Weight Reduction for Mod ℓ Bianchi Modular Forms

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Abstract

Let K be an imaginary quadratic field with class number one and ℓ be a rational prime that splits in K. We prove that mod ℓ, a system of Hecke eigenvalues occurring in the first cohomology group of some congruence subgroup Γ of $\mathrm{SL}_2(\mathcal{O}_K)$ can be realized, up to twist, in the first cohomology with trivial coefficients after increasing the level of Γ by (ℓ).

1. Motivation and Summary

Let $\mathbf{G}$ be a connected semisimple algebraic group defined over $\mathbb{Q}$. Let K be a maximal compact subgroup of the group of real points $G = \mathbf{G}(\mathbb{R})$ of $\mathbf{G}$ and denote by $X = G/K$ the associated global Riemannian symmetric space. An arithmetic subgroup Γ of G acts properly discontinuously on X. The locally symmetric space $Y := \Gamma\backslash X$ is an Eilenberg-MacLane space for Γ and the cohomology of Γ is equal to the cohomology of Y. That is

$$H^*(\Gamma, E) \simeq H^*(Y, \tilde{E})$$

where E is a rational finite dimensional representation of G over $\mathbb{C}$ and $\tilde{E}$ is the local system that E induces on Y. A theorem of Franke [Fr] describes the cohomology spaces $H^*(\Gamma, E)$ in terms of the automorphic forms attached to $\mathbf{G}$. If we take $\mathbf{G} = \mathbf{SL}_2$, then the Eichler-Shimura theorem [Sh, Chapter 8] says that the automorphic forms that appear in the cohomology spaces $H^1(\Gamma, E)$ are the classical modular forms.

Motivated by the above paragraph, we define a *Bianchi modular form* over an imaginary quadratic field K as an automorphic form attached to $G = \mathrm{Res}_{K/\mathbb{Q}}(\mathbf{SL}_2)$ that appears in some $H^1(\Gamma, E(\mathbb{C}))$ where Γ is a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ (the level) and $E(\mathbb{C})$ is a rational finite-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$ over $\mathbb{C}$ (the weight).

Harris-Soudry-Taylor, Taylor and Berger-Harcos [HST, Ta1, BH], under some hypothesis, were able to attach compatible families of λ-adic Galois representations of K to Bianchi modular forms in accordance with Langlands philosophy. In the reverse direction, it is natural to ask if mod ℓ Galois representations of K arise from mod ℓ Bianchi modular forms. We define a *mod ℓ Bianchi modular form* as a cohomology class in some $H^1(\Gamma, E \otimes \overline{\mathbb{F}}_\ell)$ where E is a rational finite-dimensional representation of $\mathrm{GL}_2(\mathcal{O}_K/(\ell))$ over $\mathbb{F}_\ell$. Unlike the case of the classical modular forms, mod ℓ Bianchi modular forms are not merely reductions of the (char 0) Bianchi modular forms. This is due to the possible torsion in the cohomology with coefficients over $\mathcal{O}_K$, see Taylor's thesis [Ta2].

Elstrod-Grunewald-Mennicke [EGM] were the first investigators of the connection between mod ℓ Bianchi modular forms and mod ℓ Galois representations of imaginary quadratic fields. In his paper [Fi], Figueiredo considered an analogue of Serre's conjecture in this setting but he only considered mod ℓ Bianchi modular forms with trivial weight. Motivated by a result of Ash and Stevens [AS2] for the classical modular forms, he assumed that a Hecke eigenvalue system attached to a mod ℓ Bianchi modular form, after increasing the level, would be attached, up to twist, to another form with trivial weight.

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In this paper, we prove that what Figueiredo assumed is true following the technique used by Ash and Stevens in [AS2]. Our main corollary is as follows

Corollary 1.1. *Let K be an imaginary quadratic field of class number one and $\mathcal{O}$ be its ring of integers. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}$ that is prime to the ideal (ℓ) where ℓ is a rational prime that is split in $\mathcal{O}$. Let Φ be a Hecke eigenvalue system occuring in $H^1(\Gamma_1(\mathfrak{a}), E)$ where E is a finite dimensional $\mathbb{F}_\ell[GL_2(\mathcal{O}/(\ell))]$-module. Then Φ occurs in $H^1(\Gamma_1(\mathfrak{a}\ell), \mathbb{F}_\ell)$, up to twist.*

As an immediate corollary of the above, we get

Corollary 1.2. *Mod ℓ, there are only finitely many eigenvalue systems with fixed level.*

Remarks Due to the possible existence of torsion in the second cohomology with integral coefficients, we cannot in general lift mod ℓ forms to characteristic 0. So our result can not directly be used to derive mod ℓ congruences between Bianchi modular forms.

While it is expected that our result holds also when the class number of h_K of K is greater than 1, a direct extension of the techniques used here runs into serious technical problems. In this case, it is best to pass to the adelic framework. Then the cohomology of the arithmetic hyperbolic 3-orbifold Y is given by $\bigoplus_{i=1}^{h_K} H^1(\Gamma_i, E)$ where Γ_i's are certain discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$, each corresponding to a class in the ideal class group of K. The main frustration for our techniques is that the Hecke operators corresponding to non-principal ideal do not fix the summands of this direct sum.

Nevertheless, it is most convenient (both theoretically and computationally) to work with class number 1 fields to investigate the arithmetic aspects of Bianchi modular forms. Thus we believe our result will be a useful addition to the literature, even though there are only nine such imaginary quadratic fields,

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Notation Once and for all, fix a quadratic imaginary field K of class number one and an ideal $\mathfrak{a}$ of $\mathcal{O} = \mathcal{O}_K$. Also fix a rational prime ℓ that is coprime to $\mathfrak{a}$ and splits in $\mathcal{O}$ as $\ell = \lambda\bar{\lambda}$. Let $\mathfrak{b}$ be an arbitrary ideal. We use the following notation:

$$\begin{aligned}
\mathrm{M}_2(\mathcal{O}) &: \quad \text{matrices in } \mathrm{GL}_2(K) \text{ with entries in } \mathcal{O} \\
\Gamma_0(\mathfrak{b}) &: \quad \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : c \equiv 0 \mod \mathfrak{b} \right\} \\
\Gamma_1(\mathfrak{b}) &: \quad \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : a-1 \equiv c \equiv d-1 \equiv 0 \mod \mathfrak{b} \right\} \\
\Delta &: \quad \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O}) : c \equiv 0 \mod \mathfrak{a} \right\} \\
\Delta(\mathfrak{b}) &: \quad \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O}) : c \equiv 0 \mod \mathfrak{a}\mathfrak{b} \right\} \\
P(\mathfrak{b}) &: \quad \left\{ g \in \Gamma_1(\mathfrak{a}) : g \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod \mathfrak{b} \right\} \\
\Gamma &: \quad \Gamma_1(\mathfrak{a}) \\
\Gamma(\mathfrak{b}) &: \quad \Gamma_1(\mathfrak{a} \cdot \mathfrak{b}) \\
\Gamma^0(\mathfrak{b}) &: \quad \Gamma_1(\mathfrak{a}) \cap \Gamma_0(\mathfrak{b})
\end{aligned}$$

2. Hecke Operators on Cohomology

In this section, we describe the Hecke operators on the cohomology. Let R be a ring and $\tilde{\alpha} = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where α is a prime element of $\mathcal{O}$. We follow the standard notations and put $\Gamma_\alpha := \Gamma \cap \tilde{\alpha}^{-1}\Gamma\tilde{\alpha}$ and $\Gamma^\alpha := \Gamma \cap \tilde{\alpha}\Gamma\tilde{\alpha}^{-1}$.

Let V be a right $R[\mathrm{M}_2(\mathcal{O})]$-module. We define the Hecke operator T_α on the cohomology as the composition

$$\begin{tikzcd}
H^1(\Gamma, V) \arrow[d, "res"] & H^1(\Gamma, V) \\
H^1(\Gamma_\alpha, V) \arrow[r, "\hat{\alpha}"] & H^1(\Gamma^\alpha, V) \arrow[u, "cores"']
\end{tikzcd}$$

where the map $\hat{\alpha}$ is defined by

$$c \mapsto (g \mapsto c(\alpha^{-1} g \alpha) \cdot \alpha^\iota)$$

where c is a cocycle in $H^1(\Gamma_\alpha, V)$ and $\alpha^\iota = \det(\alpha)\alpha^{-1}$.

One can describe Hecke operators T_α explicitly: suppose $\Gamma\alpha\Gamma = \bigsqcup_{i=1}^m \gamma_i \Gamma$. Given $g \in \Gamma$ and γ_i, there is a unique $\gamma_{j(i)}$ such that $\gamma_{j(i)}^{-1} g \gamma_i \in \Gamma$. Then

$$(T_\alpha c)(g) = \sum_{1 \le i \le m} c(\gamma_{j(i)}^{-1} g \gamma_i) \cdot \gamma_i^\iota$$

for all cocycles $c \in H^1(\Gamma, V)$ and $g \in \Gamma$. We note that this formula agrees with the one given in [AS1, p.194].

We define the *Hecke algebra* $\mathbb{H}$ as the subalgebra of the endomorphisms algebra of $H^1(\Gamma, V)$ that is generated by the T_π's where π is a prime of K. Note that $\mathbb{H}$ is a commutative algebra.

The induced module $Ind(V) = Ind(\Gamma, \Gamma(\mathfrak{b}), V)$ is the set of $\Gamma(\mathfrak{b})$-invariant maps from Γ to V, that is

$$Ind(V) = \{f : \Gamma \to V \mid f(gh) = f(g) \cdot h \text{ for all } h \in \Gamma(\mathfrak{b})\}.$$

Then $Ind(V)$ is a right Γ-module with the action $(f \cdot y)(x) = f(yx)$ for $x, y \in \Gamma$ and $f \in Ind(V)$.

We can extend the Γ-action on $Ind(V)$ to a right Δ-action in the following way. Let $\alpha \in \Delta$ and $f \in Ind(V)$ and $x \in \Gamma$, then there are $\beta \in \Delta(\mathfrak{b})$ and $y \in \Gamma$ such that $\alpha x = y\beta$. We define

$$(f \cdot \alpha)(x) = f(y) \cdot \beta.$$

A key tool is Shapiro's lemma:

Proposition 2.1. *There is an isomorphism*

$$\theta : H^1(\Gamma, Ind(V)) \to H^1(\Gamma(\mathfrak{b}), V)$$

given by $f \mapsto f(I)$ *for every cocycle* f *in* $H^1(\Gamma, Ind(V))$ *where* I *denotes the identity matrix. Moreover, the Hecke operators commute with the Shapiro map* θ.

The fact that the Hecke operators commute with the Shapiro isomorphism θ was proved in a more general setting in [AS1]. See also [Wi] for a proof in the case of $\mathrm{PSL}_2(\mathbb{Z})$ using the same construction as ours for the Hecke operators.

A *system of eigenvalues* of $\mathbb{H}$ with values in a ring R is a ring homomorphism $\Phi : \mathbb{H} \to R$. We say that an eigenvalue system Φ occurs in the $R\mathbb{H}$-module A if there is a nonzero element $a \in A$ such that $Ta = \Phi(T)a$ for all T in $\mathbb{H}$.

The following lemma is proved in [AS1, Lemma 2.1].

Lemma 2.2. *Let* F *be a field and* V *be a* $F\Delta$*-module which is finite dimensional over* F*. If an eigenvalue system* $\Phi : \mathbb{H} \to F$ *occurs in* $H^n(\Gamma, V)$*, then* Φ *occurs in* $H^n(\Gamma, W)$ *for some irreducible* $F\Delta$*-subquotient* W *of* V.

Thus it is enough to investigate the cohomology with irreducible coefficient modules if we are only interested in the eigenvalue systems. In the next two sections, we discuss the irreducible $\mathbb{F}_\ell[\mathrm{GL}_2(\mathcal{O}/(\ell))]$-modules.

3. The Irreducible Modules

For a positive integer r, let $E_r = E_r(\mathbb{F}_\ell)$ be the the space of degree r homogeneous polynomials in two variables over $\mathbb{F}_\ell$. As the quotient rings $\mathcal{O}/\lambda$ and $\mathcal{O}/\bar{\lambda}$ are canonically isomorphic to $\mathbb{F}_\ell$, we have $\mathrm{M}_2(\mathcal{O})$ acting on $E_k(\mathbb{F}_\ell)$ in two different ways: through reduction by λ and by $\bar{\lambda}$.

For $k \geq 0$, we put $E_r^k(\mathbb{F}_\ell) := \det^k \otimes_{\mathbb{F}_\ell} E_r(\mathbb{F}_\ell)$. We consider the $\mathrm{M}_2(\mathcal{O})$-modules

$$E_{r,s}^{a,b}(\mathbb{F}_\ell) := E_r^a(\mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} E_s^b(\mathbb{F}_\ell).$$

Here $\mathrm{M}_2(\mathcal{O})$ acts on $E_r^a(\mathbb{F}_\ell)$ (resp. on $E_s^b(\mathbb{F}_\ell)$) via reduction by λ (resp. by $\bar{\lambda}$). We write $E_{r,s}$ when $a = b = 0$.

It is a result of Brauer and Nesbitt [BN] that the absolutely irreducible representations of $\mathrm{GL}_2(\mathcal{O}/(\ell))$ over $\mathbb{F}_\ell$ are $E_{r,s}^{a,b}(\mathbb{F}_\ell)$'s with $0 \leq r, s \leq \ell - 1$ and $0 \leq a, b \leq l - 2$.

Let E be a $\mathbb{F}_\ell[\mathrm{M}_2(\mathcal{O})]$-module. Given $0 \leq a, b \leq l - 2$, we mean by

$$H^*(\Gamma, E)^{(a,b)}$$

the cohomology group $H^*(\Gamma, E)$ twisted as a Hecke module. More precisely, let v be an element of $H^*(\Gamma, E)$. Denote it as v' when viewed as an element of $H^*(\Gamma, E)^{(a,b)}$. Let τ_1, τ_2 be the reduction maps from $\mathcal{O}$ to $\mathbb{F}_\ell$ by λ and $\bar{\lambda}$ respectively. Given a Hecke operator T_π, we have

$$T_\pi(v') = \tau_1(\pi)^a \tau_2(\pi)^b T_\pi(v)$$

The following observation is immediate.

Lemma 3.1. *We have*

$$H^*(\Gamma, E_{r,s}^{a,b}) \simeq H^*(\Gamma, E_{r,s})^{(a,b)}$$

as Hecke modules.

4. Induced Modules

Let $\chi : \Gamma^0(\ell)/\Gamma(\ell) \to \mathbb{F}_\ell^*$ be a homomorphism. For any $\mathbb{F}_\ell[\Delta]$-module E, we define $H^*(\Gamma(\ell), \chi, E)$ as the submodule of all $v \in H^*(\Gamma(\ell), E)$ such that $v \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = v \cdot \chi(d)$ for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^0(\ell)$.

We have

$$H^1(\Gamma(\ell), \mathbb{F}_\ell) \simeq \bigoplus_\chi H^1(\Gamma(\ell), \chi, \mathbb{F}_\ell) \simeq \bigoplus_\chi H^1(\Gamma^0(\ell), (\mathbb{F}_\ell)^\chi)$$

where $(\mathbb{F}_\ell)^\chi$ is the rank one $\mathbb{F}_\ell$-module on which $\Gamma^0(\ell)$ acts via χ. The last isomorphism follows from Lemma 1.1.5 of [AS1]. Using Shapiro's lemma, we relate these to the cohomology of Γ.

$$H^1(\Gamma(\ell), \mathbb{F}_\ell) \simeq \bigoplus_\chi H^1(\Gamma, \mathrm{Ind}(\Gamma^0(\ell), \Gamma, (\mathbb{F}_\ell)^\chi)).$$

Let I be the set of $\mathbb{F}_\ell$ valued functions on $\mathbb{F}_\ell^2$ which vanish at the origin. The semigroup Δ acts on I both by reduction by λ and by $\bar{\lambda}$. The action is given by

$$(f \cdot M)(a, b) = f((a, b)M^t)$$

for $f \in I$, $(a, b) \in \mathbb{F}_\ell^2$ and $M \in \Delta$.

We have the decomposition

$$I \simeq \bigoplus_{n=0}^{\ell-2} I_n.$$

where I_n is the Δ-submodule of functions $f \in I$ such that $f((xa, xb)) = x^n f((a, b))$. Observe that $I_k = I_{k+l-1}$ and that every I_n is $\ell + 1$ dimensional.

Let $\chi_1 : (\mathcal{O}/\lambda)^* \to \mathbb{F}_\ell^*$ and $\chi_2 : (\mathcal{O}/\bar{\lambda})^* \to \mathbb{F}_\ell^*$ be the restrictions of the canonical isomorphisms to the units. We have the following isomorphisms of Δ-modules

$$\mathrm{Ind}(\Gamma^0(\lambda), \Gamma, (\mathbb{F}_\ell)^{\chi_i^k}) \simeq I_k$$

for $i = 1, 2$ and $0 \le k \le l - 2$. Here Δ acts on I_k via reduction through λ (resp. $\bar{\lambda}$) when $i = 1$ (resp. $i = 2$).

As $\Gamma^0(\ell)/\Gamma(\ell) \simeq (\mathcal{O}/\ell)^*$, any homomorphism $\chi : \Gamma^0(\ell)/\Gamma(\ell) \to \mathbb{F}_\ell^*$ can be written uniquely as a product $\chi_1^r \cdot \chi_2^s$ for some $0 \le r, s \le l - 1$. In this case, we denote χ as $\chi(r, s)$.

Lemma 4.1. *Let* $0 \le r, s, \le l - 1$. *Then*

$$H^1(\Gamma, I_r \otimes_{\mathbb{F}_\ell} I_s) \simeq H^1(\Gamma(\ell), \chi(r, s), \mathbb{F}_\ell)$$

as Hecke modules.

Proof. As before, we have

$$H^1(\Gamma(\ell), \chi(r, s), \mathbb{F}_\ell)) \simeq H^1(\Gamma^0(\ell), (\mathbb{F}_\ell)^{\chi(r,s)}) \simeq H^1(\Gamma, \mathrm{Ind}(\Gamma^0(\ell), \Gamma, (\mathbb{F}_\ell)^{\chi(r,s)})).$$

So it suffices to show that

$$\mathrm{Ind}(\Gamma^0(\ell), \Gamma, (\mathbb{F}_\ell)^{\chi(r,s)}) \simeq \mathrm{Ind}(\Gamma^0(\lambda), \Gamma, (\mathbb{F}_\ell)^{\chi_1^r}) \otimes \mathrm{Ind}(\Gamma^0(\bar{\lambda}), \Gamma, (\mathbb{F}_\ell)^{\chi_2^s}).$$

Observe that $P(\ell)$, the intersection of the principal congruence subgroup of level ℓ and Γ, acts trivially on $(\mathbb{F}_\ell)^{\chi(r,s)}$. Thus after factoring, we get

$$\mathrm{Ind}(\Gamma^0(\ell), \Gamma, (\mathbb{F}_\ell)^{\chi(r,s)}) \simeq \mathrm{Ind}(\mathrm{B}(\mathcal{O}/\ell), \mathrm{SL}_2(\mathcal{O}/\ell), (\mathbb{F}_\ell)^{\chi(r,s)})$$

where $\mathrm{B}(\mathcal{O}/\ell)$ is the subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathcal{O}/\ell)$. Notice that we have $\mathrm{B}(\mathcal{O}/\ell) \simeq \mathrm{B}(\mathcal{O}/\lambda) \times \mathrm{B}(\mathcal{O}/\bar{\lambda})$ and $\mathrm{SL}_2(\mathcal{O}/\ell) \simeq \mathrm{SL}_2(\mathcal{O}/\lambda) \times \mathrm{SL}_2(\mathcal{O}/\bar{\lambda})$. This gives

$$\mathrm{Ind}(\mathrm{B}(\mathcal{O}/\ell), \mathrm{SL}_2(\mathcal{O}/\ell), (\mathbb{F}_\ell)^{\chi(r,s)}) \simeq \mathrm{Ind}(\mathrm{B}(\mathcal{O}/\lambda), \mathrm{SL}_2(\mathcal{O}/\lambda), (\mathbb{F}_\ell)^{\chi_1^r}) \otimes \mathrm{Ind}(\mathrm{B}(\mathcal{O}/\bar{\lambda}), \mathrm{SL}_2(\mathcal{O}/\bar{\lambda}), (\mathbb{F}_\ell)^{\chi_2^s}).$$

The claim follows easily from here. □

5. Exact Sequences

We will need the following two facts, see [AS2, Section 3].

Lemma 5.1. *For* $0 \le r \le \ell - 1$, *there are* $SL_2(\mathcal{O})$*-invariant perfect pairings*

(1) $E_r \times E_r \to \mathbb{F}_\ell$

(2) $I_r \times I_{\ell-1-r} \to \mathbb{F}_\ell$

Let $0 \le r \le \ell - 1$. As in [AS2], we consider the following $SL_2(\mathcal{O})$-invariant maps. Each polynomial in E_r can be seen as a function on $\mathbb{F}_\ell^2$. This gives us a morphism $\alpha_r : E_g \to I_r$. Let $\beta_r : I_r \to E_{\ell-1-r}^r$ be given by

$$\beta_r(f) = \sum_{(a,b) \in \mathbb{F}_\ell^2} f(a, b)(bX - aY)^{\ell-1-r}.$$

Lemma 5.2. *For* $0 \le r \le \ell - 1$, *we have the following exact sequence of* Δ*-modules*

$$\begin{tikzcd} 0 \arrow[r] & E_r \arrow[r, "\alpha_r"] & I_r \arrow[r, "\beta_r"] & E_{\ell-1-r}^r \arrow[r] & 0 \end{tikzcd}$$

Remark 5.3. Lemma 5.2 shows that the semisimplification of I_r is $E_r \oplus E^r_{\ell-1-r}$. There is another way to see this. As we explained in the proof of Lemma 4.1, I_r is the induction of the one dimensional representation χ^r of the Borel subgroup of $\mathrm{SL}_2(\mathbb{F}_\ell)$ to all of $\mathrm{SL}_2(\mathbb{F}_\ell)$. One can identify the semisimplification of this $\ell+1$ dimensional representation by a calculation of Brauer characters. This has been done by Diamond in [Di, Prop 1.1.].

Definition 5.4. For given nonnegative integers r, s, we define the following Δ-modules where Δ acts on the components of every tensor product through reduction by λ and $\bar{\lambda}$ respectively.

1. $I_{r,s} := I_r \otimes I_s$;
2. $U_{r,s} := [E^r_{\ell-1-r} \otimes I_s] \oplus [I_r \otimes E^s_{\ell-1-s}]$;
3. $V_{r,s} := E^r_{\ell-1-r} \otimes E^s_{\ell-1-s}$.

We have Δ-module morphisms

$$\pi : I_{r,s} \to U_{r,s} \qquad \text{defined by} \qquad \pi := [\beta_r \otimes \mathrm{id}] \oplus [\mathrm{id} \otimes \beta_s]$$

and

$$\pi' : U_{r,s} \to V_{r,s} \qquad \text{defined by} \qquad \pi' := \mathrm{id} \otimes \beta_s - \beta_r \otimes \mathrm{id}.$$

Lemma 5.5. *Let the notation be as above. Let $0 \leq r \leq \ell - 1$ and $0 \leq s \leq \ell - 1$. We have the following exact sequence Δ-modules:*

$$\begin{tikzcd} 0 \arrow[r] & E_{r,s} \arrow[r, "\iota"] & I_{r,s} \arrow[r, "\pi"] & U_{r,s} \arrow[r, "\pi'"] & V_{r,s} \arrow[r] & 0 \end{tikzcd} .$$

Proof. Note that Δ-modules in question are flat since they are also $\mathbb{F}_\ell$-vector spaces. So, by Lemma 5.2, ι is injective. One can easily see that $Im(\iota) \subseteq Ker(\pi)$ and π' is surjective. Thus, in order to complete the proof, it suffices to show that $\dim(Im(\pi)) = (\ell+1)^2 - (r+1)(s+1)$; this is what we do below.

Identifying E_r with its image in I_r, we can write the vector space decomposition $I_r = E_r \oplus E_{\ell-1-r}$. Now, it is evident that $\dim(\pi(E_r \otimes I_s)) = (r+1)(\ell-s)$ and that $\dim(\pi(E_{\ell-1-r} \otimes I_s)) = (\ell-r)(\ell+1)$. Elementary linear algebra shows that these images have trivial intersection and this gives us the desired dimension. ☐

Setting $W_{r,s} := ker(\pi' : U_{r,s} \to V_{r,s})$, by Lemma 5.5, we get two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & E_{r,s} \arrow[r, "\iota"] & I_{r,s} \arrow[r, "\pi"] & W_{r,s} \arrow[r] & 0 \end{tikzcd} \tag{1}$$

and

$$\begin{tikzcd} 0 \arrow[r] & W_{r,s} \arrow[r, "i"] & U_{r,s} \arrow[r, "\pi'"] & V_{r,s} \arrow[r] & 0 \end{tikzcd} . \tag{2}$$

6. Invariants

For convenience, we will write $\mathbb{F}_\ell(g)$ for the module $E^{g;g}_{0,0}$ which we defined in Section 3.

Lemma 6.1. *For any nonnegative integers r, s, we we have the following isomorphism of Hecke modules*

$$H^0(\Gamma, I_{r,s}) \cong \begin{cases} \mathbb{F}_\ell(\ell-1) & \textit{if } r \equiv s \equiv 0 \textit{ (mod } \ell - 1\textit{)} \\ \textit{0} & \textit{otherwise} \end{cases}$$

Proof. By Shapiro's Lemma, we have $H^0(\Gamma, I_{r,s}) \simeq H^0(\Gamma^0(\ell), (\mathbb{F}_\ell)^{\chi(r,s)})$. In action of $\Gamma^0(\ell)$ on $\mathbb{F}_\ell$ through $\chi(r,s)$, either there are no nontrivial invariants or the whole space is fixed which means that $\chi(r,s)$ acts trivially. By the Chinese Remainder Theorem, this is possible if and only if χ_1^r and χ_2^s act trivially. Hence the congruence conditon of the claim. One can directly check that the Hecke action is as described. □

Lemma 6.2. *Assume* $0 \le r, s \le \ell - 1$. *Then, we have the following isomorphism of Hecke modules*

$$H^0(\Gamma, E_{r,s}) \quad = \begin{cases} \mathbb{F}_\ell & \text{if } r = s = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The claim is obvious when $(r,s) = (0,0)$. Assume $(r,s) \neq (0,0)$ and $(r,s) \neq (\ell-1, \ell-1)$. Then, the exact sequence (1) induces the following exact sequence

$$0 \to H^0(\Gamma, E_{r,s}) \to H^0(\Gamma, I_{r,s}).$$

By Lemma 6.1, $H^0(\Gamma, I_{r,s}) = 0$ and so is $H^0(\Gamma, E_{r,s})$.

Assume $(r,s) = (\ell-1, \ell-1)$. In [Ds], Dickson showed that Γ-invariant subring of the ring of polynomials in two variables over $\mathbb{F}_\ell$ is generated by $X^\ell Y - XY^\ell$ and $\sum_{i=0}^{\ell} (X^{\ell-i} Y^i)^{\ell-1}$. This implies that $H^0(\Gamma, E_{\ell-1,\ell-1}) = 0$. □

Lemma 6.3. *Let* $0 \le r, s \le \ell - 1$. *Then, we have the following isomorphism of Hecke modules*

$$H^0(\Gamma, U_{r,s}) \quad = \begin{cases} \mathbb{F}_\ell(\ell-1) \oplus \mathbb{F}_\ell(\ell-1) & \text{if } (r,s) = (\ell-1, \ell-1) \\ \mathbb{F}_\ell(\ell-1) & \text{if } (r,s) = (0, \ell-1) \text{ or } (\ell-1, 0) \\ 0 & \text{otherwise} \end{cases}$$

Proof. Set $U^1 := E_{\ell-1-r}(r) \otimes I_s$ and $U^2 = I_r \otimes E_{\ell-1-s}(s)$. Then, $U_{r,s} = U^1 \oplus U^2$ and $H^0(\Gamma, U^1) \oplus H^0(\Gamma, U^2)$.

Assume (r,s) is not of $(\ell-1, \ell-1)$, $(0, \ell-1)$ and $(\ell-1, 0)$. Then, tensoring the exact sequence in Lemma 5.2 with $E_{\ell-1-r}(r)$, we get the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E_{\ell-1-r}(r) \otimes E_s \arrow[r] & U^1 \arrow[r] & V_{r,s} \arrow[r] & 0 \end{tikzcd}.$$

This induces the following long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(\Gamma, E_{\ell-1-r}(r) \otimes E_s) \arrow[r] & H^0(\Gamma, U^1) \arrow[r] & H^0(\Gamma, V_{r,s}) \end{tikzcd}.$$

Since $V_{r,s} \cong E_{\ell-1-r, \ell-1-s}$ as Γ-modules, by Lemma 6.2, $H^0(\Gamma, V_{r,s}) = 0$. On the other hand, by Lemma 6.2, $H^0(\Gamma, E_{\ell-1-r}(r) \otimes E_s) = 0$ and $H^0(\Gamma, U^1) = 0$. Likewise, one tensors the exact sequence in Lemma 5.2 with $E_{\ell-1-s}(s)$ and gets $H^0(\Gamma, U^2) = 0$, hence the vanishing of $H^0(\Gamma, U_{r,s})$.

One establishes the result for the remaining cases using Lemma 6.2 and the exact sequence of cohomology groups above, noting that $\pi'|_{U^1} : U^1 \to V_{r,s}$ is surjective. Checking the action of the Hecke algebra completes the proof. □

Lemma 6.4. *Let* $0 \le r, s \le \ell - 1$. *Then, we have the following isomorphism of Hecke modules*

$$H^0(\Gamma, W_{r,s}) \quad = \begin{cases} \mathbb{F}_\ell & \text{if } (r,s) = (\ell-1, \ell-1), (0, \ell-1) \text{ or } (\ell-1, 0) \\ 0 & \text{otherwise} \end{cases}$$

Proof. First of all, the exact sequence (2) above induces the following long exact sequence of Hecke modules in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(\Gamma, W_{r,s}) \arrow[r, "i_*"] & H^0(\Gamma, U_{r,s}) \arrow[r, "\pi'_*"] & H^0(\Gamma, V_{r,s}) \arrow[r] & H^1(\Gamma, W_{r,s}) \end{tikzcd} .$$

Assume $(r, s) = (\ell - 1, \ell - 1)$. Then, by Lemma 6.2, $H^0(\Gamma, V_{r,s}) \cong \mathbb{F}_\ell$ and, by Lemma 6.3, $H^0(\Gamma, U_{r,s}) \cong \mathbb{F}_\ell \oplus \mathbb{F}_\ell$. Using the definition, one can easily see that π'_* is surjective and gets the desired result using the exact sequence of cohomology groups above.

The remaining cases follow similarly from Lemma 6.3. Checking the action of the Hecke algebra completes the proof. □

Remark 6.5. One can compute the above invariants using the following approach which was suggested by Gebhard Boeckle. As $P(\ell)$ acts trivially on $E_{r,s}$, $I_{r,s}$ and the direct summands of $U_{r,s}$, we get, for instance, $H^0(\Gamma, E_{r,s}) \simeq H^0(\mathrm{SL}_2(\mathcal{O}/\lambda) \times \mathrm{SL}_2(\mathcal{O}/\bar{\lambda}), E_{r,s})$. Since we are taking invariants, we get

$$H^0(\Gamma/(P(\ell) \cap \Gamma), E_{r,s}) \simeq H^0(\mathrm{SL}_2(\mathcal{O}/\lambda), E_r) \otimes H^0(\mathrm{SL}_2(\mathcal{O}/\bar{\lambda}), E_s).$$

This gives

$$H^0(\Gamma, E_{r,s}) \simeq H^0(\Gamma, E_r) \otimes H^0(\Gamma, E_s).$$

Now one can follow the proof of Lemma 3.3 of [AS2] to compute these invariants. Same approach applies to Lemmas 6.3 and 6.4 as well.

7. Proof Of The Theorem

We are now ready to prove our main result:

Theorem 7.1. *Let Φ be a Hecke eigenvalue system occuring in $H^1(\Gamma, E^{a,b}_{r,s})$ for some $0 \leq a, b \leq l - 2$ and $0 \leq r, s \leq l - 1$. Then Φ occurs in $H^1(\Gamma(\ell), \chi(r, s), \mathbb{F}_\ell)^{(a,b)}$.*

Proof. By Lemma 3.1, we have $H^1(\Gamma, E^{a,b}_{r,s}) \simeq H^1(\Gamma, E_{r,s})^{(a,b)}$. Exact sequence (1) induces the following long exact sequence of $\mathbb{H}$-modules

$$0 \to H^0(\Gamma, E_{r,s})^{(a,b)} \xrightarrow{\iota_*} H^0(\Gamma, I_{r,s})^{(a,b)} \xrightarrow{\pi'_*} H^0(\Gamma, W_{r,s})^{(a,b)} \to H^1(\Gamma, E_{r,s})^{(a,b)} \to H^1(\Gamma, I_{r,s})^{(a,b)}$$

We claim that the map $H^1(\Gamma, E_{r,s}) \to H^1(\Gamma, I_{r,s})$ is injective. Assume that (r, s) is equal to one of the tuples $(0, \ell - 1), (\ell - 1, 0)$ or $(\ell - 1, \ell - 1)$. Then, by Lemma 6.2, $H^0(\Gamma, E_{r,s}) = 0$; by Lemma 6.1, $H^0(\Gamma, I_{r,s}) \cong \mathbb{F}_\ell$ and, by Lemma 6.4, $H^0(\Gamma, W_{r,s}) \cong \mathbb{F}_\ell$ (as vector spaces). By the definition, π'_* is surjective and thus we get the claim. Otherwise, by Lemma 6.4, $H^0(\Gamma, W_{r,s}) = 0$. Now the result follows from Proposition 4.1 □

Let Φ be a Hecke eigenvalue system occuring in $H^1(\Gamma, E)$ where E is some rational finite dimensional $\mathbb{F}_\ell[\mathrm{GL}_2(\mathcal{O}/(\ell))]$-module. Then Lemma 2.2 tells us that Φ can be realized in some $H^1(\Gamma, E^{a,b}_{r,s})$ with $0 \leq a, b \leq l - 2$ and $0 \leq r, s \leq l - 1$. Thus our main theorem implies the followings as we announced in the introduction.

Corollary 7.2. *Let Φ be a Hecke eigenvalue system occuring in $H^1(\Gamma_1(\mathfrak{a}), E)$ where E is a finite dimensional $\mathbb{F}_\ell[GL_2(\mathcal{O}/(\ell))]$-module. Then Φ occurs in $H^1(\Gamma_1(\mathfrak{a}\ell), \mathbb{F}_\ell)$, up to determinant twist.*

For congruence subgroups of $SL_2(\mathbb{Z})$, the following was first proved by Tate-Serre for level 1 (unpublished), by Jochnowitz [J] for prime levels less than 19 and for arbitrary levels by Ash-Stevens [AS2].

Corollary 7.3. *The set of Hecke eigenvalue systems occuring in* $H^1(\Gamma_1(\mathfrak{a}), E)$ *for fixed* $\mathfrak{a}$ *and varying* E*, where* E *is a rational finite dimensional* $\mathbb{F}_\ell[GL_2(\mathcal{O}/(\ell))]$*-module, is finite.*

It is natural to ask whether increasing the level by (ℓ) is optimal. In other words, are there eigenvalue systems with nontrivial weight which have no twists that occur with trivial weight when the level is increased by (λ) or $(\bar{\lambda})$. One can see, by the methods we used in this note, that the answer to this question is positive if $r = 0$ or $s = 0$. It looks like this is the only case where the answer is positive. We present a numerical example to support this speculation.

Example 7.4. Let $\mathcal{O} = \mathbb{Z}(\omega)$ where $\omega = \sqrt{-2}$. Using the programs developed by the first author in his doctoral thesis [Sen], we find an eigenform v in $H^1(\mathrm{PSL}_2(\mathcal{O}), E_{10,10}(\mathbb{F}_{11}))$. The following table gives eigenvalues Φ_α of v for the first few Hecke operators T_α.

α	ω	$1+\omega$	$1-\omega$	$3+2\omega$	$3-2\omega$	$1+3\omega$	$1-3\omega$	$3+4\omega$	$3-4\omega$
Φ_α	9	10	10	9	9	0	0	5	5

Note that we have $11 = (3+\omega)(3-\omega)$. The spaces $H^1(\Gamma_0(3+\omega), \mathbb{F}_{11})$ and $H^1(\Gamma_0(3-\omega), \mathbb{F}_{11})$ are isomorphic and they are two dimensional. Our eigenvalue system Φ does not occur in these spaces. Next, we examine $H^1(\Gamma_0(11), \mathbb{F}_{11})$. We find an eigenvalue system that is the reduction of a characteristic 0 system. Indeed, we find an eigenvector in $H^1(\Gamma_0(11), \mathcal{O})$ with the following eigenvalues Ψ_α.

α	ω	$1+\omega$	$1-\omega$	$3+2\omega$	$3-2\omega$	$1+3\omega$	$1-3\omega$	$3+4\omega$	$3-4\omega$
Ψ_α	-2	-1	-1	-2	-2	0	0	-6	-6

Reducing these eigenvalues mod 11, we get an eigenvalue system in $H^1(\Gamma_0(11), \mathbb{F}_{11})$ that matches (we computed only the first 20 split primes) our level 1 weight $(10, 10)$ eigenvalue system Φ.

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