

Cohomological mod ℓ modular forms over $\mathbb{Q}(i)$

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Goal

To investigate the expected correspondence between mod ℓ modular forms and mod ℓ Galois representations over imaginary quadratic fields.

Serre's Conjecture

In 1987, Serre made the following conjecture in [7]. Suppose

$$\rho : G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\bar{\mathbb{F}}_{\ell})$$

is continuous, irreducible and odd. Then ρ is “modular”, i.e., ρ arises from a normalized cuspidal eigenform f in the sense that

1. $\text{tr}(\rho(\text{Frob}_p)) = \text{eigenvalue of } f \text{ at } p$; and
2. $\det(\rho(\text{Frob}_p)) = \varepsilon(p)p^{k-1}$

for all $p \nmid N\ell$, where N , k and ε are the level, weight and character (respectively) of f .

Serre then gives a refinement of this conjecture in which he specifies the minimal level $N(\rho)$ and weight $k(\rho)$ of f (each depending only on ρ) and such that $\ell \nmid N(\rho)$ and $k(\rho) \geq 2$.

Generalization to number fields

Buzzard, Diamond and Jarvis give an extension of Serre's conjecture to totally real fields in their forthcoming paper [3]. They provide an explicit recipe detailing for which weights V one expects to find a mod ℓ Galois representation. Given a representation ρ , they define, for each prime \mathfrak{p} dividing ℓ , a set of representations $W_{\mathfrak{p}}(\rho)$ depending only on ρ restricted to the inertia group at \mathfrak{p} . They then set

$$W(\rho) = \{\otimes_{\bar{\mathbb{F}}_{\ell}} V_{\mathfrak{p}} \mid V_{\mathfrak{p}} \in W_{\mathfrak{p}}(\rho)\}.$$

They conjecture that if ρ is modular, then

$$W(\rho) = \{V \mid \rho \text{ is modular of weight } V\}.$$

In [5], Dembélé, Diamond and Roberts provide computational evidence for this conjecture.

I am investigating what happens in the imaginary quadratic case. I hope to provide computational evidence for an analogous relationship in this situation. To that end, I have written code to compute cohomological mod ℓ modular forms over $\mathbb{Q}(i)$.

Modular Forms

We define a mod ℓ modular form for $K = \mathbb{Q}(i)$ of level n and Serre weight V to be a non-zero cohomology class $v \in H^2(\Gamma_0(n), V)$, which is a simultaneous eigenvector for all the Hecke operators $T_{\mathfrak{q}}$.

Borel-Serre duality [2] gives an isomorphism

$$H^2(\Gamma_0(n), V) \xrightarrow{\sim} H_0(\Gamma_0(n), St \otimes V),$$

where St denotes the Steinberg module. My code is actually computing simultaneous eigenvectors in the homology group $H_0(\Gamma_0(n), St \otimes V)$.

Steinberg module and modular symbols

Following Ash in [1], we define the Steinberg module in terms of universal minimal modular symbols. Consider the set of formal $\bar{\mathbb{F}}_{\ell}$ -linear sums of symbols $[v] = [v_1, v_2]$ where the v_i are unimodular vectors in \mathcal{O}_K^2 . Mod out by the $\bar{\mathbb{F}}_{\ell}$ -module generated by the following elements:

1. $[v_2, v_1] + [v_1, v_2]$;
2. $[v] = [v_1, v_2]$ whenever $\det(v) = 0$; and
3. $[v_1, v_3] - [v_1, v_2] - [v_2, v_3]$,

where the v_i again run over all unimodular vectors in \mathcal{O}_K^2 . We take this quotient module as the definition of the Steinberg module St .

Computation of modular forms

With the Steinberg module written in terms of modular symbols, we can use methods of Cremona [?] et al. to compute our mod ℓ modular forms.

Manin symbols provide a computationally friendly description of modular symbols. In order to see the equivalence between these spaces, we follow the algebraic approach used by Wiese [8], adjusting some particular arguments to adapt the strategy to the $K = \mathbb{Q}(i)$ case.

My code uses these Manin symbols to compute the homology space $H_0(\Gamma_0(n), St \otimes V)$. The action of the Hecke operators on this space is then computed by converting back to modular symbols, computing the action of $T_{\mathfrak{q}}$ and then converting back to Manin symbols via the usual continued fraction convergents method.

Serre weights

For simplicity, we assume ℓ is inert and let \mathfrak{p} be the prime above ℓ . We let $k_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$ and $S_{\mathfrak{p}}$ be the set of embeddings $k_{\mathfrak{p}} \hookrightarrow \bar{\mathbb{F}}_{\ell}$. A Serre Weight is an irreducible $\bar{\mathbb{F}}_{\ell}$ -representation of

$$G = \text{GL}_2(\mathcal{O}_K/\ell\mathcal{O}_K).$$

Such representations are of the form

$$V_{\vec{a}, \vec{b}} = \bigotimes_{\tau \in S_{\mathfrak{p}}} \left(\det^{a_{\tau}} \otimes_{k_{\mathfrak{p}}} \text{Sym}^{b_{\tau}} k_{\mathfrak{p}}^2 \right) \otimes_{\tau} \bar{\mathbb{F}}_{\ell},$$

where each of the a_{τ} and b_{τ} are integers and $0 \leq b_{\tau} \leq \ell - 1$. Furthermore, we may assume that $0 \leq a_{\tau} \leq \ell - 1$ for each $\tau \in S_{\mathfrak{p}}$ and (to guarantee the representations are inequivalent) that $a_{\tau} < \ell - 1$ for some τ .

In the code, we represent $\text{Sym}^{b_{\tau}} k_{\mathfrak{p}}^2$ as the space of homogeneous polynomials of degree b_{τ} in two variables with coefficients in $k_{\mathfrak{p}}$, equipped with the natural action of $\text{GL}_2(\mathcal{O}_K)$.

Modular of weight V

We say that a continuous, irreducible representation

$$\rho : G_K \rightarrow \text{GL}_2(\bar{\mathbb{F}}_{\ell})$$

is modular of Serre weight V if $H^2(\Gamma_0(n), V)[\mathfrak{m}_{\rho}] \neq 0$, where

$$\mathfrak{m}_{\rho} = \langle T_{\mathfrak{q}} - \text{tr}(\rho(\text{Frob}_{\mathfrak{q}})) \mid \mathfrak{q} \text{ prime} \rangle$$

is the maximal ideal of the Hecke algebra associated to ρ . So a representation ρ is modular of weight V if the corresponding system of eigenvalues shows up in our computed forms for that weight.

References

- [1] A. Ash, “Unstable Cohomology of $\text{SL}(n, \mathcal{O})$ ”, Journal of Algebra 167 (1994) 330-342.
- [2] A. Borel and J.-P. Serre, “Corners and Arithmetic Groups”, Commentarii Mathematici Helvetici 48 (1973) 436-491.
- [3] K. Buzzard, F. Diamond and F. Jarvis, “On Serre's Conjecture for Mod ℓ Galois Representations Over Totally Real Fields”, preprint.
- [4] J. Cremona, “Hyperbolic Tessellations, Modular Symbols, and Elliptic Curves over Complex Quadratic Fields”, Compositio Mathematica 51 (1984), 275-323.
- [5] L. Dembélé, F. Diamond and D. Roberts, “Numerical Examples and Evidence for Serre's Conjecture over Totally Real Fields”, preprint.
- [6] PARI/GP, version 2.3.1, Bordeaux, 2005, <http://pari.math.u-bordeaux.fr/>.
- [7] J.-P. Serre, “Sur les représentations modulaires de degré 2 de $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ ”, Duke Math. J. 54 (1987) 179-230.
- [8] G. Wiese, “Modular Forms of Weight One Over Finite Fields”, Ph.D. thesis, Universiteit Leiden (2005).

Implementation Specifications

The code is written in C, using the PARI [6] library.

The code computes mod ℓ modular forms over $\mathbb{Q}(i)$ for:

- ℓ a rational prime which is inert in $K = \mathbb{Q}(i)$
- congruence subgroup $\Gamma = \Gamma_0(n)$ for level $n \subset \mathcal{O}_K$
- arbitrary “Serre weight” V

The code takes as input:

- level n
- weight $V_{\vec{a}, \vec{b}}$ (or will optionally compute all weights for a given level)
- maximum prime \mathfrak{q} for Hecke operators $T_{\mathfrak{q}}$

The code outputs the system(s) of eigenvalues for any simultaneous eigenvectors it finds.

Planned extensions of the code

- Support primes ℓ which split in K (currently the code will only compute for inert primes).
- Implement congruence subgroups $\Gamma_1(n)$ and accompanying character.
- Allow for other class number one imaginary quadratic fields.
- Compute forms for imaginary quadratic fields with higher class number.
- Look for eigenvalues in larger extensions of $\bar{\mathbb{F}}_{\ell}$.