ON THE COMPUTATION OF ALGEBRAIC MODULAR FORMS

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ABSTRACT. In this paper, we revisit the theory of Brandt matrices for algebraic modular forms. We then apply this to the computation of Hilbert-Siegel modular forms of genus 2 over totally real number fields of narrow class number one. Some numerical examples of eigensystems of Hilbert-Siegel modular forms of genus 2 over $\mathbb{Q}(\sqrt{2})$ are provided.

Introduction

Let $G/\mathbb{Q}$ be a connected reductive group such that $G(\mathbb{R})$ is compact modulo its center. In [5], Gross developed a theory of algebraic modular forms on G and conjectured that they parametrize certain Galois representations. In many cases, the groups studied by Gross are inner forms of linear reductive groups that are intimately related to arithmetic geometry. For instance, let $B/\mathbb{Q}$ be a definite quaternion algebra that is ramified at infinity and let $G/\mathbb{Q}$ be the algebraic group whose $\mathbb{Q}$-rationals points are given by the (quaternionic) unitary similitude group $\mathbf{GU}_2(B)$. Then, G is an inner form of $\mathbf{GSp}_4/\mathbb{Q}$ such that $G(\mathbb{R})$ is compact modulo its center. In that case, the Langlands philosohpy predicts that there is a transfer map between automorphic forms on G and Siegel modular forms on $\mathbf{GSp}_4/\mathbb{Q}$, thus providing theoretical support for the conjectures in [5]. In [10], Lansky and Pollack computed examples of such algebraic modular forms and determined their Satake parameters. The significance of their data is pretty clear as they were able to provide numerical evidence for the existence of a number field with Galois group $G_2(\mathbb{F}_5)$ that is ramified only at 5, for the exceptional Lie group G_2, and also for the existence of symmetric cube liftings from $\mathbf{PGL}_2$ to $\mathbf{PGSp}_4$. Unfortunately, the computations in [10] use the classical approach to Brandt matrices, and their algorithm does not appear to be very efficient. Since the conjectures in [5] have rather important number theoretic consequences, we think that it would be useful to provide a systematic computational approach to algebraic modular forms so that experimenting with them can be much easier. In this paper, we give an alternative approach to the theory of Brandt matrices that is based on the use of integral models of G, and which extends the ideas in [2] and [3]. As an application,

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we explain how to compute Hilbert-Siegel modular forms over a totally real number field of narrow class number one.

The outline of the paper is as follows. In Section 1, we relate the space of algebraic modular forms of a given level U on G to spaces of algebraic modular forms on integral models of G. In Section 2, we present a refined version of the Jacquet-Langlands correspondence for Hilbert-Siegel modular forms stated in [1]. Then in Section 3, we explain how the results of the two previous sections translate into a more efficient algorithm for the computation of Hilbert-Siegel modular forms over totally real number fields than the classical approach to Brandt matrices. Finally, in Section 4, we give numerical examples over the real quadratic field $\mathbb{Q}(\sqrt{2})$.

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1. **Algebraic modular forms and Hecke action**

In this section, we let $G/\mathbb{Q}$ be a connected reductive group such that $G(\mathbb{R})$ is connected and compact modulo its center, and which admits an integral model $\underline{G}/\mathbb{Z}$ in the sense of Gross [6]. We also fix an irreducible algebraic representation (ρ, V) of G that is defined over a number field E. We let $\hat{\mathbb{Z}}$ and $\hat{\mathbb{Q}}$ be the finite adèles of $\mathbb{Z}$ and $\mathbb{Q}$ respectively. We fix a compact open subgroup U of $\underline{G}(\hat{\mathbb{Z}})$.

The space of *algebraic modular forms* on $G(\hat{\mathbb{Q}})$ of weight V and level U is defined by

$$M_G(U, V) := \left\{ f : G(\hat{\mathbb{Q}})/U \to V : \ f|_\rho \gamma = f, \text{ for all } \gamma \in G(\mathbb{Q}) \right\},$$

where $f|_\rho\gamma(x) = f(\gamma x)\rho(\gamma)$, for all $x \in G(\hat{\mathbb{Q}})$ and $\gamma \in G(\mathbb{Q})$. The Hecke algebra acting on this space is defined as follows. For any $u \in G(\hat{\mathbb{Q}})$, write the finite disjoint union $UuU = \coprod_i u_i U$ and put

$$f|_\rho[UuU](x) = \sum_i f(xu_i).$$

The space $M_G(U, V)$ can be related to the integral structures on $G/\mathbb{Q}$ as follows. Let h be the class number of $G/\mathbb{Q}$ in the genus of $\underline{G}$. Then, we have

$$G(\hat{\mathbb{Q}}) = \coprod_{\sigma=1}^{h} G(\mathbb{Q}) g_\sigma \underline{G}(\hat{\mathbb{Z}}),$$

and each g_σ determines an integral structure $\underline{G}_\sigma/\mathbb{Z}$ such that $\underline{G}_\sigma(\mathbb{Z}) = G(\mathbb{Q}) \cap g_\sigma \underline{G}(\hat{\mathbb{Z}}) g_\sigma^{-1}$ and $U_\sigma = g_\sigma U g_\sigma^{-1}$ is a compact open subgroup of $\underline{G}_\sigma(\hat{\mathbb{Z}})$.

We define the space of *algebraic modular forms* on $\underline{G}_\sigma(\hat{\mathbb{Z}})$ of weight V and level U_σ by

$$M_{\underline{G}_\sigma}(U_\sigma,\, V) := \left\{ f : \underline{G}_\sigma(\hat{\mathbb{Z}})/U_\sigma \to V :\ f|_\rho \gamma = f,\ \text{for all } \gamma \in \underline{G}_\sigma(\mathbb{Z}) \right\}.$$

For any $u \in G(\hat{\mathbb{Q}})$ and any $\sigma,\, \tau = 1,\, \ldots,\, h$, we define a Hecke transformation as follows. Write the disjoint union $U_\sigma u U_\tau = \coprod_\nu u_\nu U_\tau$ and, for any $x \in \underline{G}_\sigma(\hat{\mathbb{Z}})$, let

$$\Theta(x,\, u,\, \sigma,\, \tau) := \left\{ u_\nu : x u_\nu = \gamma_\nu x_\nu,\ \text{for some}\, \gamma_\nu \in G(\mathbb{Q}),\ x_\nu \in \underline{G}_\tau(\hat{\mathbb{Z}}) \right\}.$$

For any $f \in M_{\underline{G}_\tau}(U_\tau,\, V)$, put

$$f|_\rho [U_\sigma u U_\tau](x) = \sum_{\Theta(x,\, u,\, \sigma,\, \tau)} f(x_\nu)\gamma_\nu^{-1}.$$

Then, it is not hard to see that $f|_\rho [U_\sigma u U_\tau](x)$ is well-defined. Furthermore, multiplying x to the right by an element in U_σ permutes the elements in $\Theta(x,\, u,\, \sigma,\, \tau)$, while multiplication to the left by an element in $\underline{G}_\sigma(\mathbb{Z})$ preserves the elements in the set. Therefore, $f|_\rho [U_\sigma u U_\tau]$ belongs to $M_{\underline{G}_\sigma}(U_\sigma,\, V)$, and we have a linear map

$$\begin{array}{rcl} [U_\sigma u U_\tau] :\ M_{\underline{G}_\tau}(U_\tau,\, V) & \longrightarrow & M_{\underline{G}_\sigma}(U_\sigma,\, V) \\ f & \longmapsto & f|_\rho [U_\sigma u U_\tau], \end{array}$$

which we call the Hecke transform $[U_\sigma u U_\tau]$.

Theorem 1. *There is an isomorphism of Hecke modules*

$$\begin{array}{rcl} M_G(U,\, V) & \to & \displaystyle\bigoplus_{\sigma=1}^{h} M_{\underline{G}_\sigma}(U_\sigma,\, V) \\ f & \mapsto & (f_\sigma)_\sigma, \end{array}$$

where the Hecke action on the right hand side is given by the family of Hecke transforms $([U_\sigma u U_\tau])_{\sigma,\tau}$.

Proof. Let f be an element of $M_G(U,\, V)$ and $\sigma = 1,\, \ldots,\, h$. We define the function f_σ by

$$f_\sigma(x) = f(x g_\sigma),\ \ x \in \underline{G}_\sigma(\hat{\mathbb{Z}}).$$

It is not hard to see that f_σ lies in $M_{\underline{G}_\sigma}(U_\sigma,\, V)$, and that the map $f \mapsto (f_\sigma)_\sigma$ is an isomorphism of complex vector spaces. So it only remains to show that this isomorphism is compatible with the Hecke module structure on each side. This amounts to showing that each Hecke operator $[UuU]$ on $M_G(U,\, V)$ can be decomposed into a family of Hecke transforms $([U_\sigma u U_\tau])_{\sigma,\tau}$ on the right hand side. This is done by using the same argument as in [3, Theorem 2]. □

The translation of Theorem 1 into global terms is what provides us with a more efficient way of computing algebraic modular forms. The best analogy for this translation is the passage from the adelic to the classical setting in the case of modular forms. This was done already in the case of Hilbert modular forms in [2] and [3], and in the case of Hilbert-Siegel modular forms of genus 2 over real quadratic fields under the restriction that the class number in the principal genus of G is one [1]. In the rest of this paper, we remove this restriction on the class number of the principal genus. The discussion should readily apply to other groups such as the unitary groups $\mathbf{U}(3)$.

2. The Jacquet-Langlands correspondence for Hilbert-Siegel modular forms

In this section, we let F be a totally real number field of narrow class number one, and $\mathcal{O}_F$ its ring of integers. We let I be the set of real embeddings of F. For any $a \in F$ and $\tau \in I$, we let a^τ be the image of a in $\mathbb{R}$ under τ. We say that a is totally positive if $a^\tau > 0$ for all $\tau \in I$; and we denote this by $a \gg 0$. Let B be a totally definite quaternion algebra over F, and $u \mapsto \bar{u}$ its involution. We let S be the set of finite primes at which B is ramified, and $\mathrm{disc}(B)$ the discriminant of B. We fix a maximal order $\mathcal{O}_B$ in B. We choose a finite extension K/F that is Galois over $\mathbb{Q}$ and such that there is an isomorphism $j : B \otimes_{\mathbb{Q}} K \cong \mathbf{M}_2(K)^I$ with $j(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_K) = \mathbf{M}_2(\mathcal{O}_K)^I$. For any prime $\mathfrak{p}$ in F, we denote the completions of F and $\mathcal{O}_F$ at $\mathfrak{p}$ by $F_\mathfrak{p}$ and $\mathcal{O}_{F_\mathfrak{p}}$, respectively. Similarly, we denote the completions of B and $\mathcal{O}_B$ at $\mathfrak{p}$ by $B_\mathfrak{p}$ and $\mathcal{O}_{B_\mathfrak{p}}$, respectively. For any finite prime $\mathfrak{p} \notin S$, we fix an isomorphism $B_\mathfrak{p} \cong \mathbf{M}_2(F_\mathfrak{p})$ such that $\mathcal{O}_{B_\mathfrak{p}} = \mathbf{M}_2(\mathcal{O}_{F_\mathfrak{p}})$. Let $G/\mathbb{Q}$ be the reductive group defined by

$$G(A) = \left\{\gamma \in \mathbf{M}_2(B \otimes_{\mathbb{Q}} A) :\ \gamma\bar{\gamma}^t = \nu_G(\gamma)\mathbf{1}_2,\ \nu_G(\gamma) \in (A \otimes_{\mathbb{Q}} F)^\times\right\},$$

for any $\mathbb{Q}$-algebra A. The maximal order $\mathcal{O}_B$ defines an integral structure $\underline{G}/\mathbb{Z}$ on G by

$$\underline{G}(A) = \left\{\gamma \in \mathbf{M}_2(\mathcal{O}_B \otimes_{\mathbb{Z}} A) :\ \gamma\bar{\gamma}^t = \nu_G(\gamma)\mathbf{1}_2,\ \nu_G(\gamma) \in (A \otimes_{\mathbb{Z}} \mathcal{O}_F)^\times\right\},$$

for any $\mathbb{Z}$-algebra A. The group $G/\mathbb{Q}$ is an inner form of $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{GSp}_4)$. By combining the isomorphism j with conjugation by some permutation matrix, we obtain an isomorphism $G(K) \cong \mathbf{GSp}_4(K)^I$ such that $\underline{G}(\mathcal{O}_K) \cong \mathbf{GSp}_4(\mathcal{O}_K)^I$, which we fix from now on. For any finite prime $\mathfrak{p} \notin S$, we choose an isomorphism $\mathbf{GU}_2(\mathcal{O}_{B_\mathfrak{p}}) \cong \mathbf{GSp}_4(\mathcal{O}_{F_\mathfrak{p}})$ and fix the maximal compact open subgroup

$$\underline{G}(\hat{\mathbb{Z}}) = \prod_{\mathfrak{p} \in S} \mathbf{GU}_2(\mathcal{O}_{B_\mathfrak{p}}) \times \prod_{\mathfrak{p} \notin S} \mathbf{GSp}_4(\mathcal{O}_{F_\mathfrak{p}}).$$

Let $N = \prod_{\mathfrak{p}|N} \mathfrak{p}^{e_\mathfrak{p}}$ be an integral ideal of F such that $(\mathrm{disc}(B), N) = 1$, and set $M = \mathrm{disc}(B)N$. We define the open compact subgroup $K_0^B(N)$ of $\underline{G}(\hat{\mathbb{Z}})$ as follows. We let

$$K_0^B(N) := \prod_{\mathfrak{p}|M} K_0^B(\mathfrak{p}^{e_\mathfrak{p}}) \times \prod_{\mathfrak{p}\nmid M} \mathbf{GSp}_4(\mathcal{O}_{F_\mathfrak{p}}),$$

where, for any $\mathfrak{p} \in S$, we let

$$K_0^B(\mathfrak{p}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GU}_2(\mathcal{O}_{B_\mathfrak{p}}) : \ c \equiv 0 \mod \mathfrak{p} \right\},$$

and for any $\mathfrak{p} \mid N$, we let

$$K_0^B(\mathfrak{p}^{e_\mathfrak{p}}) := \left\{ \gamma \in \mathbf{GSp}_4(\mathcal{O}_{F_\mathfrak{p}}) : \ \gamma \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \mod \mathfrak{p}^{e_\mathfrak{p}} \right\}.$$

Let $\underline{k} \in \mathbb{Z}^I$ be an integer vector such that $k_\tau \geq 3$, and all components have the same residue modulo 3. Analogously to the Hilbert modular case, we normalize the weight vector as follows. We set $\underline{t} = (1, \ldots, 1) \in \mathbb{Z}^I$ and $\underline{m} = \underline{k} - 3\underline{t}$. We then choose a vector $\underline{v} \in \mathbb{Z}^I$ such that each $v_\tau \geq 0$, some $v_\tau = 0$, and $\underline{m} + 3\underline{v} = n\underline{t}$ for some integer $n \geq 0$. For any integer $a \geq b \geq 0$, we let $V_{a,\,b}$ be the representation of $\mathbf{GSp}_4(\mathbb{C})$ of highest weight vector $(a,\, b)$. We define the weight representation $(\rho_{\underline{k}},\, V_{\underline{k}})$ by setting

$$V_{\underline{k}} := \bigotimes_{\tau \in I} V_{m_\tau,\, v_\tau},$$

and letting $G(\mathbb{R})$ acts via the embedding $G(\mathbb{R}) \hookrightarrow \mathbf{GSp}_4(\mathbb{C})^I$. To simplify notations, we let

$$\begin{aligned} M_{\underline{k}}^B(N) &= M_G(K_0^B(N),\, V_{\underline{k}}), \\ M_{\underline{k}}^{B,\,\sigma}(N) &= M_{\underline{G}_\sigma}(K_0^B(N)_\sigma,\, V_{\underline{k}}),\ \sigma = 1,\, \ldots,\, h. \end{aligned}$$

When $\underline{k} = (3, \ldots, 3)$, we let $I_{\underline{k}}^B(N)$ be the subspace of $M_{\underline{k}}^B(N)$ that consists of the constant functions. Then, we define

$$S_{\underline{k}}^B(N) := \begin{cases} M_{\underline{k}}^B(N), & \underline{k} \neq 3\underline{t} \\ M_{\underline{k}}^B(N)/I_{\underline{k}}^B(N), & \underline{k} = 3\underline{t}. \end{cases}$$

For any prime $\mathfrak{p} \nmid M$, the local Hecke algebra at $\mathfrak{p}$ is generated by the two Hecke operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ corresponding to the double cosets of the diagonal matrices $\mathrm{diag}(1, 1, \varpi_\mathfrak{p},\, \varpi_\mathfrak{p})$ and $\mathrm{diag}(1, \varpi_\mathfrak{p}, \varpi_\mathfrak{p}^2,\, \varpi_\mathfrak{p})$ respectively, where $\varpi_\mathfrak{p}$ is a uniformizer at $\mathfrak{p}$. The Hecke algebra $\mathbf{T}_{\underline{k}}^B(N)$ acting on $S_{\underline{k}}^B(N)$ is the $\mathbb{Z}$-algebra generated by all the operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ for all primes $\mathfrak{p} \nmid M$.

We define the congruence subgroup

$$\Gamma_I^{(2)}(M) := \left\{ \gamma \in \mathbf{GSp}_4(\mathcal{O}_F) : \ \nu_G(\gamma) \gg 0;\ \gamma \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \mod M \right\},$$

and let $S_{\underline{k}}(\Gamma_I^{(2)}(M))$ be the space of holomorphic cuspidal Hilbert-Siegel modular forms of weight $\underline{k}$ and level $\Gamma_I^{(2)}(M)$. As above, for any prime $\mathfrak{p} \nmid M$, the local Hecke algebra at $\mathfrak{p}$ is generated by the two Hecke operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$. The Hecke algebra $\mathbf{T}_{\underline{k}}(M)$ acting on $S_{\underline{k}}(\Gamma_I^{(2)}(M))$ is the $\mathbb{Z}$-algebra generated by all the operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ for $\mathfrak{p} \nmid M$. (For more details on some of the terminology we use below, we refer to Ibukiyama [8], Rastegar [11] and Roberts and Schmidt [12]). The following conjecture is a restatement of a conjecture in Ibukiyama [8] (that was first discussed in [9]) in the case of totally real number fields.

Conjecture 2. *There is an isomorphism of Hecke modules*

$$S_{\underline{k}}^B(N) \xrightarrow{\sim} S_{\underline{k}}(\Gamma_I^{(2)}(M))^{S\text{-}new},$$

where the latter space is the space of Hilbert-Siegel cusp forms that are new at all primes $\mathfrak{p} \in S$.

Remark 1. Although we have stated Conjecture 2 for levels of the type $\Gamma_I^{(2)}(M)$, the statement is still valid when the level structure is of Siegel type away from the set of ramification S of the quaternion algebra B.

3. Computing Hilbert-Siegel modular forms

Conjecture 2 establishes a connection between certain spaces of algebraic modular forms and Hilbert-Siegel modular forms. In this section, we will explain how combining this with Theorem 1 allows us to compute spaces of the latter forms.

In the remaining of this section, we fix a prime ideal $\mathfrak{q}$ in F, and let $\pi_{\mathfrak{q}}$ be a totally positive generator of $\mathfrak{q}$. (When the class number of G is one, we simply let $\mathfrak{q} = (1)$). Keeping the notations of the previous sections, we recall that the elements g_σ, $\sigma = 1, \ldots, h$, parametrize the isomorphism classes of $\mathcal{O}_B$-lattices in the genus of $\mathcal{O}_B^{\oplus 2}$. For each $\sigma = 1, \ldots, h$, let L_σ be the corresponding lattice. Then, it is easy to see that $\underline{G}_\sigma(\mathbb{Z}) = \mathrm{Stab}_{G(\mathbb{Q})}(L_\sigma)$, and we set $\Gamma_\sigma = \underline{G}_\sigma(\mathbb{Z})/\mathcal{O}_F^\times$. By Shimura [13], $\mathbf{GL}_2(B)$ has class number one. Combining this with the strong approximation theorem, we can choose $\alpha_\sigma \in \mathbf{GL}_2(\mathcal{O}_B[\frac{1}{\pi_{\mathfrak{q}}}])$ such that $L_\sigma = \mathcal{O}_B^{\oplus 2} \cdot \alpha_\sigma$. There is an algorithm in [7] for finding representatives of the equivalence classes of lattices when $F = \mathbb{Q}$. It is not hard to see that this algorithm extends to any totally real number field of narrow class number one. In this case, we have the following lemma.

Lemma 1. *a) Let* $L = \mathcal{O}_B^{\oplus 2} \cdot \alpha$ *be a lattice in* $B^{\oplus 2}$*, for some* $\alpha \in \mathbf{GL}_2(B)$*. Then* L *belongs to the principal genus if and only if*

$$\alpha\bar{\alpha}^t = m \begin{pmatrix} s & r \\ \bar{r} & t \end{pmatrix}, \text{ with } m \in F^{\times +},\ s,\ t \in \mathcal{O}_F^+,\ r \in \mathcal{O}_B \text{ and } st - \mathbf{nr}(r) \in \mathcal{O}_F^{\times +}.$$

b) Let $L_1 = \mathcal{O}_B^{\oplus 2} \cdot \alpha_1$ *and* $L_2 = \mathcal{O}_B^{\oplus 2} \cdot \alpha_2$ *be two* $\mathcal{O}_B$*-lattices in the principal genus. Then,* L_1 *and* L_2 *are equivalent if and only if there exist a matrix*

$\beta \in \mathbf{GL}_2(\mathcal{O}_B)$ and an element $n \in F^{\times +}$ such that

$$\beta(\alpha_1\bar{\alpha}_1^t)\bar{\beta}^t = n\alpha_2\bar{\alpha}_2^t.$$

Proof. This is easily adapted from [7, Proposition 22]. □

The ideal generated by the number $m \in F^{\times +}$ in Lemma 1 a) is called the *norm* of the lattice L. It is shown in Shimura [13] that every equivalence class in the principal genus contains a lattice whose norm is the unit ideal. So, by Lemma 1 and without loss of generality, we choose $\gamma_\sigma \in \mathbf{GL}_2(\mathcal{O}_B)$ and $\alpha_\sigma \in \mathbf{GL}_2(\mathcal{O}_B[\frac{1}{\pi_\mathfrak{q}}])$ such that $\gamma_\sigma = \alpha_\sigma\bar{\alpha}_\sigma^t$, $\sigma = 1, \ldots, h$. Then, for any $\sigma = 1, \ldots, h$, there is a canonical isomorphism

$$\begin{array}{ccc} \Gamma_\sigma & \xrightarrow{\sim} & \left\{u \in \mathbf{M}_2(\mathcal{O}_B) \middle| \, u\gamma_\sigma\bar{u}^t = \gamma_\sigma\right\}/\{\pm 1\} \\ \gamma & \longmapsto & \alpha_\sigma\gamma\alpha_\sigma^{-1}. \end{array}$$

Let $\mathfrak{p} \notin S$ be a prime ideal in F and choose a totally positive generator $\pi_\mathfrak{p}$ of $\mathfrak{p}$. Let $\mathbb{F}_\mathfrak{p} = \mathcal{O}_F/\mathfrak{p}$ be the residue field at $\mathfrak{p}$, and define the reduction map

$$\begin{array}{ccc} \mathbf{M}_2(\mathcal{O}_B) & \to & \mathbf{M}_4(\mathbb{F}_\mathfrak{p}) \\ u & \mapsto & \tilde{u}. \end{array}$$

Then let $\Theta_1^{(\mathfrak{q})}(\mathfrak{p}, \sigma, \tau)$ and $\Theta_2^{(\mathfrak{q})}(\mathfrak{p}, \sigma, \tau)$ be the subsets of $G(\mathbb{Q})$ defined by the bijections

$$\begin{array}{ccc} \alpha_\tau\underline{G}_\tau(\mathbb{Z})\alpha_\tau^{-1}\backslash\left\{u \in \mathbf{M}_2(\mathcal{O}_B) \middle| \begin{array}{c} u\gamma_\sigma\bar{u}^t = \pi_\mathfrak{p}\gamma_\tau \\ \mathrm{rank}_{\mathbb{F}_\mathfrak{p}}(\tilde{u}) = 2 \end{array}\right\} & \xrightarrow{\sim} & \Theta_1^{(\mathfrak{q})}(\mathfrak{p}, \sigma, \tau) \\ \alpha_\tau\underline{G}_\tau(\mathbb{Z})\alpha_\tau^{-1}\backslash\left\{u \in \mathbf{M}_2(\mathcal{O}_B) \middle| \begin{array}{c} u\gamma_\sigma\bar{u}^t = \pi_\mathfrak{p}^2\gamma_\tau \\ \mathrm{rank}_{\mathbb{F}_\mathfrak{p}}(\tilde{u}) = 1 \end{array}\right\} & \xrightarrow{\sim} & \Theta_2^{(\mathfrak{q})}(\mathfrak{p}, \sigma, \tau) \\ u & \longmapsto & \alpha_\tau^{-1}u\alpha_\sigma. \end{array}$$

Let

$$G(\mathbb{Q})_{(\mathfrak{q})} := \left\{\gamma \in G(\mathbb{Q}) : \nu_G(\gamma) \in \mathcal{O}_{F,(\mathfrak{q})}^\times\right\},$$

where $\mathcal{O}_{F,(\mathfrak{q})}^\times$ is the set of $\mathfrak{q}$-unit in $\mathcal{O}_F$. Then by construction, the sets Γ_σ, $\Theta_1^{(\mathfrak{q})}(\mathfrak{p}, \sigma, \tau)$ and $\Theta_2^{(\mathfrak{q})}(\mathfrak{p}, \sigma, \tau)$ are all subsets of $G(\mathbb{Q})_{(\mathfrak{q})}$. The group $G(\mathbb{Q})_{(\mathfrak{q})}$ acts on the quotient space $\underline{G}(\hat{\mathbb{Z}})/K_0^B(N)$, which is a flag variety we denote by $\mathcal{F}_0^B(N)$. This induces an action on the space of functions $f : \mathcal{F}_0^B(N) \to V_{\underline{k}}$ by

$$f|_{\underline{k}}\gamma(x) = f(\gamma x)\rho_{\underline{k}}(\gamma), \; x \in \mathcal{F}_0^B(N), \gamma \in G(\mathbb{Q})_{(\mathfrak{q})}.$$

It is not hard to see that there is a canonical isomorphism of complex spaces

$$M_{\underline{k}}^{B,\sigma}(N) \xrightarrow{\sim} \left\{f : \mathcal{F}_0^B(N) \to V_{\underline{k}} \middle| \, f|_{\underline{k}}\gamma = f, \text{ for all } \gamma \in \Gamma_\sigma\right\}.$$

Proposition 3. *Let N be an integral ideal of F such that $(\mathfrak{q}, N) = 1$. Then, for any prime $\mathfrak{p} \notin S$, the Hecke transforms $T_1^{\sigma,\tau}(\mathfrak{p})$ and $T_2^{\sigma,\tau}(\mathfrak{p})$ from*

$M_{\underline{k}}^{B,\tau}(N)$ to $M_{\underline{k}}^{B,\sigma}(N)$ are given by

$$T_1^{\sigma,\tau}(\mathfrak{p})f := \sum_{u\in\Theta_1^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)} f|_{\underline{k}}u,$$

$$T_2^{\sigma,\tau}(\mathfrak{p})f := \sum_{u\in\Theta_2^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)} f|_{\underline{k}}u.$$

Proof. It is not hard to see that the action of Hecke on $M_{\underline{k}}^B(1)$, in terms of lattices, is determined by the sets $\Theta_1^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)$ and $\Theta_2^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)$ for any prime $\mathfrak{p}\notin S$. By Theorem 1, the same data suffices for $M_{\underline{k}}^B(N)$ if $\mathfrak{q}\nmid N$. And so, Proposition 3 is simply a translation of Theorem 1 into global terms that uses the isomorphisms above. □

By combining Theorem 1 and Proposition 3, we get an algorithm that can be used for the systematic computation of Hilbert-Siegel modular forms. This algorithm is essentially the same as the one developed in [3] for the genus one case. We summarize the main steps below, and refer to [1] and [3] for further details on the implementation.

Precomputations.

(1) Choose a prime $\mathfrak{q}$ and find coset representatives $\gamma_\sigma\in\mathbf{M}_2(\mathcal{O}_B)$ by using Lemma 1.
(2) Choose a bound $b\in\mathbb{N}$, and precompute the unit groups Γ_σ, and the sets $\Theta_1^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)$ and $\Theta_2^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)$, for all primes $\mathfrak{p}\notin S$ such that $\mathbf{N}(\mathfrak{p})\le b$.

For any integral ideal N such that $\mathfrak{q}\nmid N$, the computation of the space $S_{\underline{k}}^B(N)$ now proceeds as follows.

Algorithm.

(1) Compute a splitting isomorphism $\mathcal{O}_B\otimes_{\mathcal{O}_F}(\mathcal{O}_F/N)\cong\mathbf{M}_2(\mathcal{O}_F/N)$.
(2) Compute the orbits of Γ_σ acting on $\mathcal{F}_0^B(N)$ and a fundamental domain $\mathcal{F}_\sigma$, for each $\sigma=1,\ldots,h$.
(3) Compute the stabilizers of the elements in $\mathcal{F}_\sigma$ and the corresponding invariants spaces, for each $\sigma=1,\ldots,h$.
(4) For each prime $\mathfrak{p}\notin S$, with $\mathbf{N}(\mathfrak{p})\le b$, compute the families of Hecke transforms $(T_1^{\sigma,\tau}(\mathfrak{p}))$ and $(T_2^{\sigma,\tau}(\mathfrak{p}))$. They provide the Hecke operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ as block matrices.

Remark 2. The computations of the groups Γ_σ, and the sets $\Theta_1^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)$ and $\Theta_2^{(\mathfrak{q})}(\mathfrak{p},\sigma,\tau)$ amount to representing totally positive elements in $\mathcal{O}_F$ by totally positive definite quadratic forms subject to some constraints. The efficiency of the algorithm depends crucially on how those precomputations are done.

Remark 3. In [10], Lansky and Pollack already made use of the flag variety $\mathcal{F}_0^B(N)$ in order to find double coset representatives for the quotient space $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K_0^B(N)$. But, they then returned to the adelic setting in which they used the classical definition of Brandt matrices. The first advantage of our approach is that we combine Theorem 1 and Proposition 3 to define the Brandt matrices directly on the flag variety $\mathcal{F}_0^B(N)$, thus eliminating several unnecessary steps from their algorithm. The second one is that our algorithm is asymptotically more efficiency than the classical Brandt matrices. Indeed, assume that we would like to compute the spaces $S_{\underline{k}}^B(N)$ up to a bound $c \in \mathbb{N}$ for the norms of the levels N. Then, asymptotically, we can find a prime $\mathfrak{q} \nmid N$, with $\mathbf{N}(\mathfrak{q}) \leq \log c$. As a result, we only have to do relative few pre-computations. From our presentation, it is not hard to see that the same approach will work for other algebraic groups such as the unitary groups $\mathbf{U}(3)$.

4. A numerical example: Hilbert-Siegel modular forms over $\mathbb{Q}(\sqrt{2})$

In this section, we explain how the results of the previous sections can be implemented into an algorithm for $F = \mathbb{Q}(\sqrt{2})$. Let B be the Hamilton quaternion algebra over F, i.e., the quaternion algebra over F determined by the relations $i^2 = -1$, $j^2 = -1$ and $k = ij = -ji$. Then $\mathrm{disc}(B) = (1)$ and we choose the maximal order $\mathcal{O}_B = \mathbb{Z}[\sqrt{2}][e_1, e_2, e_3, e_4]$, where

$$e_1 = \frac{1+i}{\sqrt{2}},\ e_2 = \frac{1+j}{\sqrt{2}},\ e_3 = e_1e_2,\ e_4 = e_2e_1.$$

By working with the prime $\mathfrak{q} = (\sqrt{2})$ above 2, it is not hard to see that the following matrices determine different equivalence classes of lattices in the principal genus of G:

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 2 & \frac{(2+\sqrt{2})+(2-\sqrt{2})i}{2} \\ \frac{2+\sqrt{2}+(-2+\sqrt{2})i}{2} & 2 \end{pmatrix}.$$

We compute the stabilizers and obtain $\mathrm{Card}(\Gamma_1) = 2304$ and $\mathrm{Card}(\Gamma_2) = 1920$. And by using the mass formula in W. T. Gan, J. Hanke and J.-K. Yu [4, Proposition 9.3], we get

$$\begin{aligned} \mathrm{Mass}(\mathcal{O}_B^{\oplus 2}) &= \frac{1}{2^4}\zeta_{\mathbb{Q}(\sqrt{2})}(-1)\zeta_{\mathbb{Q}(\sqrt{2})}(-3) = \frac{1}{2^4}\frac{1}{12}\frac{11}{120} = \frac{1}{2}\left(\frac{1}{2304} + \frac{1}{1920}\right) \\ &= \frac{1}{2}\left(\frac{1}{\mathrm{Card}(\Gamma_1)} + \frac{1}{\mathrm{Card}(\Gamma_2)}\right). \end{aligned}$$

(The correction factor $1/2$ in the last two equalities come from the fact that $\Gamma_\sigma = \underline{G}_\sigma(\mathbb{Z})/\mathcal{O}_F^\times$). Therefore, the class number in the principal genus of G is $h = 2$.

Example 1. The data in the preliminary computations determine the space of Hilbert-Siegel modular forms of parallel weight 3 and full level structure.

Here we get $\dim M_3^B(1) = 2$ and $\dim S_3^B(1) = 1$. The Brandt matrices $\mathcal{B}_1(\mathfrak{p})$ and $\mathcal{B}_2(\mathfrak{p})$ of the Hecke operators acting on $M_3^B(1)$ for the first four prime ideals $\mathfrak{p}$ are given below.

$\mathbf{N}(\mathfrak{p})$	$\mathfrak{p}$	$\mathcal{B}_1(\mathfrak{p})$	$\mathcal{B}_2(\mathfrak{p})$
2	$(\sqrt{2})$	$\begin{pmatrix} 9 & 6 \\ 5 & 10 \end{pmatrix}$	$\begin{pmatrix} 12 & 18 \\ 15 & 15 \end{pmatrix}$
7	$(3+\sqrt{2})$	$\begin{pmatrix} 208 & 192 \\ 160 & 240 \end{pmatrix}$	$\begin{pmatrix} 1264 & 1536 \\ 1280 & 1520 \end{pmatrix}$
7	$(3-\sqrt{2})$	$\begin{pmatrix} 208 & 192 \\ 160 & 240 \end{pmatrix}$	$\begin{pmatrix} 1264 & 1536 \\ 1280 & 1520 \end{pmatrix}$
9	(3)	$\begin{pmatrix} 436 & 384 \\ 320 & 500 \end{pmatrix}$	$\begin{pmatrix} 3540 & 3840 \\ 3200 & 4180 \end{pmatrix}$

There is only one Hilbert-Siegel eigenform of level $\Gamma_I^{(2)}(1)$ and parallel weight 3. The first few Hecke eigenvalues of that form are listed in Table 1.

For every integral ideal N whose norm is odd, we can now compute the space $S_{\underline{k}}^B(N)$, which in this case is isomorphic to the full space of Hilbert-Siegel modular forms $S_{\underline{k}}(\Gamma_I^{(2)}(N))$ since $S = \emptyset$. In Tables 1 and 2, we list all the eigenforms of parallel weight 3 and Siegel type levels $\Gamma_0^{(2)}(\mathfrak{r})$ that are defined over the rationals or a quadratic field, for all primes $\mathfrak{r}$ such that $7 \le \mathbf{N}(\mathfrak{r}) \le 31$. Here are the conventions we use in the tables.

(1) For a quadratic field K of discriminant D, we let ω_D be a generator of the ring of integers $\mathcal{O}_K$ of K.
(2) The first row contains the level N, given in the format $(\mathrm{Norm}(N), \alpha)$ for some generator $\alpha \in F$ of N, and the dimensions of the relevant spaces.
(3) The second and third rows list the Hecke operators that have been computed.
(4) For each eigenform f, the Hecke eigenvalues are given in a row.
(5) The levels and the eigenforms are both listed up to Galois conjugation.

$N = (1, 1)$, $\dim M_3^B(N) = 2$, $\dim S_3^B(N) = 1$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	4	−3	48	−16	48	−16	116	340

$N = (7, 3+\omega_8)$, $\dim M_3^B(N) = 6$, $\dim S_3^B(N) = 5$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	10	15	−7	0	60	80	80	−20
f_2	−4	1	7	0	32	80	−60	148

$N = (9, 3)$, $\dim M_3^B(N) = 12$, $\dim S_3^B(N) = 11$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	−6	7	−22	64	−22	64	−9	0
f_2	4	5	−16	16	−16	16	9	0

$N = (17, 5+2\omega_8)$, $\dim M_3^B(N) = 23$, $\dim S_3^B(N) = 22$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	−2	3	6	32	−36	80	8	28
f_2	$4-\omega_{40}$	$-3-3\omega_{40}$	$38-4\omega_{40}$	$-96-32\omega_{40}$	$58+6\omega_{40}$	$64+48\omega_{40}$	$66+4\omega_{40}$	$-160+40\omega_{40}$

TABLE 1. **Hilbert-Siegel eigenforms of parallel weight 3 over $\mathbb{Q}(\sqrt{2})$**

$N = (23,\ 5+\omega_8)$, $\dim M_3^B(N) = 32$, $\dim S_1^B(N) = 31$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	-8	10	-46	119	-54	145	-54	153
f_2	10	15	72	176	56	48	124	420

$N = (25,\ 5)$, $\dim M_3^B(N) = 48$, $\dim S_3^B(N) = 47$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	2	-9	62	96	62	96	40	-420
f_2	$-4-2\omega_{12}$	$5+2\omega_{12}$	$-15+7\omega_{12}$	$56-8\omega_{12}$	$-15+7\omega_{12}$	$56-8\omega_{12}$	$44-18\omega_{12}$	$132-52\omega_{12}$
f_3	8	9	$44-10\omega_{24}$	$-48-80\omega_{24}$	$44+10\omega_{24}$	$-48+80\omega_{24}$	76	-60

$N = (31,\ 7+3\omega_8)$, $\dim M_3^B(N) = 65$, $\dim S_3^B(N) = 64$								
$\mathbf{N}(\mathfrak{p})$	2		7		7		9	
	$T_1(2+\omega_8)$	$T_2(2+\omega_8)$	$T_1(3+\omega_8)$	$T_2(3+\omega_8)$	$T_1(3-\omega_8)$	$T_2(3-\omega_8)$	$T_1(3)$	$T_2(3)$
f_1	-4	4	-60	174	-20	-2	-60	130
f_2	-4	5	20	16	-8	48	8	28
f_3	-6	7	$44-2\omega_{204}$	$112-8\omega_{204}$	$-4\omega_{204}$	48	$-32-2\omega_{204}$	$108+4\omega_{204}$
f_4	2	-9	$72-2\omega_{204}$	$176-16\omega_{204}$	$56-4\omega_{204}$	$48-32\omega_{204}$	$76-2\omega_{204}$	$-60-20\omega_{204}$

TABLE 2. **Hilbert-Siegel eigenforms of parallel weight 3 over $\mathbb{Q}(\sqrt{2})$ (cont'd)**

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