Seminar on Wild Ramification

Summer 2010
Time: Tuesday 2 pm (first talk 27.4.)
Hélène Esnault, Moritz Kerz

Abstract

In this seminar, which we plan to run at least every two weeks, we want to learn the local and global aspects of the theory of wild ramification of Galois extensions of complete discretely valued fields with imperfect residue fields. This theory was developed by K. Kato, A. Abbes and T. Saito since the late 80s and extends the classical theory in the perfect residue field case. It is related to higher dimensional class field theory and there are analogies with the theory of irregular singularities of differential equations over the complex numbers. The latter topic will show up in the seminar as a motivation.

In the seminar we would like to understand the following constructions:

- For a henselian discrete valuation ring R, $K = Q(T)$, $F = R/\mathfrak{m}$ we study a descending ramification filtration

  $$G^r_{K,log} \subset \mathrm{Gal}(\bar{K}/K), \qquad r \in \mathbb{Q}$$

  which generalizes the usual ramification filtration in the perfect residue field case.

- For a smooth variety X over a perfect field, an effective simple normal crossing divisor $D \hookrightarrow X$ and a reasonable smooth l-adic sheaf $\mathcal{F}$ on $X - D$ (whose ramification along $|D|$ is bounded by the multiplicities of D) we study the characteristic $\dim(X)$-cycle $CC(\mathfrak{F})$ on the logarithmic cotangent bundle $T^*X(\log D)$.

- For $D \hookrightarrow X$ and $\mathcal{F}$ as above we study the Swan class $\mathrm{Sw}(\mathcal{F}) \in \mathrm{CH}_0(|D|)$, which allows us to formulate a generalization of the Grothendieck-Ogg-Shafarevich formula for higher dimensional varieties.

Talks

The following list of talks is just meant to be a rough schedule! Especially several people might share one topic.

1) Perfect residue field case: (Nguyen Duy Tan, Oliver Bültel)
Recall the classical definition of ramification groups in the perfect residue field case [11, Ch. IV], the Hasse-Arf Theorem (without proof), the Artin and Swan representations [11, Ch. VI], [8, Sec. V.5] and the Grothendieck-Ogg-Shararevich formula [8, Sec. V.6].

2) Rank one case: (Hélène Esnault)
In this talk we define the above invariants in the case of abelian extensions and sheaves of rank one. Before we do this we motivate the whole construction by the theory of irregular singularities of $\mathcal{D}$-modules over the complex numbers. Follow [5] but restrict to the equicharacteristic case, do not give proofs for Sections 4 and 5. Quote the necessary results from [4] without proofs.

3) Higher dimensional class field theory: (Ivan Barrientos)
Explain the class field theory of henselian local rings with residue fields, which are finitely generated over finite fields, [6, Sec. 7+8]. Explain very roughly how this is applied to the proof of the main theorem of global class field theory over finite fields, [6, Sec. 9].

4) Ramification filtration: (Christian Kappen)
Explain the details of the (log) ramification filtration in the imperfect residue field case following [9, Sec. 1]. Skip the proof of Theorem 1.24.

5) Global constructions: (Kay Rülling)
Explain the construction of the characteristic cycle, the Swan class and the higher dimensional generalization of the Grothendieck-Ogg-Shafarevich Theorem due to Abbes, Kato and Saito [9, Sec. 2+3].

Further details or alternative approaches to the last two topics can be found in [1], [2], [7], [3]. For an overview see [10].

References

[1] Abbes, Ahmed; Saito, Takeshi *Ramification of local fields with imperfect residue fields.* Amer. J. Math. 124 (2002), no. 5, 879–920.

[2] Abbes, Ahmed; Saito, Takeshi *Ramification of local fields with imperfect residue fields. II.* Kazuya Kato's fiftieth birthday. Doc. Math. 2003, Extra Vol., 5–72.

[3] Abbes, Ahmed; Saito, Takeshi *The characteristic class and ramification of an l-adic étale sheaf.* Invent. Math. 168 (2007), no. 3, 567–612.

[4] Kato, Kazuya *Swan conductors for characters of degree one in the imperfect residue field case.* Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), 101–131, Contemp. Math., 83, Amer. Math. Soc., Providence, RI, 1989.
Remark: We have a papercopy for distribution.

[5] Kato, Kazuya *Class field theory, $\mathcal{D}$-modules, and ramification on higher-dimensional schemes.* Amer. J. Math. 116 (1994), no. 4, 757–784.

[6] Kato, Kazuya; Saito, Shuji *Global class field theory of arithmetic schemes.* Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 255–331, Contemp. Math., 55, Amer. Math. Soc., Providence, RI, 1986.

[7] Kato, Kazuya; Saito, Takeshi *Ramification theory for varieties over a perfect field.* Ann. of Math. (2) 168 (2008), no. 1, 33–96.

[8] Kiehl, Reinhardt; Weissauer, Rainer *Weil conjectures, perverse sheaves and l-adic Fourier transform.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 2001.

[9] Saito, Takeshi *Wild ramification and the characteristic cycle of an l-adic sheaf.* J. Inst. Math. Jussieu 8 (2009), no. 4, 769–829.

[10] Saito, Takeshi *Wild ramification of schemes and sheaves.* Proceedings of the International Congress of Mathematicians, Hyderabad, India (2010). (to appear)

[11] Serre, Jean-Pierre *Local fields.* Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979.