A numerical study of mod p Bianchi modular forms and Galois representations

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1. Introduction

Bianchi modular forms are modular forms for GL_2 over an imaginary quadratic field K. In the early nineties, Harris-Soudry-Taylor [14] and Taylor [22], under some hypothesis, attached compatible families of p-adic Galois representations of $G_K := \mathrm{Gal}(\overline{K}/K)$ to cuspidal Bianchi modular forms, in accordance with Langlands philosophy. Their results have recently been strengthened by Berger-Harcos [1].

Explicit computations done by Elstrodt-Grunewald-Mennicke [9], Figuieredo [10], Sengun [18] and Torrey [23] suggest that there might be a connection between certain classes in the mod p cohomology of congruence subgroups of $\mathrm{GL}_2(\mathcal{O}_K)$ and mod p Galois representations of G_K in the sense of Serre.

In this note, we present further explicit computations strongly supporting the existence of such a connection. We first focus on results of Moon-Taguchi [16] and Sengun [19] on the nonexistence of certain mod p Galois representations. These results together with our mod p cohomology computations give further evidence to the above mentioned connection. Secondly, we study a certain mod 2 Galois representation of $G_{\mathbb{Q}(i)}$. Our mod 2 cohomology computations suggest that the representation is modular. The novelty of our study is that it provides the first example in the literature in which the image of the Galois representation is nonsolvable.

2. The Connection

2.1. the cohomology

Let K be an imaginary quadratic field, for convenience, of class number one. Let $\mathcal{O}$ be its ring of integers. Let $\Gamma_1(N)$ denote the congruence subgroup of level $N \lhd \mathcal{O}$, defined in the usual way, of the Bianchi group $\mathrm{GL}_2(\mathcal{O})$.

Given a rational prime p, a *(mod p) weight* E is an $\overline{\mathbb{F}}_p-$representation of $\mathrm{GL}_2(\mathcal{O}/(p))$. If E is irreducible, we call it a *Serre weight*. Serre weights can be explicitly described via a result of Brauer-Nesbitt [2].

We define mod p Bianchi modular forms as mod p cohomology classes.

Definition 2.1. A *mod p Bianchi modular form* (for K) of level N and weight E is a cohomology class in

$$H^1(\Gamma_1(N), E).$$

There is a well-known construction of Hecke operators acting on this cohomology group. This can be found, for example, in [5] page 116.

A mod p Bianchi modular form f of level $N(f)$ is called a (Hecke) *eigenform* if it is a simultaneous eigenvector for all the Hecke operators T_π where $(\pi) \not| N(f)$.

If one is interested in eigenvalue systems, it is enough to consider the Serre weights, see [20].

2.2. Galois representations

We will be interested in 2-dimensional mod p Galois representation of G_K. These are continuous linear representations

$$\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p).$$

Continuity forces the image to be finite. Thus the representation factors through a finite Galois extension L of K:

$$\rho : \mathrm{Gal}(L/K) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_{p^a}).$$

Following Serre, we define the conductor $N(\rho)$ of a mod p Galois representation to be the Artin conductor away from p. That is,

$$N(\rho) = \prod_{\pi} (\pi)^{n(\pi)}$$

where the product runs over primes π of K which do *not* lie over p. Here the exponent $n(\pi)$ are defined as follows:

$$n(\pi) = \sum_{i \geq 0} \frac{|G_{\pi,i}|}{|G_{\pi,0}|} \dim(V/V_i)$$

where $G_{\pi,0} \supseteq G_{\pi,1} \supseteq \dots$ are the higher ramification subgroups of the prime π in $\mathrm{Gal}(L/K)$. For each i, we denote by V_i the subspace of V fixed by $G_{\pi,i}$ where V is the 2-dimensional $\mathbb{F}_{p^a}$-vector space that G_K acts through ρ. The followings are well-known:

- $n(\pi) = 0$ if and only if π is unramified in L/K.
- $n(\pi)$ is a non-negative integer.
- $n(\pi) = \dim(V/V_0)$ if and only if π is tamely ramified.

Definition 2.2. A mod p Galois representation ρ of G_K and a mod p Bianchi eigenform f are said to *match* if

$$\mathrm{trace}(\rho(\mathrm{Frob}_\pi)) = a_f(\pi)$$

for almost all finite prime π of K. Here Frob_π is a Frobenius element for π in G_K and $a_f(\pi)$ is the eigenvalue of f for the Hecke operator T_π.

2.3. two questions

In this note, our interest is in the following questions.

Question 2.3. Given an irreducible mod p Galois representation of G_K with conductor N, is there a matching mod p Bianchi eigenform over K whose level is N?

A natural next question is to ask for which Serre weights one finds such matching Bianchi eigenforms. R.Torrey investigated this question (see [23]) using the weight recipe formulated by Buzzard-Diamond-Jarvis [4] for totally real fields. We will not consider the question of weights here.

Question 2.4. Given a mod p Bianchi eigenform with level N, is there a matching mod p Galois representation of G_K whose conductor is N ?

The eigenvalue system attached to a mod p Bianchi eigenform f may not lift to char. 0 (the obstruction is in the p-torsion of the second cohomology with integral coefficients $E(\mathcal{O}_K)$). If the system does lift to char. 0 and can be realized by a char. 0 Bianchi eigenform which satisfies the hypothesis of the result of Taylor et al. then one knows the existence of a mod p Galois representation of K attached to f.

3. Nonexistence

Let us list some nonexistence of Galois representations results related to quadratic fields:

Theorem 3.1. *Every continuous representation of G_K into $GL_2(\overline{\mathbb{F}}_p)$ that is unramified away fom $\{p, \infty\}$ with*

A. $p = 2$ and $K = \mathbb{Q}(\sqrt{d})$ for $d = 6, 5, 3, 2, -1, -2, -3, -5, -6$,

B. $p = 3$ and $K = \mathbb{Q}(\sqrt{d})$ for $d = -3$.

is reducible. Moreover, up to conjugation, its image will lie in $\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{F}_{p^a}\}$ for some a.

This is proved in [16, 19]. See [17] for extensive finiteness results in this direction.

Let us examine the implications of the above theorem in conjuntion with our two questions of interest. Let $p = 2, 3$ and K be one of the above fields according to p.

If Question 2.4 has a positive answer, then there should be no eigenvalue systems occuring in $H^1(\Gamma_0(N), E(\mathbb{F}_p))$ with $\mathrm{Norm}(N) = p^r$ for any weight E, except perhaps the constant eigenvalue system $(2)_\pi$, that is $a_\pi = 2 \mod p$ for every $\pi \not| (p)$.

One can construct reducible mod p Galois representations of the above type using class field theory or by looking at the p-torsion of suitable elliptic curves. Thus if Question 2.3 has a positive answer, then the constant eigenvalue system $(2)_\pi$ should occur in $H^1(\Gamma_0(N), \mathbb{F}_p)$

for every N with $\mathrm{Norm}(N) = p^r$.

With the computer programs we have developed, we examined the cohomology groups $H^1(\Gamma_0(N), E_i(\mathbb{F}_2))$ with $\mathrm{Norm}(N) = 2^m$ and $i = 0, 1$ for the fields $d = -1, -2$. For $0 \leq m \leq 10$, we have observed that there is a unique eigenvalue system. Thus, by the above remark, this system has to be the constant system $(0)_\pi$.

This is in perfect agreement with the above implications.

4. A Modular Representation

4.1. the representation

Let $K = \mathbb{Q}(i)$ and let

$$f(x) = x^5 + (-1+2i)x^4 + (-6+2i)x^2 + (-4-7i)x - 3i \in K[x].$$

Let M be the splitting field of f over $\mathbb{Q}(i)$. We found this polynomial from Eric Driver's table [7] of decic extensions of $\mathbb{Q}$ containing $\mathbb{Q}(i)$ which ramify only over $\{2, 5\}$.

Proposition 4.1. *The Galois group G of $M/Q(i)$ is isomorphic to A_5. The only primes of $\mathbb{Q}(i)$ that ramify in M are $(1+i), (2-i), (2+i)$ with ramification degrees $4, 3, 10$ and inertia degrees $3, 2, 1$ respectively. The ramification groups for these primes (starting with the decomposition group) are*

- *for (1+i): A_4, D_2, D_2, D_2.*
- *for (2-i): S_3, C_3.*
- *for (2+i): D_5, D_5, C_5.*

Here S_n, A_n are the symmetric and alternating groups on n letters, D_n, C_n are the dihedral and the cyclic groups of orders $2n$ and n respectively.

One way to compute the above data is to use Buhler's table on page 46 of [3]. We performed a more direct calculation, explained to us by Jürgen Klüners [15], which uses the very efficient "Montes package" of Guardia-Montes-Nart (see [11, 12, 13]).

Fix an identification $A_5 \simeq \mathrm{SL}_2(\mathbb{F}_4)$ (there are two such identifications, isomorphic by the Frobenius automorphism of $\mathbb{F}_4$). Via the surjection $G_K \to \mathrm{Gal}(L/K)$ and fixing an injection $\mathrm{GL}_2(\mathbb{F}_4) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_2)$, we get an irreducible mod 2 Galois representation

$$\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$$

with image is $\mathrm{SL}_2(\mathbb{F}_4)$ and thus has trivial determinant $det(\rho)$.

Below is a table of conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_4)$ with their order, trace and the dimension of fixed subspace of $\mathbb{F}_4^2$ under the action of a member of the class:

Class	#1	#2	#3	# 4	#5
Order	1	2	3	5	5
Trace	0	0	1	w^2	w
Dim	2	1	0	0	0

w is a root of $x^2 + x + 1$ in $\mathbb{F}_4$

To compute the Artin conductor away from 2 of ρ, we need to consider the higher ramification groups of $2 - i$ and $2 + i$ only. We need to look at the codimension of the fixed spaces of the higher ramification groups, which is read off from the "Dim" row of the above table. Thus we find that $N(\rho) = (2 - i)^2(2 + i)^3$.

To calculate the traces of the Frobenius elements attached to primes of K that are unramified in M, we need to compute their inertia degrees in M/K, which can be calculated from L/K using the above lemma. Once that is known, the trace can be read off from the table except when the inertia degree is 5. In this case, the trace can be w or w^2 and we use an idea of Serre, as explained by Buhler ([3] p.53), to distinguish the two cases.

We list now the Frobenius orders and traces for a few split primes.

Prime	Order	Trace	Prime	Order	Trace
$3+2i$	3	1	$10-i$	5	w
$3-2i$	5	w^2	$10+3i$	5	w
$4+i$	5	w	$10-3i$	3	1
$4-i$	2	0	$8+7i$	5	w
$5+2i$	5	w^2	$8-7i$	5	w
$5-2i$	3	1	$11+4i$	3	1
$6+i$	3	1	$11-4i$	2	0
$6-i$	5	w	$10+7i$	2	0
$5+4i$	5	w^2	$10-7i$	2	0
$5-4i$	5	w	$11+6i$	2	0
$7+2i$	3	1	$11-6i$	3	1
$7-2i$	3	1	$13+2i$	5	w
$6+5i$	5	w	$13-2i$	5	w^2
$6-5i$	5	w	$10+9i$	3	1
$8+3i$	5	w^2	$10-9i$	5	w
$8-3i$	5	w	$12+7i$	2	0
$8+5i$	2	0	$12-7i$	3	1
$8-5i$	5	w^2	$14+i$	5	w^2
$9+4i$	3	1	$14-i$	3	1
$9-4i$	5	w^2			
$10+i$	2	0			

4.2. the mod p eigenvalue system

In this section we will present the mod 2 cohomology computations we performed. Our data suggests that the above representation is modular.

We found a promising eigenvalue system in $H^1(\Gamma_0((2-i)^2(2+i)^3), \mathbb{F}_2)$. It is listed as System 1 in the table below. System 2 is the Galois conjugate of System 1 which also lives in this space.

below w is a root of $x^2 + x + 1$ in $\mathbb{F}_4$

Hecke op.	system 1	system 2
$3+2i$	1	1
$3-2i$	w^2	w
$4+i$	w	w^2
$4-i$	0	0
$6+i$	1	1
$6-i$	w	w^2
$5+4i$	w^2	w
$5-4i$	w	w^2
$7+2i$	1	1
$7-2i$	1	1
$6+5i$	w	w^2
$6-5i$	w	w^2
$8+3i$	w^2	w
$8-3i$	w	w^2
$8+5i$	0	0
$8-5i$	w^2	w
$9+4i$	1	1
$9-4i$	w^2	w

We asked John Cremona to compute some Hecke operators on the space

$$H^1\big(\Gamma_0((2-i)^2(2+i)^3), \mathbb{C}\big),$$

as our programs needed much more memory than we had at our disposal. With the data he kindly provided, we found out that there are 4 non-rational cuspidal eigenforms. For each eigenform, the attached eigenvalues generate the field $\mathbb{Q}(\sqrt{5})$. The mod 2 reductions of the eigenvalue systems attached to these eigenforms give our Systems 1 and 2. For the sake of completeness, we list these eigenvalue systems attached to these for newforms below.

eigenvalue systems in $H^1\big(\Gamma_0((2-i)^2(2+i)^3),\mathbb{C}\big)$
here $z=\sqrt{5}$

Hecke op.	system 1	system 2	system 3	system 4
2+i	[-5]	[-5]	[-5]	[-5]
1+2i	[5]	[5]	[-5]	[-5]
1+i	[1/2*(-5*z + 5)]	[1/2*(5*z + 5)]	[1/2*(5*z - 5)]	[1/2*(-5*z - 5)]
3	[-5*z]	[5*z]	[-5*z]	[5*z]
3+2i	[-10*z - 5]	[10*z - 5]	[10*z + 5]	[-10*z + 5]
2+3i	[1/2*(15*z + 15)]	[1/2*(-15*z + 15)]	[1/2*(-15*z - 15)]	[1/2*(15*z - 15)]
4+i	[1/2*(5*z + 55)]	[1/2*(-5*z + 55)]	[1/2*(-5*z - 55)]	[1/2*(5*z - 55)]
1+4i	[-5*z + 15]	[5*z + 15]	[5*z - 15]	[-5*z - 15]
5+2i	[1/2*(5*z - 75)]	[1/2*(-5*z - 75)]	[1/2*(5*z - 75)]	[1/2*(-5*z - 75)]
2+5i	[5*z]	[-5*z]	[5*z]	[-5*z]
6+i	[15]	[15]	[-15]	[-15]
1+6i	[1/2*(-15*z + 55)]	[1/2*(15*z + 55)]	[1/2*(15*z - 55)]	[1/2*(-15*z - 55)]
5+4i	[1/2*(25*z - 55)]	[1/2*(-25*z - 55)]	[1/2*(25*z - 55)]	[1/2*(-25*z - 55)]
4+5i	[1/2*(-25*z + 45)]	[1/2*(25*z + 45)]	[1/2*(-25*z + 45)]	[1/2*(25*z + 45)]
7	[1/2*(-25*z + 25)]	[1/2*(25*z + 25)]	[1/2*(-25*z + 25)]	[1/2*(25*z + 25)]
7+2i	[45]	[45]	[-45]	[-45]
2+7i	[5*z + 20]	[-5*z + 20]	[-5*z - 20]	[5*z - 20]
6+5i	[1/2*(-25*z - 55)]	[1/2*(25*z - 55)]	[1/2*(-25*z - 55)]	[1/2*(25*z - 55)]
5+6i	[1/2*(25*z - 5)]	[1/2*(-25*z - 5)]	[1/2*(25*z - 5)]	[1/2*(-25*z - 5)]
8+3i	[1/2*(45*z - 35)]	[1/2*(-45*z - 35)]	[1/2*(-45*z + 35)]	[1/2*(45*z + 35)]
3+8i	[1/2*(35*z + 65)]	[1/2*(-35*z + 65)]	[1/2*(-35*z - 65)]	[1/2*(35*z - 65)]
8+5i	[10*z]	[-10*z]	[10*z]	[-10*z]
5+8i	[1/2*(25*z + 25)]	[1/2*(-25*z + 25)]	[1/2*(25*z + 25)]	[1/2*(-25*z + 25)]
9+4i	[35*z - 10]	[-35*z - 10]	[-35*z + 10]	[35*z + 10]
4+9i	[1/2*(-15*z + 5)]	[1/2*(15*z + 5)]	[1/2*(15*z - 5)]	[1/2*(-15*z - 5)]
10+i	[25*z + 35]	[-25*z + 35]	[25*z + 35]	[-25*z + 35]
1+10i	[1/2*(-25*z + 45)]	[1/2*(25*z + 45)]	[1/2*(-25*z + 45)]	[1/2*(25*z + 45)]
10+3i	[1/2*(45*z + 75)]	[1/2*(-45*z + 75)]	[1/2*(45*z + 75)]	[1/2*(-45*z + 75)]
3+10i	[30*z + 25]	[-30*z + 25]	[30*z + 25]	[-30*z + 25]
8+7i	[1/2*(45*z + 15)]	[1/2*(-45*z + 15)]	[1/2*(-45*z - 15)]	[1/2*(45*z - 15)]
7+8i	[1/2*(-25*z - 35)]	[1/2*(25*z - 35)]	[1/2*(25*z + 35)]	[1/2*(-25*z + 35)]
11	[-15]	[-15]	[-15]	[-15]
11+4i	[10*z + 15]	[-10*z + 15]	[-10*z - 15]	[10*z - 15]
4+11i	[-5*z - 35]	[5*z - 35]	[5*z + 35]	[-5*z + 35]
10+7i	[10*z]	[-10*z]	[10*z]	[-10*z]
7+10i	[-10*z - 50]	[10*z - 50]	[-10*z - 50]	[10*z - 50]
11+6i	[-15*z + 15]	[15*z + 15]	[15*z - 15]	[-15*z - 15]
6+11i	[15*z - 10]	[-15*z - 10]	[-15*z + 10]	[15*z + 10]
13+2i	[1/2*(25*z - 85)]	[1/2*(-25*z - 85)]	[1/2*(-25*z + 85)]	[1/2*(25*z + 85)]
2+13i	[1/2*(-45*z + 15)]	[1/2*(45*z + 15)]	[1/2*(45*z - 15)]	[1/2*(-45*z - 15)]

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