^{-1}B)^2 = 1 \end{array} \right\rangle$$

where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$. Here we represent elements of $\mathbf{PSL}(\mathbb{C})$ with preimages from $\mathbf{SL}_2(\mathbb{C})$.

$$\mathbf{PSL}_2(\mathcal{O}_2) = \left\langle A, B, U \mid (AB)^3 = B^2 = AUA^{-1}U^{-1} = (BU^2BU^{-1})^2 = 1 \right\rangle$$

where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ \sqrt{-2} & 1 \end{pmatrix}$.

We now define the congruence subgroups. Given an ideal $\mathfrak{a} \subset \mathcal{O}$, the principal congruence subgroup of level \mathfrak{a} is defined by

$$\Gamma(\mathfrak{a}) = \left\{ M \in \mathbf{PSL}_2(\mathcal{O}) \mid M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{a}} \right\}$$

A subgroup of $\mathbf{PSL}_2(\mathcal{O})$ is a *congruence subgroup* if it contains $\Gamma(\mathfrak{a})$ for some $\mathfrak{a} \trianglelefteq \mathcal{O}$, otherwise it is a *noncongruence subgroup*. It is a well known theorem by Bass-Milnor-Serre that Bianchi groups have many noncongruence subgroups.

We end this section with a group theoretic description of $\mathbf{PSL}_2(\mathcal{O}_1)$ which is again derived from an explicit fundamental domain. Note the similarity with the fact that $\mathbf{PSL}_2(\mathbb{Z}) \simeq C_2 \star C_3$.

$$\mathbf{PSL}_2(\mathcal{O}_1) \simeq ((C_2 \times C_2) \star_{C_2} \mathbf{S}_3) \star_{(C_2 \star C_3)} (\mathbf{A}_4 \star_{C_3} \mathbf{S}_3)$$

where C_n is the cyclic group of order n .

2.4. Cohomology and Hecke Action

In this section we will work with \mathbf{SL}_2 instead of \mathbf{PSL}_2 for convenience. For convenience, we assume that K has class number one. This allows us to associate Hecke operators with generators of the ideals instead of the ideals themselves. We start by constructing the Hecke operators explicitly on the first cohomology group with arbitrary coefficient modules. Next, we describe the specific coefficient modules that we need.

2.4.1 Hecke Operators

Let R be a commutative ring with 1 and $\alpha = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ where π is a prime element of \mathcal{O} . Let $\Gamma \leq \mathbf{SL}_2(\mathcal{O})$ be a congruence subgroup of level \mathfrak{a} . We follow the standard notations and put $\Gamma_\alpha := \Gamma \cap \alpha^{-1}\Gamma\alpha$ and $\Gamma^\alpha := \Gamma \cap \alpha\Gamma\alpha^{-1}$.

Let V be a right $R[\mathbf{Mat}_2(\mathcal{O})_{\neq 0}]$ -module where $\mathbf{Mat}_2(\mathcal{O})_{\neq 0}$ is the semi-group of 2×2 matrices of non-zero determinant with entries in \mathcal{O} . We define the Hecke operator T_π on the cohomology as the composition

$$\begin{array}{ccc} H^m(\Gamma, V) & & H^m(\Gamma, V) \\ \downarrow \text{res} & & \uparrow \text{cores} \\ H^m(\Gamma_\alpha, V) & \xrightarrow{\hat{\alpha}} & H^m(\Gamma^\alpha, V) \end{array}$$

where the map $\hat{\alpha}$ is defined by

$$c \mapsto (g \mapsto c(\alpha^{-1}g\alpha) \cdot \alpha^t)$$

where c is a cocycle in $H^m(\Gamma_\alpha, V)$ and $\alpha^t = \det(\alpha)\alpha^{-1}$.

One can describe Hecke operator T_π explicitly: suppose $\Gamma\alpha\Gamma = \bigsqcup_{i=1}^m \gamma_i\Gamma$. Given $g \in \Gamma$ and γ_i , there is a unique $\gamma_{j(i)}^{-1}g\gamma_i \in \Gamma$. Then

$$(T_\pi c)(g) = \sum_{1 \leq i \leq m} c(\gamma_{j(i)}^{-1}g\gamma_i) \cdot \gamma_i^t$$

for all cocycles c in $H^m(\Gamma, V)$ and $g \in \Gamma$. We note that this formula agrees with the one given in [1, p.194].

For $(\pi, \mathfrak{a}) = 1$, going through the same construction with the matrix $\beta = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ instead of $\alpha = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$, we get the Hecke operators S_π .

We define the *Hecke algebra* \mathbf{H} as the (commutative) \mathbb{Z} -algebra generated by the T_π 's and S_π 's.

2.4.2 Shapiro's Lemma

For an ideal \mathfrak{a} of \mathcal{O} , set

$$\Delta_1(\mathfrak{a}) := \{M \in \mathbf{Mat}_2(\mathcal{O})_{\neq 0} : M \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{a}}\}$$

We define the induced module $\text{Ind}(V) = \text{Ind}(\Gamma_1(\mathfrak{a}), \Gamma_1(\mathfrak{a}\mathfrak{b}), V)$ as the set of $\Gamma_1(\mathfrak{a}\mathfrak{b})$ -invariant maps from $\Gamma_1(\mathfrak{a})$ to V , that is

$$\text{Ind}(V) = \{f : \Gamma_1(\mathfrak{a}) \rightarrow V \mid f(gh) = f(g) \cdot h \text{ for all } h \in \Gamma_1(\mathfrak{a}\mathfrak{b})\}.$$

Then $\text{Ind}(V)$ is a right $\Gamma_1(\mathfrak{a})$ -module with the action $(f \cdot y)(x) = f(yx)$ for $x, y \in \Gamma_1(\mathfrak{a})$ and $f \in \text{Ind}(V)$.

We can extend the $\Gamma_1(\mathfrak{a})$ -action on $\text{Ind}(V)$ to a right $\Delta_1(\mathfrak{a})$ -action in the following way. Let $\alpha \in \Delta_1(\mathfrak{a})$ and $f \in \text{Ind}(V)$ and $x \in \Gamma_1(\mathfrak{a})$, then there are $\beta \in \Delta_1(\mathfrak{ab})$ and $y \in \Gamma_1(\mathfrak{a})$ such that $\alpha x = y\beta$ where $\Delta_1(\mathfrak{ab})$. We define

$$(f \cdot \alpha)(x) = f(y) \cdot \beta.$$

Shapiro's Lemma asserts that there is an isomorphism

$$\theta : H^m(\Gamma_1(\mathfrak{a}), \text{Ind}(V)) \rightarrow H^m(\Gamma_1(\mathfrak{ab}), V)$$

given by $f \mapsto f(I)$ for every non-homogeneous cocycle f in $H^m(\Gamma, \text{Ind}(V))$ where I denotes the identity matrix. The fact that the Hecke operators commute with the Shapiro isomorphism θ was proved in a more general setting in [1]. See also [18] for a proof in the case of $\text{PSL}_2(\mathbb{Z})$ using the same construction as ours for the Hecke operators.

Proposition 2.7. *The Hecke operators commute with the Shapiro map θ .*

2.4.3 Eigenvalue Systems

A *system of eigenvalues* of \mathbf{H} with values in a ring R is a set-theoretic a map $\Phi : \mathbf{H} \rightarrow R$. We say that an eigenvalue system Φ occurs in the $R\mathbf{H}$ -module A if there is a nonzero element $a \in A$ such that $Ta = \Phi(T)a$ for all T in \mathbf{H} .

The following lemma is proved in [1, Lemma 2.1].

Lemma 2.8. *Let F be a field and V be a $F\Delta_1(\mathfrak{a})$ -module which is finite dimensional over F . If an eigenvalue system $\Phi : \mathbb{H} \rightarrow F$ occurs in $H^n(\Gamma_1(\mathfrak{a}), V)$, then Φ occurs in $H^n(\Gamma_1(\mathfrak{a}), W)$ for some irreducible $F\Delta_1(\mathfrak{a})$ -subquotient W of V .*

Thus it is enough to investigate the cohomology with irreducible coefficient modules if we are only interested in the eigenvalue systems.

2.4.4 The Coefficient Modules

In this section, we describe certain modules that we will use as the coefficient modules in the cohomology of Bianchi groups in the next section.

For a commutative ring R with 1, let $E_k(R)$ denote the homogeneous polynomials of degree k in two variables with coefficients in R . The set $\{X^{k-i}Y^i : 0 \leq i \leq k\}$ is an R -basis of $E_k(R)$.

We can give $E_k(\mathbb{C})$ a right $\mathbf{SL}_2(\mathcal{O})$ -module structure as follows. For a polynomial $P(X, Y)$ in $E_k(K)$ and a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathcal{O})$, we set

$$(P \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix})(X, Y) = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}\right) = P(aX + bY, cX + dY).$$

When λ is a prime ideal of \mathcal{O} , we may view $E_k(\mathbb{F}_\lambda)$, where \mathbb{F}_λ is the residue field of λ , as an $\mathbf{SL}_2(\mathcal{O})$ module via the above formula by reducing each entry of the matrix modulo λ .

A result of Steinberg [15] says that the irreducible representations of $\mathbf{SL}_2(K)$ over \mathbb{C} are of the form

$$E_{k,l}(\mathbb{C}) := E_k(\mathbb{C}) \otimes \overline{E}_l(\mathbb{C})$$

with $k, l \geq 0$ integers where the bar on the second module means that the action is twisted with complex conjugation. Note that $-I$ acts trivially when $k + l$ is even, thus in this case the action factors through $\mathbf{PSL}_2(\mathcal{O})$.

Let ℓ be a rational prime which splits as $\lambda\bar{\lambda}$ in the ring of integers \mathcal{O} of K . Note that then the residue fields of both λ and $\bar{\lambda}$ are isomorphic to \mathbb{F}_ℓ . In this project, we are interested in the (absolutely) irreducible representations of $\mathbf{SL}_2(\mathcal{O}/(\ell)) = \mathbf{SL}_2(\mathcal{O}/\lambda) \times \mathbf{SL}_2(\mathcal{O}/\bar{\lambda})$ over \mathbb{F}_ℓ . Results of Steinberg [15] and Brauer-Nesbitt [5] show that these are of the form

$$E_{k,l}(\mathbb{F}_\ell) := E_k(\mathbb{F}_\ell) \otimes \bar{E}_l(\mathbb{F}_\ell) \quad 0 \leq k, l \leq \ell.$$

Here, $\mathbf{SL}_2(\mathcal{O})$ acts on the first module through reduction by λ and on the second through reduction by $\bar{\lambda}$. Note that $-I$ acts trivially when $k + l$ is even, thus in this case the action factors through $\mathbf{PSL}_2(\mathcal{O})$.

2.5. Bianchi Modular Forms

In this section, we discuss the automorphic forms on \mathbb{H} that are of cohomological type.

Let K be an imaginary quadratic field of class number 1 and let \mathcal{O} be its ring of integers. Let $\mathbf{G} = \text{Res}_{K/\mathbb{Q}}(\mathbf{SL}_2)$ be the algebraic group over \mathbb{Q} that is obtained from \mathbf{SL}_2 by restriction the scalars from K to \mathbb{Q} , see Platonov and Rapunchik [13][pg.49].

The group of real points $\mathbf{G}(\mathbb{R}) = \mathbf{SL}_2(K \otimes_{\mathbb{Q}} \mathbb{R}) = \mathbf{SL}_2(\mathbb{C})$ of \mathbf{G} acts transitively on \mathbb{H} as we have discussed in Section 2.1.. The stabilizer $\mathbf{SU}(2)$ of $j = (0, 0, 1) \in \mathbb{H}$ is a maximal compact subgroup of $\mathbf{SL}_2(\mathbb{C})$. Thus the global symmetric space $\mathbf{SL}_2(\mathbb{C})/\mathbf{SU}(2)$ of \mathbf{G} can be identified with \mathbb{H} via the map $M \mapsto M \cdot j$.

Let Γ be a torsion-free subgroup of $\mathbf{G}(\mathbb{Z}) = \mathbf{SL}_2(\mathcal{O})$. Then the quotient $\Gamma \backslash \mathbb{H}$ is a smooth 3-manifold which is noncompact but is of finite volume. Since \mathbb{H} is contractible, $\Gamma \backslash \mathbb{H}$ is an Eilenberg-Mac Lane space for Γ . That is, $\pi_1(\Gamma \backslash \mathbb{H}) \simeq \Gamma$ and higher homotopy groups vanish. Let E be a Γ -module and let \tilde{E} be the local system of coefficients induced from E . These can be defined as the bundle $\mathbb{H} \times_{\Gamma} E$ with discrete structure group Γ . We have

$$H^m(\Gamma, E) \simeq H^m(\Gamma \backslash \mathbb{H}; \tilde{E})$$

for all $m \in \mathbb{N}$.

In [8], Franke proves the Borel Conjecture [3] which says that the cohomology group $H^m(\Gamma, E)$ can be directly computed in terms of certain automorphic forms. Moreover, he obtains a direct sum decomposition

$$H^m(\Gamma, E) = H_{\text{cusp}}^m(\Gamma, E) \oplus H_{\text{Eis}}^m(\Gamma, E)$$

where the first summand is called the *cuspidal cohomology* and it is represented by cuspidal automorphic forms. The second summand is called the *Eisenstein cohomology* and it is constructed using Eisenstein series attached to certain cuspidal automorphic forms on lower rank groups.

The most well known example of the above decomposition is given by the theorem of Eichler and Shimura (see Theorem ??).

Due to Borel and Serre [4], there is a compactification $\overline{\Gamma \backslash \mathbb{H}}$ with boundary such that the inclusion $\Gamma \backslash \mathbb{H} \hookrightarrow \overline{\Gamma \backslash \mathbb{H}}$ is a homotopy equivalence. Thus after a suitable extension of the sheaf \tilde{E} , we have $H^m(\Gamma \backslash \mathbb{H}; \tilde{E}) = H^m(\overline{\Gamma \backslash \mathbb{H}}; \tilde{E})$.

Let $\delta(\overline{\Gamma \backslash \mathbb{H}})$ denote the boundary of $\overline{\Gamma \backslash \mathbb{H}}$. Then Harder [9, 10] shows that the kernel of the restriction map

$$H^m(\overline{\Gamma \backslash \mathbb{H}}, \tilde{E}) \longrightarrow H^m(\delta(\overline{\Gamma \backslash \mathbb{H}}), \tilde{E})$$

can be identified with the cuspidal cohomology $H_{\text{cusp}}^m(\Gamma, E)$ and the image of this map can be identified with the Eisenstein cohomology $H_{\text{Eis}}^m(\Gamma, E)$. The cuspidal cohomology can also be algebraically described as the kernel of the restriction map in

$$0 \longrightarrow H_{\text{cusp}}^m(\Gamma, E) \longrightarrow H^m(\Gamma, E) \xrightarrow{\text{res}} \prod_{h \in C_\Gamma} H^m(\Gamma \cap \Gamma_h, E). \quad (1)$$

Recall that C_Γ is the set of cusps of Γ and Γ_h is the stabilizer of $h \in \mathbb{H}$ inside $\mathbf{SL}_2(\mathcal{O})$.

We note that the virtual cohomological dimension of $\mathbf{SL}_2(\mathcal{O})$ is 2, see Serre [14]. Thus for torsion-free Γ , $H^m(\Gamma, E) = \{0\}$ for all $m > 2$.

We summarize from [17] what is known about the nature of the first two cohomology groups. Recall the definition of the irreducible $\mathbf{SL}_2(K)$ -modules $E_{k,l}(\mathbb{C})$ from Section 2.4.4.

- $H_{\text{cusp}}^m(\Gamma, E_{k,l}(\mathbb{C})) = \{0\}$, unless $k = l$ for $m = 1, 2$.
- $H_{\text{cusp}}^1(\Gamma, E_{k,l}(\mathbb{C})) \simeq H_{\text{cusp}}^2(\Gamma, E_{k,l}(\mathbb{C}))$
- $\dim H_{\text{Eis}}^1(\Gamma, E_{k,l}(\mathbb{C})) = \frac{1}{2} \dim H^1(\delta(\overline{\Gamma \backslash \mathbb{H}}), \tilde{E}_{k,l}(\mathbb{C}))$
- $H_{\text{Eis}}^2(\Gamma, E_{k,l}(\mathbb{C})) \simeq H^2(\delta(\overline{\Gamma \backslash \mathbb{H}}), \tilde{E}_{k,l}(\mathbb{C}))$ unless $k = l = 0$

We have seen that the cohomology groups $H^1(\Gamma, E)$ and $H^2(\Gamma, E)$ can be identified with certain automorphic forms. These automorphic forms (also known as automorphic forms of cohomological type) are what we want to call Bianchi modular forms. But as their cohomological identification is what we are going to use for computational purposes, we go ahead and call the cohomology classes themselves ‘‘Bianchi modular forms’’. The cuspidal automorphic forms are very important for us and the cuspidal part of H^1 and H^2 are isomorphic. So we solely focus on the first cohomology.

For an ideal \mathfrak{a} of \mathcal{O} , we define

$$\Gamma_0(\mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{O}) : c \equiv 0 \pmod{\mathfrak{a}} \right\}$$

and

$$\Gamma_1(\mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{O}) : c \equiv d - 1 \equiv 0 \pmod{\mathfrak{a}} \right\}$$

Definition 2.9. Let \mathfrak{a} be an ideal of \mathcal{O} .

- A Bianchi modular form of level \mathfrak{a} and weight (k, l) is a cohomology class in $H^1(\Gamma_1(\mathfrak{a}), E_{k,l}(\mathbb{C}))$. It is cuspidal if it is in the cuspidal part $H_{\text{cusp}}^1(\Gamma_1(\mathfrak{a}), E_{k,l}(\mathbb{C}))$.
- Similarly, we define a mod ℓ Bianchi modular form of level \mathfrak{a} and weight (k, l) as a cohomology class in $H^1(\Gamma_1(\mathfrak{a}), E_{k,l}(\overline{\mathbb{F}}_\ell))$. It is cuspidal, if it is in the cuspidal part that is given algebraically by the exact sequence (1).

3. Algorithm

In this chapter, we discuss the algorithm that we used to compute the cohomology groups and the Hecke operators on them. Although our algorithm works for any of the five Euclidean imaginary quadratic fields $K_1, K_2, K_3, K_7, K_{11}$, we work specifically with K_2 in this chapter and expose more details.

So set $K = K_2 = \mathbb{Q}(\sqrt{-2})$ and $\mathcal{O} = \mathcal{O}_2 = \mathbb{Z}[\sqrt{-2}]$.

3.1. Congruence Subgroups

We will only work with congruence subgroups of prime level although this is not necessary.

For a subgroup G of $PSL(2, \mathcal{O})$ and a nonzero element α in \mathcal{O} , define

$$G_\alpha = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : \alpha | c \right\} \quad \text{and} \quad G^\alpha = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : \alpha | b \right\}$$

For every prime element π in \mathcal{O} , we define the congruence subgroup $\Gamma(\pi)$ of $PSL(2, \mathcal{O})$ as $PSL(2, \mathcal{O})_\pi$. We have $\Gamma(1) = PSL(2, \mathcal{O})$.

If $N(\pi) = p$, a prime integer, then $[\Gamma(1) : \Gamma(1)_\pi] = [\Gamma(1) : \Gamma(1)^\pi] = p + 1$. The set

$$R = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} : 0 \leq x \leq p - 1 \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a complete set of representatives of coclasses modulo $\Gamma(1)_\pi$ and also modulo $\Gamma(1)^\pi$.

For a congruence subgroup Γ and a prime element π , one can check that

$$\Gamma_\pi = \Gamma \cap \alpha^{-1} \Gamma \alpha \quad \text{and} \quad \Gamma^\pi = \Gamma \cap \alpha \Gamma \alpha^{-1}$$

where $\alpha = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$. Note that α, α^{-1} are not in $PSL(2, K)$ but in $PGL(2, K)$. It is easy to see that

$$\alpha^{-1} \Gamma^\pi \alpha = \Gamma_\pi$$

3.2. Computing H^1 for $PSL_2(\mathcal{O})$

3.2.1 Finite Presentation

Recall that $PSL_2(\mathcal{O}_2)$ has the following presentation.

$$PSL_2(\mathcal{O}_2) = \left\langle A, B, U \mid (AB)^3 = B^2 = AUA^{-1}U^{-1} = (BU^2BU^{-1})^2 = 1 \right\rangle$$

where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ \sqrt{-2} & 1 \end{pmatrix}$.

3.2.2 H^1 for $PSL_2(\mathcal{O})$

Let $\Gamma = PSL_2(\mathcal{O}_K)$ and let $V = E_{k,l}(K)$. Take a cocycle $c \in H^1(\Gamma, V)$. By definition, $c : \Gamma \rightarrow V$ such that

$$c(hg) = c(h) \cdot g + c(g)$$

for all $g, h \in \Gamma$. If $g = h_1 \dots h_m$, then by induction on m , we get

$$c(g) = \sum_{1 \leq j \leq m} c(h_j) \cdot h_1 \dots h_{j-1} \quad (2)$$

Also,

$$c(g^{-1}) = -c(g) \cdot g^{-1} \quad (3)$$

Since any element g can be written as a product of A, A^{-1}, B, U, U^{-1} , a cocycle c can be determined by its values $c(A), c(B), c(U)$. So we identify a cocycle with the triple $(c(A), c(B), c(U))$ inside V^3 .

Any cocycle c satisfies the equations

$$c((AB)^3) = c(B^2) = c(AUA^{-1}U^{-1}) = c((BU^2BU^{-1})^2) = c(1) = 0 \quad (4)$$

which are given by the relations that the generators A, B, U are satisfying.

Conversely, a function c defined on the generators A, B, U can be extended to whole Γ if it satisfies these equations. Since these equations are linear in $c(A), c(B), c(U)$, we can identify the space of cocycles $Z(\Gamma, V)$ as the null space of the system of equations (4). Using the identities (2) and (3) on the equations (4), we get the following system

- $c(B) \cdot [B + 1] = 0$
- $c(A) \cdot [B((AB)^2 + AB + 1)] + c(B) \cdot [(AB)^2 + AB + 1] = 0$
- $c(A) \cdot [A^{-1}(1 - U^{-1})] + c(U) \cdot [(A^{-1} - 1)U^{-1}] = 0$
- $c(B) \cdot [(U^{-1}BU + U)(BU^{-1}BU + 1)] + c(U) \cdot [(1 - U^{-1}BU)(BU^{-1}BU + 1)] = 0$

The subspace of coboundaries $B(\Gamma, V)$ inside the space $Z(\Gamma, V)$ of cocycles is given as the image of the linear map from V to V^3

$$v \longmapsto (v \cdot (A - 1), v \cdot (B - 1), v \cdot (U - 1))$$

Finally the quotient $Z(\Gamma, V)/B(\Gamma, V)$ gives us $H^1(\Gamma, V)$.

3.2.3 Hecke Action Revisited

Now we want to describe the action of Hecke operators on the vector space $H^1(\Gamma, V)$ that we computed in the previous section. Given a cocycle c , which is identified with the triple $(c(A), c(B), c(U))$, and a Hecke operator T , we want to describe $T(c)$, which is identified with $(T(c)(A), T(c)(B), T(c)(U))$. So the goal of this section is to describe $T(c)(A), T(c)(B)$ and $T(c)(U)$ in terms of $c(A), c(B)$ and $c(U)$.

Let π be a prime element of \mathcal{O} such that $N(\pi) = p$, a rational prime. Let c be a cocycle in $H^1(\Gamma, V)$. We will compute $T_\pi(c)$. We need to compute $T_\pi(c)(A)$, $T_\pi(c)(B)$ and $T_\pi(c)(U)$. Let $\alpha = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$. We fix the transversal R for Γ_π in Γ

$$\left\{ R_i = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} : 0 \leq x \leq p-1 \right\} \cup \left\{ R_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Then for every $g \in \Gamma$, we have

$$(T_\pi c)(g) = \sum_{1 \leq i \leq m} c(\alpha^{-1} h_i(g) \alpha) \cdot \alpha^i \cdot R_i^{-1}$$

where $h_i(g) = R_{\sigma_g(i)}^{-1} g R_i$ with $R_{\sigma_g(i)}$ the unique element in R such that $h_i(g) \in \Gamma^\pi$.

Thus for $g = A, B, U$, we need to describe $\sigma_g(i)$ and $h_i(g)$. This is done below.

(a) $T_\pi(c)(A)$

We have

$$\sigma_A(i) = \begin{cases} \overline{i-1} \pmod{p} & , 0 \leq i \leq p-1 \\ \infty & , i = \infty \end{cases}$$

This gives

$$h_i(A) = \begin{cases} I & , 1 \leq i \leq p-2 \\ \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} & , i = 0 \\ A & , i = \infty \end{cases}$$

Since $c(I) = 0$, the middle $p-2$ terms of $T_\pi(c)(A)$ vanish.

$$T_\pi(c)(A) = c(\alpha^{-1} h_0(A) \alpha) \cdot \alpha^\ell \cdot R_0^{-1} + c(\alpha^{-1} h_\infty(A) \alpha) \cdot \alpha^\ell \cdot R_\infty^{-1}$$

Note that

$$\begin{aligned} \alpha^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha &= \begin{pmatrix} a & b/\pi \\ c\pi & d \end{pmatrix} \\ T_\pi(c)(A) &= c(\alpha^{-1} \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} \alpha) \cdot \alpha^\ell \cdot B + c(\alpha^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \alpha) \cdot \alpha^\ell \\ &= c(\begin{pmatrix} 1 & -p/\pi \\ 0 & 1 \end{pmatrix}) \cdot \alpha^\ell \cdot B + c(\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}) \cdot \alpha^\ell \\ &= c(\begin{pmatrix} 1 & -\bar{\pi} \\ 0 & 1 \end{pmatrix}) \cdot \alpha^\ell \cdot B + c(\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}) \cdot \alpha^\ell \\ &= c(B \begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix} B) \cdot \alpha^\ell \cdot B + c(\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}) \cdot \alpha^\ell \end{aligned}$$

Let $a, b \in \mathbb{Z}$ be such that $\pi = a + b\omega$. Then

$$\begin{aligned} T_\pi(c)(A) &= c(BA^aU^{-b}B) \cdot \alpha^\ell \cdot B + c(A^aU^b) \cdot \alpha^\ell \\ &= [c(B)A^aU^{-b}B + c(A^a)U^{-b}B + c(U^{-b})B + c(B)] \cdot \alpha^\ell \cdot B \\ &\quad + [c(A^a)U^b + c(U^b)] \cdot \alpha^\ell \end{aligned}$$

For an element $g \in \Gamma$,

$$c(g^n) = c(g)(g^{n-1} + \dots + g^2 + g + 1)$$

Thus

$$c(g^{-n}) = -c(g^n)g^{-n} = -c(g)(g^{n-1} + \dots + g^2 + g + 1)(g^{-n})$$

For abbreviation, put

$$P_M(n) = \begin{cases} M^{n-1} + \dots + M^2 + M + 1 & \text{if } n > 0 \\ -(M^{n-1} + \dots + M^2 + M + 1)(M^{-n}) & \text{if } n < 0 \end{cases}$$

so that $c(M^k) = c(M)P_M(k)$ for all $k \in \mathbb{Z}$ and $M \in \Gamma$.

Then we have

$$T_\pi(c)(A) = [c(B)A^aU^{-b}B + c(A)P_A(a)U^{-b}B + c(U)P_U(-b)B + c(B)] \cdot \alpha^t \cdot B \\ + [c(A)P_A(a)U^b + c(U)P_U(b)] \cdot \alpha^t$$

Putting like terms together, we get

$$T_\pi(c)(A) = c(A) [P_A(a)(U^{-b}B\alpha^tB + U^b\alpha^t)] \\ = + c(B) [(A^aU^{-b}B + 1)(\alpha^tB)] \\ = + c(U) [P_U(-b)B\alpha^tB + P_U(b)\alpha^t]$$

(b) $T_\pi(c)(B)$

We have

$$\sigma_B(i) = \begin{cases} \overline{-1/i} \pmod{p} & , 1 \leq i \leq p-1 \\ \infty & , i = 0 \\ 0 & , i = \infty \end{cases}$$

This gives

$$h_i(B) = \begin{cases} \begin{pmatrix} \sigma_B(i) & 1 + i\sigma_B(i) \\ -1 & -i \end{pmatrix} & , 1 \leq i \leq p-2 \\ I & , i = 0, \infty \end{cases}$$

Observe also that $R_i = A^{-i}B$ for $0 \leq i \leq p-1$. Thus we have

$$(T_\pi c)(B) = \sum_{1 \leq i \leq p-1} c(\alpha^{-1}h_i(B)\alpha) \cdot \alpha^t \cdot R_i^{-1} \\ = \sum_{1 \leq i \leq p-1} c\left(\begin{pmatrix} \sigma_B(i) & \frac{1+i\sigma_B(i)}{\pi} \\ -\pi & -i \end{pmatrix}\right) \cdot \alpha^t \cdot BA^i$$

We need to write those matrices in terms of A, B, U and we can do this effectively as our field is Euclidean. We describe a word decomposition algorithm to this effect in the next section.

(c) $T_\pi(c)(U)$

We have

$$\sigma_U(i) = \begin{cases} \overline{i - \omega} \pmod{\pi} & , 0 \leq i \leq p-1 \\ \infty & , i = \infty \end{cases}$$

This gives

$$h_i(U) = \begin{cases} \begin{pmatrix} 1 & (i - \omega) - \sigma_U(i) \\ 0 & 1 \end{pmatrix} & , 0 \leq i \leq p-1 \\ \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} & , i = \infty \end{cases}$$

We get

$$\begin{aligned}
(T_\pi c)(U) &= \sum_{0 \leq i \leq p-1} c\left(\begin{pmatrix} 1 & \frac{(i-\omega)-\sigma_U(i)}{\pi} \\ 0 & 1 \end{pmatrix}\right) \cdot \alpha^i \cdot BA^i + c\left(\begin{pmatrix} 1 & 0 \\ \omega\pi & 1 \end{pmatrix}\right) \alpha^t \\
&= \sum_{0 \leq i \leq p-1} c\left(B \begin{pmatrix} 1 & 0 \\ \frac{\sigma_U(i)-(i-\omega)}{\pi} & 1 \end{pmatrix} B\right) \cdot \alpha^i \cdot BA^i + c(A^{-2b}U^a) \alpha^t
\end{aligned}$$

For every i , put $(\sigma_U(i) - (i - \omega))/\pi = \operatorname{Re}(i) + \operatorname{Im}(i)\omega$. Then

$$(T_\pi c)(U) = \sum_{0 \leq i \leq p-1} c(BA^{\operatorname{Re}(i)}U^{\operatorname{Im}(i)}B) \cdot \alpha^i \cdot BA^i + c(A^{-2b}U^a) \alpha^t$$

Observe that

$$\begin{aligned}
c(BA^{\operatorname{Re}(i)}U^{\operatorname{Im}(i)}B) &= c(B)A^{\operatorname{Re}(i)}U^{\operatorname{Im}(i)}B + c(A^{\operatorname{Re}(i)}U^{\operatorname{Im}(i)}B) + c(U^{\operatorname{Im}(i)})B + c(B) \\
&= c(A)[P_A(\operatorname{Re}(i))U^{\operatorname{Im}(i)}B] + c(B)[A^{\operatorname{Re}(i)}U^{\operatorname{Im}(i)}B + 1] \\
&\quad + c(U)[P_U(\operatorname{Im}(i))B]
\end{aligned}$$

Moreover

$$c(A^{-2b}U^a) = c(A)[P_A(-2b)U^a] + c(U)[P_U(a)]$$

Hence

$$\begin{aligned}
(T_\pi c)(U) &= c(A) \left[\left(\sum_{0 \leq i \leq p-1} P_A(\operatorname{Re}(i))U^{\operatorname{Im}(i)}B \alpha^i \cdot BA^i \right) + P_A(-2b)U^a \alpha^t \right] \\
&\quad + c(B) \left[\left(\sum_{0 \leq i \leq p-1} (A^{\operatorname{Re}(i)}U^{\operatorname{Im}(i)}B + 1) (\alpha^i \cdot BA^i) \right) \right] \\
&\quad + c(U) \left[\left(\sum_{0 \leq i \leq p-1} P_U(\operatorname{Im}(i))B \alpha^i \cdot BA^i \right) + P_U(a) \alpha^t \right]
\end{aligned}$$

3.2..4 Word Decomposition

In this section we describe an effective algorithm to write a given element M of $\mathbf{PSL}_2(\mathcal{O})$ in terms of the generators A, B, U .

We first write M as product of lower triangular matrices.

Proposition 3.1. *Given a matrix in $\mathbf{PSL}_2(\mathcal{O})$, let $M \in \mathbf{SL}_2(\mathcal{O})$ be a preimage of M . There exists upper triangular matrices Q_1, \dots, Q_m such that*

$$M = \pm Q_m B Q_2 \dots B Q_1 \quad \text{or} \quad M = \pm B Q_m B Q_2 \dots B Q_1.$$

Proof. Say $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Without loss of generality, we may assume that $N(d) > N(b)$, because if not, we can replace M with $BM = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$.

We will proceed by induction on the norm $N(b)$ of b . If $b \neq 0$, using the Euclidean algorithm we find q and r such that

$$d = qb + r \quad , N(r) < N(b)$$

In the inductive step, set $Q = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}$ and we take

$$M' = BQM = \begin{pmatrix} c' & -r \\ a & b \end{pmatrix}$$

Since $N(-r) < N(b)$, in a finite number of steps the top right corner will become 0, forcing the diagonal entries to be units ± 1 . This gives the desired result. \square

To finish the word decomposition, we need the following straightforward observation. Let $z = a + b\sqrt{-2} \in \mathcal{O}$. Then we have

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = A^a U^b$$

3.3. Computing H^1 for $\Gamma < \mathbf{PSL}_2(\mathcal{O})$

Let $\Gamma = \Gamma_0(\mu)$ for some element μ of \mathcal{O} with rational prime norm q and $V = E_{k,l}$. To compute $H^1(\Gamma, V)$, we will not use a presentation of Γ like we did for $\mathbf{PSL}_2(\mathcal{O})$. This is feasible but very inefficient. Instead, we will use Shapiro's Lemma and express cohomology of Γ as a cohomology of $\mathbf{PSL}_2(\mathcal{O})$. More precisely, we have

$$H^1(\Gamma, V) \simeq H^1(\mathbf{PSL}_2(\mathcal{O}), \text{Ind}(V))$$

where

$$\text{Ind}(V) = \{f : \mathbf{PSL}_2(\mathcal{O}) \rightarrow V \mid f(gh) = f(g) \cdot h \text{ for all } h \in \Gamma\}.$$

To compute $H^1(\mathbf{PSL}_2(\mathcal{O}), \text{Ind}(V))$, we first need to describe the action of the generators A, B, U of $\mathbf{PSL}_2(\mathcal{O})$ on the induced module $\text{Ind}(V)$. We do this using another description of $\text{Ind}(V)$ which is as follows.

Let $f : \mathbf{PSL}_2 \rightarrow V$ be an element of $\text{Ind}(V)$. As f is Γ -invariant by definition, f is determined by its values on some (equivalently, any) transversal (that is, a set of coset representatives) of Γ in $\mathbf{PSL}_2(\mathcal{O})$. Thus, $\text{Ind}(V)$ can be seen as direct sum of copies of V indexed by a transversal $\{\gamma_i\}$ of Γ in $\mathbf{PSL}_2(\mathcal{O})$. Then the action of $\mathbf{PSL}_2(\mathcal{O})$ can be described as follows. Let $g \in \mathbf{PSL}_2(\mathcal{O})$ and $v_{\gamma_i} \in \text{Ind}(V)$ be an element indexed by σ_i . When $v \in V$ is indexed by γ_i , let us use v_i to denote this. We have

$$g \cdot v_i = v_{\sigma(i)} \cdot h_i(g)$$

where $g\gamma_i = \gamma_{\sigma(i)}h_i(g)$ with $h_i(g) \in \Gamma$. In other words, the element $v \in V$ which is contained in the copy indexed by γ_i goes to $v \cdot h_i(g)$ that is contained in the copy of V indexed by $\gamma_{\sigma(i)}$.

So we need to describe the action of A, B, U and $\alpha^t = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ on the induced module. This is done below.

For convenience, we choose to work with the transversal

$$\left\{ R_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : 0 \leq x \leq q-1 \right\} \cup \left\{ R_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

for Γ in $\mathbf{PSL}_2(\mathcal{O})$.

(a) Action of A on the induced module

We have

$$\sigma_A(i) = \begin{cases} \overline{i+1} \pmod{q} & , 0 \leq i \leq q-1 \\ \infty & , i = \infty \end{cases}$$

This gives

$$h_i(A) = \begin{cases} I & , 0 \leq i \leq q-2 \\ A^q & , i = q-1 \\ BAB & , i = \infty \end{cases}$$

(b) Action of B on the induced module

We have

$$\sigma_B(i) = \begin{cases} \overline{-1/i} \pmod{q} & , 1 \leq i \leq p-1 \\ \infty & , i = 0 \\ 0 & , i = \infty \end{cases}$$

This gives

$$h_i(B) = \begin{cases} A^{-\sigma_B(i)}BA^i & , 1 \leq i \leq q-2 \\ I & , i = 0, \infty \end{cases}$$

(c) Action of U on the induced module

We have

$$\sigma_U(i) = \begin{cases} \overline{i+\omega} \pmod{\mu} & , 0 \leq i \leq p-1 \\ \infty & , i = \infty \end{cases}$$

This gives

$$h_i(U) = \begin{cases} A^{-\sigma_U(i)}UA^i & , 0 \leq i \leq p-1 \\ BUB & , i = \infty \end{cases}$$

(d) Action of α^t on the induced module

We have

$$\sigma_U(i) = \begin{cases} \overline{i\pi} \pmod{\mu} & , 0 \leq i \leq p-1 \\ \infty & , i = \infty \end{cases}$$

This gives

$$h_i(U) = \begin{cases} \alpha^t & , 0 \leq i \leq p-1 \\ \alpha & , i = \infty \end{cases}$$

3.4. Computing Cuspidal Cohomology

3.5. Computing The Plus-Subspace

A very important subspace is the space of so called *plus-forms* that we define now.

Let ϵ be a generator of \mathcal{O}^* . Then conjugation by the element $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{PGL}_2(\mathcal{O})$ leaves Γ invariant and induces an involutory automorphism $\tilde{\epsilon}$ on $H^1(\Gamma, E(\mathbb{F}))$. We put

$$H^1(\Gamma, E(\mathbb{F}))^\pm$$

for the ± 1 eigenspaces of $\tilde{\epsilon}$. If \mathbb{F} is of char. 2, we denote by $H^1(\Gamma, E(\mathbb{F}))^+$ the subspace that is invariant under $\tilde{\epsilon}$. This involution preserves the cuspidal and Eisenstein subspaces.

Remark : Note that taking $\tilde{\epsilon}$ invariants equals to working with $\mathbf{PGL}_2(\mathcal{O})$ instead of $\mathbf{PSL}_2(\mathcal{O})$. One sees by the inflation-restriction sequence that

$$H^1(\mathbf{PGL}_2(\mathcal{O}), E) \simeq H^1(\mathbf{PSL}_2(\mathcal{O}), E)^+.$$

4. Implementation

We implemented the above algorithm in MAGMA. For $K = \mathbb{Q}(\sqrt{-2})$, we compute

$$H^1(\Gamma_0(\mathfrak{p}), E_{k,l}(K))$$

for split primes \mathfrak{p} (that is, norm of \mathfrak{p} is a rational prime) and the Hecke operators $T(\mathfrak{a})$ for split primes \mathfrak{a} such that $\mathfrak{a} \neq \mathfrak{p}$.

We discuss here two of the functions we wrote: DIM and HECKE. The scripts are available on the website of the author.

The function DIM takes 4 integers (a, b, k, l) as input where $\mathfrak{p} = a + b\sqrt{-2}$ gives the level and k, l give the weight. As output, it we give two ordered pairs of integers. The first pair gives the dimensions of full cohomology and and its cuspidal subspace respectively. The second pair gives the dimensions of plus-subspace and cuspidal plus-subspace respectively.

The function HECKE takes 6 integers (a, b, c, d, k, l) as input where $\mathfrak{a} = c + d\sqrt{-2}$ gives the Hecke operator $T(\mathfrak{a})$ and $\mathfrak{p} = a + b\sqrt{-2}$, k, l as above. As output, it gives the matrix, the characteristic polynomial and the eigenspaces of the eigenvalues

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