MONOMIAL CYCLE BASIS ON KOSZUL HOMOLOGY MODULES

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Abstract. It gives a class of p-Borel principal ideals of a polynomial algebra over a field K for which the graded Betti numbers do not depend on the characteristic of K and the Koszul homology modules have monomial cyclic basis. Also it shows that all principal p-Borel ideals have binomial cycle basis on Koszul homology modules.

Introduction

Let K be an infinite field, $S = K[x_1, \ldots, x_n]$, $n \geq 2$ the polynomial ring over a field K and $I \subset S$ a graded ideal. Consider the reverse lexicographical order on the monomials of S. Let M be a graded S-module and $\beta_{ij}(M) = \beta_{ij}$ the graded Betti numbers of M. The Castelnuovo-Mumford regularity of M is $\text{reg}(M) = \max\{j - i : \beta_{ij}(M) \neq 0\}$. By a theorem of Bayer and Stillman [5] we have $\text{reg}(\text{Gin}(I)) = \text{reg}(I)$. If char $K = 0$ then $\text{Gin}(I)$ is strongly stable, that is, it is monomial and for each monomial u of $\text{Gin}(I)$ and $1 \leq j < i \leq n$ such that $x_i|u$ it follows $x_j(u/x_i) \in \text{Gin}(I)$. Then $\text{reg}(\text{Gin}(I)$ is the highest degree of minimal generators of $\text{Gin}(I)$ by Eliahou and Kervaire [9]. If char $K = p > 0$ then Borel fixed ideals are just the so called p-Borel ideals and they are not necessarily strongly stable and it is hard to give a formula for the regularity of these ideals. Let I be a monomial ideal of S, u a monomial of I and $\nu_i(u)$ be the highest power of x_i dividing u. Let a, b be two integers and $a = \Sigma_{i\geq 0} a_i p^i$, $b = \Sigma_{i\geq 0} b_i p^i$ be the p-adic expansion of a, respectively b. We say that $a \leq_p b$ if $a_i \leq b_i$ for all i. It is well known that a monomial ideal I is p-Borel if for any monomial $u \in I$ and $1 \leq j < i \leq n$ and a positive integer t such that $t \leq_p \nu_i(u)$ it holds $x_j^t(u/x_i^t) \in I$. This is a pure combinatorial description of the p-Borel ideals which can be given independently of the characteristic of K. Let u be a monomial of S and $J = \langle u \rangle$ the smallest monomial ideal containing u. J is called principal p-Borel ideal. For such ideals there exists a complicated formula for regularity in terms of u conjectured by Pardue [13] and proved in two papers [2], [10] (another proof is given in [11]).

In general it is hard to bound the regularity of a graded ideal I. If char $K = 0$ and $d(I)$ is the highest degree of a minimal monomial generator of I then D. Bayer and D. Mumford [4] showed that $\text{reg}(I) \leq (2d(I))^{2^{n-2}}$. Caviglia and Sbarra [7] showed that the same bound holds for all characteristic of K. This bound seems to be sharp

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since Mayr and Meyer [12] gave an example with $d(I) = 4$ and $\mathrm{reg}(I) \geq 2^{2^{n-1}} + 1$. Thus in general a bound for the regularity is very high, but what about if we restrict to some classes of ideals? In [14] it is showed that $d(I) \leq \mathrm{reg}(I) \leq nd(I)$ if I is a p-Borel ideal. The proof uses the formula conjectured by Pardue. The Betti numbers and the regularity of a graded ideal I can depend of characteristic of the field K even when I is monomial. This is not the case when I is strongly stable as follows from Eliahou and Kervaire resolution [9]. Using again the formula conjectured by Pardue we get that the regularity of principal p-Borel ideals does not depend on the characteristic of K (see [14]).

Let $Syz_t(M) = \mathrm{Ker}(F_t \to F_{t-1})$ be the t-th syzygy module of M. The module M is called (r,t)-*regular* if $Syz_t(M)$ is $(r+t)$-regular in the sense that all generators of F_j for $t \leq j \leq s$ have degrees $\leq j + r$. The t-*regularity* of M, that is, the regularity of $Syz_t(M)$ is given by

$$(t - \mathrm{reg})(M) = \min\{r \ : M \text{ is } (r,t) - \text{regular}\}.$$

Obviously, we have $(t-\mathrm{reg})(M) \leq ((t-1)-\mathrm{reg})(M)$. If the inequality is strict and $r = (t - \mathrm{reg})(M)$, then (t,r) is called a *corner* of M and $\beta_{t,r+t}(M)$ is an *extremal Betti number* of M [3]. As the regularity of principal p-Borel ideals is completely determined by their extremal Betti numbers it is natural to ask if these numbers depend on the characteristic of K. They do not depend indeed in a more general frame explained bellow.

Bayer and Stillman proved that if I is Borel-fixed then it satisfies the following property:

$$(I : x_j^\infty) = (I : (x_1, \ldots, x_j)^\infty)$$

for $j = 1, \ldots, n$. A monomial ideal $I \subset S$ satisfying the above condition is said to be of *Borel type* [11]. For a monomial u, let $\nu_i(u)$ be the highest power of x_i which divides u and $m(u) = \max\{i : \nu_i(u) \neq 0\}$. For a monomial ideal $I \neq 0$, let $G(I)$ be the unique set of monomial minimal generators of I and $m(I) = \max\{m(u) : u \in G(I)\}$. We define recursively an ascending chain of monomial ideals:

$$I = I_0 \subset I_1 \subset \ldots$$

as follows: We let $I_0 = I$ and if $I_e \neq S$ is already defined then set $I_{e+1} = (I_e : x_{n_e}^\infty)$ for $n_e = m(I_e)$, $n_0 > n_1 > \ldots > n_e > \ldots$. If $I_e = S$ then the chain ends. Let $(I_e)_{0 \leq e \leq q}$, (n_e) be the sequences obtained above, $I_q \neq 0$ and $I_{q+1} = S$. Let $S_i = K[x_1, \ldots, x_{n_i}]$, $J_i = I_i \cap S_i$ and J_i^{sat} the saturation of J_i made in S_i.

Theorem 0.1 ([14]). *Let $I \subset S$ be a Borel type ideal. Then S/I has at most $q+1$-corners among $(n_i, s(J_i^{sat}/J_i))$, $0 \leq i \leq q$ and the corresponding extremal Betti numbers are*

$$\beta_{n_i, s(J_i^{sat}/J_i)+n_i} \ (S/I) = \dim_K (J_i^{sat}/J_i)_{s(J_i^{sat}/J_i)},$$

where $s(N) = \max\{i : N_i \neq 0\}$ for a graded S-module N of finite length. In particular the corners of S/I and their corresponding extremal Betti numbers do not depend on the characteristic of K. Moreover the regularity of S/I does not depend of characteristic of K.

Remains to ask what about the general Betti numbers? Bellow we remind you a nice case when this is true. The principal p-Borel ideals $I \subset S$ such that S/I is Cohen-Macaulay have the form $I = \Pi_{j=0}^{s}(m^{[p^j]})^{\alpha_j}$, $0 \le \alpha_j < p$, where $m^{[p^j]} = (x_1^{p^j}, \ldots, x_n^{p^j})$. For these ideals is well known the description of a canonical monomial cycle basis of the Koszul homology module $H_i(x; S/I)$ given by Aramova and Herzog [2] (see details in 4.1). One can easily see from this description that $\beta_{ij}(S/I)$ does not depend on the characteristic of the field K for all i, j.

Now let I be the p-Borel ideal generated by the monomial $x_{n-1}^{\gamma} x_n^{\alpha}$ for some integer $\gamma, \alpha \ge 0$, that is

$$I = \Pi_{j=0}^{s}((m_{n-1}^{[p^j]})^{\gamma_j}(m^{[p^j]})^{\alpha_j}),$$

where $m_{n-1} = (x_1, \ldots, x_{n-1})$, and γ_j, α_j are defined by the p-adic expansion of γ, respectively α. Suppose that $\alpha_j + \gamma_j < p$ for all $0 \le j \le s$. Then $H_i(x; S/I)$ has a monomial cycle basis for all $i \ge 2$, and $\beta_{ij}(S/I)$ does not depend on the characteristic of K for all i, j (see 4.7).

We saw that in some cases of principal p-Borel ideals there exist a monomial cycle basis for the homology modules of S/I. How it is in general? If $I \subset S$ is a monomial ideal then $H_2(x; S/I)$ has a monomial cycle basis (see 1.5). Unfortunately, in general there are no monomial cycle basis even on the Koszul homology modules of principal p-Borel ideals as shows our Example 2.2. However if I is a principal p-Borel ideal then $H_3(x; S/I)$ has a binomial cycle basis (see 3.10). In general our Example 2.7 shows that there are reduced monomial ideals which have not even a trinomial cycle basis. Perhaps in general there exist monomial reduced ideals I in n-variables such that there exists non-zero cycles of length $n-2$ which are not modulo bounds sum of cycles of length $< n-2$.

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1. Cycles of Koszul homology modules of monomial ideals

Let $S = K[x_1, \ldots, x_n]$ be a polynomial algebra over a field K and $I \subset S$ a monomial ideal. A cycle $z \in K_i(x; S/I)$ has the form $z = \Sigma_{j=1}^{s} \gamma_j u_j e_{\sigma_j}$, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \ldots n\}$, $|\sigma_j| = i$ for all $1 \le j \le s$. Since I is monomial the Koszul antiderivation ∂ is multigraded and each cycle is a sum of multigraded cycles. The cycle z is multigraded if $u_j x_{\sigma_j} = u_1 x_{\sigma_1}$ for all $1 \le j \le s$, here $x_{\sigma_1} = \Pi_{k \in \sigma_1} x_k$. We denote $m(u_j) = \max\{i; x_i | u_j\}$ and $m(\sigma_j) = m(x_{\sigma_j})$. Note that in z we may suppose $\sigma_j \ne \sigma_t$ for $j \ne t$ because otherwise it follows $u_j = u_t$ (z is multigraded) and so we may reduce the sum. The element $u_j e_{\sigma_j}$ is a monomial cycle if $\partial(u_j e_{\sigma_j}) = 0$, that is $x_t u_j \in I$ for all $t \in \sigma_j$.

We introduce a totally order on the monomial elements ue_σ of $K_i(x; S/I)$ (u monomial) by "$ue_\sigma \ge ve_\tau$" if either "$u >_{rlex} v$" or "$u = v$" and "$x_\sigma \ge_{rlex} x_\tau$", here rlex denotes the reverse lexicographical order on the monomials of S. As usually we denote $in(z) = u_1 e_{\sigma_1}$ if $u_1 e_{\sigma_1} > u_j e_{\sigma_j}$ for all $j > 1$. A σ_j is called a *neighbour* in z of σ_1 if $|\sigma_j \setminus \sigma_1| = 1$.

Lemma 1.1. *If σ_1 has no neighbour in z then $u_1 e_{\sigma_1}$ is a monomial cycle.*

Proof. Since z is a cycle all the terms of $\partial(u_1e_{\sigma_1})$ should be reduced with terms of some $\partial(u_je_{\sigma_j})$, $j>1$. But this is possible only if σ_j is a neighbour of σ_1.

Lemma 1.2. *Let $z=\Sigma_{j=1}^s\gamma_ju_je_{\sigma_j}$ be a multigraded cycle, $\gamma_j\in K^*$, u_j monomials, $\sigma_j\subset\{1,\dots n\}$, $|\sigma_j|=i$ for all $1\leq j\leq s$. Then the following statements hold:*

1. *If $in(z)=u_1e_{\sigma_1}$ and $m(u_1)>m(\sigma_1)$ then there exists a multigraded element $w\in B_i(x;S/I)$ such that $in(w)=in(z)$.*
2. *For every multigraded cycle w there exist a multigraded cycle z of the above form in the same multigraded homology class with w such that $m(u_j)\leq m(\sigma_j)$ for all $1\leq j\leq s$.*
3. *If z is in the form given by (2) it follows $m(\sigma_j)=m(\sigma_1)$ for all $1\leq j\leq s$.*

Proof. (1) Take a $q>m(\sigma_1)$ such that $x_q|u_1$ and set $y=(u_1/x_q)e_{\sigma_1\cup\{q\}}$. We have ∂y is the sum of $(u_1/x_q)(\partial e_{\sigma_1})\wedge e_{\{q\}}$ with + or - $u_1e_{\sigma_1}$. Thus $in(\partial y)=in(z)$.

(2)+(3) Substracting from z such elements w of $B_i(x;S/I)$ we may arrive to the case $m(u_j)\leq m(\sigma_j)$. Since z is multigraded we get then $m(\sigma_j)=m(\sigma_1)$.

Lemma 1.3. *Let $z=\Sigma_{j=1}^s\gamma_ju_je_{\sigma_j}$ be a multigraded cycle as in the above Lemma. Suppose that $m(u_j)\leq m(\sigma_j)$ for all $1\leq j\leq s$. Then $x_ru_j\in I$ for all $1\leq j\leq s$.*

Proof. By Lemma 1.2 (3) we get $r=m(\sigma_1)=m(\sigma_j)$. The terms $x_ru_je_{\sigma_j\setminus\{r\}}$ of $\partial(u_je_{\sigma_j})$ cannot be reduced since $\sigma_j\setminus\{r\}$ are all different. It follows necessarily $x_ru_j\in I$ since z is a cycle. □

Let $\mathcal{M}_i(x;S/I)$ be the subspace of $K_i(x;S/I)$ generated by all monomial cycles.

Lemma 1.4. *Let $z=\Sigma_{j=1}^s\gamma_ju_je_{\sigma_j}$ be a multigraded 2-cycle, $\gamma_j\in K^*$, u_j monomials, $\sigma_j\subset\{1,\dots,n\}$, $|\sigma_j|=i$ for all $1\leq j\leq s$. Suppose that $m(u_j)\leq m(\sigma_j)$ for all j, $s>1$ and $in(z)=u_1e_{\sigma_1}$. Then one of the following conditions holds:*

1. *$in(z)$ is a monomial cycle,*
2. *$in(z)\equiv u_je_{\sigma_j}$ mod $(B_2(x;S/I)+\mathcal{M}_2(x;S/I))$ for some $1<j\leq s$.*

Proof. By Lemma 1.2 (3) and our hypothesis we get $r=m(\sigma_1)=m(\sigma_j)$ for all $1\leq j\leq s$. Let $\sigma_1=\{a,r\}$. If $x_au_1\in I$ then $u_1e_{\sigma_1}$ is a monomial cycle. Otherwise by Lemma 1.1 there exists a neighbour $\sigma_j=(\sigma_1\setminus\{a\})\cup\{b\}$ for some $1\leq b<r$. As $in(z)=u_1e_{\sigma_1}$ we have $\sigma_1>\sigma_j$ and so $a<b$. From $x_au_1=x_bu_j$ (z is multigraded!) it follows $x_b|u_1$. Set $y=(u_1/x_b)e_{\{a,b,r\}}\in K_3(x;S/I)$. We have

$$\partial y=-u_1e_{\sigma_1}+u_je_{\sigma_j}+(x_ru_1/x_b)e_{\{a,b\}}.$$

Using Lemma 1.3 we have $x_ru_t\in I$ for all $1\leq t\leq s$. This shows that the last term of ∂y is a monomial cycle, which is enough. □

Theorem 1.5. *Every 2-cycles of $K_2(x;S/I)$ belongs to $B_2(x;S/I)+\mathcal{M}_2(x;S/I)$, that is coincides modulo $B_2(x;S/I)$ with a sum of monomial cycles. In particular, $H_2(x;S/I)$ has a monomial cycle basis.*

Proof. Note that given a 2-cycle z in the form from Lemma 1.2 (2) $in(z)$ can be substitute in z modulo $B_2(x;S/I)$ with one monomial term smaller than $in(z)$ and some monomial cycles which can be removed from z (see 1.4). By recurrence we arrive finally to the case when z has just one term which must be then monomial cycle. $\square$

2. Some useful examples

The purpose of this paper is to study when the Koszul homology modules of principal p-Borel ideals I have monomial cycle basis. This is the case when I is the smallest p-Borel ideal containing a power u of one variable x_r (the so called p-Borel ideal generated by u), that is $I = \Pi_{j\geq 0}(\mathfrak{m}_r^{\alpha_j})^{[p^j]}$, where $\mathfrak{m}_r = (x_1, \dots, x_r)$ and $\alpha_j < p$ are non-negative integers (see [2]). For an ideal J we denote by $J^{[p^j]}$ the ideal generated in S by $\varphi(J)$, φ being the K-automorphism of S given by $x \to x^{p^j}$. An interesting and promising example is the following:

Example 2.1. Let $n = 4$, $S = K[x_1, \dots, x_4]$ and I the p-Borel ideal generated by the monomial $\{x_3x_4^p\}$, that is $I = (x_1, x_2, x_3)(x_1^p, \dots, x_4^p)$. Then $z = x_1^{p-1}x_3x_4^{p-1}e_{124} - x_1^{p-1}x_2x_4^{p-1}e_{134}$ is a cycle. Take $y = x_1^{p-1}x_4^{p-1}e_{1234} \in K_4(x; S/I)$. We have $\partial y = z + x_1^px_4^{p-1}e_{234} - x_1^{p-1}x_4^pe_{123}$. Note that $x_1^{p-1}x_4^p \in I$ but $x_1^px_4^{p-1} \notin I$. Thus z coincides with $x_1^px_4^{p-1}e_{234}$ modulo $B_3(x; S/I)$. As z and ∂y are cycles we get $x_1^px_4^{p-1}e_{234}$ monomial cycle.

Unfortunately, in general there are not monomial cycle basis even on the Koszul homology modules of principal p-Borel ideals as shows the following examples:

Example 2.2. Let $n = 4$, that is $S = K[x_1, \dots, x_4]$ and I the p-Borel ideal generated by the monomial $\{x_2x_4^p\}$, that is $I = (x_1, x_2)(x_1^p, \dots, x_4^p)$. Consider the element $z = x_2x_3^{p-1}x_4^{p-1}e_{134} - x_1x_3^{p-1}x_4^{p-1}e_{234} \in K_3(x; S/I)$. We see that z is binomial cycle but $in(z) = x_2x_3^{p-1}x_4^{p-1}e_{134}$ is not a monomial cycle because $x_1x_2x_3^{p-1}x_4^{p-1} \notin I$. Note that z is multigraded and in its multigraded homology class take another element of the form $z + \partial y$, $y \in K_4(x; S/I)$. Since y must be multigraded from the same multigraded class with z we see that the only possibility is to take $y = x_3^{p-1}x_4^{p-1}e_{1234}$. It follows that there exist no monomial cycle in the homology class of z.

Remark 2.3. Let S and I as in Example 2.2. The element

$$z = x_1^{p-1}x_3x_4^{p-1}e_{124} - x_1^{p-1}x_2x_4^{p-1}e_{134} + x_1^px_4^{p-1}e_{234} \in K_3(x; S/I)$$

is a cycle in the form given by Lemma 1.2 (2) but belong to $B_3(x; S/I)$ because $z = z - x_1^{p-1}x_4^pe_{123} = \partial(x_1^{p-1}x_4^{p-1}e_{1234})$, $x_1^{p-1}x_4^p$ being an element in I.

Example 2.2 suggests the following:

Lemma 2.4. *Let I be an arbitrary monomial ideal. Then $H_{n-1}(x; S/I)$ has a basis given by cycles of length $\leq \lfloor n/2 \rfloor$.*

Proof. Suppose y is a multigraded $(n-1)$-cycle in the form given by Lemma 1.2 (2). Then there exist $1 \leq k_s < \ldots < k_1 < n$ such that $y = \Sigma_{j=1}^s \gamma_j u_j e_{\sigma_j}$, $\gamma_j \in K^*$, $u_j \notin I$ monomials and $\sigma_j = \{1, \dots, n\} \setminus \{k_j\}$. Suppose that y cannot be written as a sum of cycles of length $< s$. We will show that we may choose a cycle y' of length $\leq \lfloor n/2 \rfloor$ which coincides with y modulo $B_{n-1}(x; S/I)$. This is enough because then these cycles will give a system of generators of $H_{n-1}(x; S/I)$ from which we may choose a basis. We claim that $\gamma_r = (-1)^{k_1-k_r}\gamma_1$ for all $1 \leq r \leq s$. Let $E = \{r : 1 \leq r \leq s, \gamma_r = (-1)^{k_1-k_r}\gamma_1\}$. If $E \neq \{1,\dots,s\}$ then the element $q = \Sigma_{j\in E}\gamma_j u_j e_{\sigma_j}$ of $K_{n-1}(x;S/I)$ is different from y. Thus y cannot be a cycle because otherwise $y = q + (y-q)$ is a decomposition of y in a sum of two cycles of smaller length which is false $(1 \in E)$ by our assumption. So one of σ_j, $j \in E$ has a neighbour σ_t in z which is not in q, that is $t \notin E$, $x_{k_t}u_j \notin I$ and $((-1)^{k_t-1}\gamma_j - (-1)^{k_j-1}\gamma_t)x_{k_t}u_j e_{\sigma_j\setminus\{k_t\}} = 0$ because z is a cycle (actually the above equation is written for the case $k_j > k_t$, otherwise all the signs changed but the equation is not really affected). Thus $\gamma_t = (-1)^{k_j-k_t}\gamma_j = (-1)^{k_1-k_t}\gamma_1$, that is $t \in I$ which is false. Hence $E = \{1,\dots,s\}$, that is our claim holds.

As y is multigraded note that $x_{k_j}|u_{k_j}$. We have the following cycle

$$y' = \partial(\gamma_1(u_1/x_{k_1})e_{1\dots n}) - y = \gamma_1\Sigma(-1)^{k-1}(x_k u_1/x_{k_1})e_{\sigma_k},$$

where $\sigma_k = \{1,\dots,n\}\setminus\{k\}$ and the sum is made over all $k \in \{1,\dots,n\}\setminus\{k_1,\dots,k_s\}$. If $y' = 0$ then $y \in B_{n-1}(x;S/I)$ and there exist nothing to show. If $y' \neq 0$ then $\text{length}(y)+\text{length}(y') \leq n$. Thus $\min\{\text{length}(y), \text{length}(y')\} \leq \lfloor n/2 \rfloor$. As $y \equiv -y'$ mod $B_{n-1}(x;S/I)$ we are done.

Example 2.5. Let $I = (x_3x_4x_5, x_2x_4x_5, x_1x_2x_4, x_1x_2x_3, x_1x_3x_5)$ be an ideal in $S = K[x_1,\dots,x_5]$. We claim that there exists no monomial cycles or binomial cycles in the homology class of the following multigraded cycle

$$z = x_3x_4e_{125} - x_2x_4e_{135} + x_1x_3e_{245}.$$

We adopt the following notation: for a monomial element ue_{abc} we will write let us say $\bar{a}$, that is $ue_{\bar{a}bc}$ if $x_a u \notin I$. So we may write

$$z = x_3x_4e_{\bar{1}\bar{2}5} - x_2x_4e_{1\bar{3}5} + x_1x_3e_{2\bar{4}5}$$

and now we can see easily that z is indeed a cycle. We list all monomial elements of $K_3(x;S/I)$, which are in the multigraded class of z:

$$x_4x_5e_{\bar{1}23}, x_3x_5e_{1\bar{2}4}, x_3x_4e_{\bar{1}\bar{2}5}, x_2x_5e_{\bar{1}34}, x_2x_4e_{1\bar{3}5}, x_2x_3e_{1\bar{4}\bar{5}}, x_1x_5e_{\bar{2}3\bar{4}},$$

$$x_1x_4e_{2\bar{3}\bar{5}}, x_1x_3e_{2\bar{4}5}, x_1x_2e_{34\bar{5}}.$$

Clearly no one is monomial cycle. A binomial cycle should have the form $\gamma_1 u_1 e_{a\bar{b}c} + \gamma_2 u_2 e_{ac\bar{d}}$, $\gamma_i \in K^*$, u_i monomials. Thus we might find such pairs $(a\bar{b}c), (ac\bar{d})$ among above. But there are no such pairs. For example $e_{\bar{1}23}$ could make such a pair only with $e_{\bar{2}3\bar{4}}$, $e_{2\bar{3}\bar{5}}$ but they are not of the necessary type because each one has two numbers in bold. In this way we see that our claim holds. It follows that z cannot be written modulo $B_3(x;S/I)$ as a sum of cycles of smaller length.

Remark 2.6. In [6, Exercise 5.5.4] we see that for monomial ideals I the Betti numbers $\beta_2, \beta_{n-2}, \beta_{n-1}$ do not depend on the characteristic of K. Since in the previous section we see that $H_2(x;S/I)$ has monomial cycle basis we may ask by analogy if $H_{n-2}(x;S/I)$ or $H_{n-1}(x;S/I)$ have monomial cycle basis. Example 2.2 shows that this is not true. On the other hand note that in $K_{n-2}(x;S/I)$ there are cycles which cannot be written modulo $B_{n-2}(x;S/I)$ as a sum of cycles of length $\leq \lfloor n/2 \rfloor$ (see Example 2.5) as happens in the $K_{n-1}(x;S/I)$ (see Lemma 2.4).

We might ask which is the minimal possible positive integer r such that $H_3(x; S/I)$ has a basis given by 3-cycles of length $\leq r$. The following example shows that r could be even 4.

Example 2.7. Let $I = (x_3x_4x_5x_6, x_2x_4x_5x_6, x_1x_2x_4x_6, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_3x_5x_6)$ be an ideal in $S = K[x_1, \dots, x_6]$. We claim that there exists no monomial cycles or binomial cycles in the homology class of the following multigraded cycle

$$z = x_3x_4x_5e_{\bar{1}\bar{2}6} - x_2x_4x_5e_{\bar{1}\bar{3}6} + x_1x_3x_5e_{2\bar{4}6} - x_1x_2x_4e_{3\bar{5}6}$$

We list all monomial elements of $K_3(x; S/I)$, which are in the multigraded class of z:

$$x_4x_5x_6e_{\bar{1}23}, x_3x_5x_6e_{1\bar{2}4}, x_3x_4x_6e_{\bar{1}\bar{2}5}, x_3x_4x_5e_{\bar{1}\bar{2}6}, x_2x_5x_6e_{\bar{1}\bar{3}4}, x_2x_4x_6e_{1\bar{3}5}, x_2x_4x_5e_{\bar{1}\bar{3}6},$$

$$x_2x_3x_6e_{\bar{1}\bar{4}5}, x_2x_3x_5e_{1\bar{4}\bar{6}}, x_2x_3x_4e_{1\bar{5}\bar{6}}, x_1x_5x_6e_{\bar{2}3\bar{4}}, x_1x_4x_6e_{2\bar{3}\bar{5}}, x_1x_4x_5e_{\bar{2}\bar{3}\bar{6}},$$

$$x_1x_3x_6e_{\bar{2}\bar{4}5}, x_1x_3x_5e_{2\bar{4}6}, x_1x_3x_4e_{2\bar{5}\bar{6}}, x_1x_2x_6e_{\bar{3}4\bar{5}}, x_1x_2x_5e_{3\bar{4}\bar{6}}, x_1x_2x_4e_{3\bar{5}6}, x_1x_2x_3e_{45\bar{6}}.$$

Clearly no one is monomial cycle. There are only 6 elements from the above list (with just one "bar"), which can be used to construct binomial cycles in the multigraded homology class of z. As in Example 2.5 we see that there are no binomial cycles. Now a cycle of length 3 should have the form $\gamma_1u_1e_{a\bar{b}c} + \gamma_2u_2e_{a\bar{c}\bar{d}} + \gamma_3u_3e_{ad\bar{t}}$, $\gamma_i \in K*$, u_i monomials, for some $a, b, c, d, t \in \{1, \dots, n\}$, we may also have $t \in \{a, b\}$. We claim that this is not possible. For example if $a = 1$, $b = 2$, $c = 4$ then $d \in \{3, 5, 6\}$. If $d = 5$ note that in fact we have $e_{\bar{a}\bar{c}\bar{d}} = e_{1\bar{4}\bar{5}}$ which is not possible to be term in a cycle of length 3. If $d = 6$ then $t \in \{2, 3, 5\}$. If for example $t = 5$ then we have $e_{ad\bar{t}} = e_{1\bar{5}\bar{6}}$ which is not possible again. In this way we can show that our claim holds. It follows that z cannot be written modulo $B_3(x; S/I)$ as a sum of cycles of smaller length.

3. Cycles of Koszul homology modules of principal p-Borel ideals

Let $u = \Pi_{q=1}^n x_q^{\lambda_q}$ be a monomial and $J = \Pi_{q=1}^n (x_1, \dots, x_q)^{\alpha_q}$ for some integers $\lambda_q, \alpha_q \geq 0$. Set $u_{\leq a} = \Pi_{q=1}^a x_q^{\lambda_q}$, $J_{\leq a} = \Pi_{q=1}^a (x_1, \dots, x_q)^{\alpha_q}$ for some integer $1 \leq a < n$ and $u_{>a} = \Pi_{q>a}^n x_q^{\lambda_q}$, $J_{>a} = \Pi_{q>a}^n (x_1, \dots, x_q)^{\alpha_q}$.

Lemma 3.1. *Suppose that $ux_a \in J$ and $ux_{a+1} \notin J$. Then $u_{>a} \in J_{>a}$ and $u_{\leq a} \notin J_{\leq a}$.*

Proof. By hypothesis we have $ux_a = vw$ for some monomials $v \in J_{\leq a}$ and $w \in J_{>a}$. If there exists $1 \leq j \leq a$ with $x_j | w$ then

$$ux_{a+1} = (x_jv/x_a)(x_{a+1}w/x_j) \in J_{\leq a}J_{>a} = J.$$

Contradiction! It follows $w | u_{>a} = (ux_a)_{>a}$ and so $u_{>a} \in J_{>a}$. If $u_{\leq a} \in J_{\leq a}$ then $u \in J$ and so $ux_{a+1} \in J$ which is false. □

Lemma 3.2. *Let v, w be some monomials in $(x_q)_{q>a}$ such that $v | u$. Suppose that $wu/v \in J$ and $ux_a^r \in J$ for some integer $r > 0$. Then $ux_{a+1}^r \in J$.*

Proof. Suppose that $ux_{a+1}^r \notin J$. We may suppose that there exists an integer $0 \leq d < r$ such that for $u' = ux_a^d x_{a+1}^{r-d-1}$ we have $u'x_a \in J$ and $u'x_{a+1} \notin J$. By Lemma 3.1 we get $u'_{\leq a} \notin J_{\leq a}$. Since $wu/v = u_{\leq a}(wu_{>a}/v) \in J \subset J_{\leq a}$ it follows $u_{\leq a} \in J_{\leq a}$ because the variables from $wu_{>a}/v$ are regular on $S/J_{\leq a}$. Thus $u'_{\leq a} \in J$ because $u'_{\leq a} = x_a^d u_{\leq a}$. Contradiction! □

Lemma 3.3. *Let $t > a$ be an integer and v, w be some monomials in $(x_q)_{q\geq t}$ such that $v|u$. Suppose that $wu/v \in J$ and $ux_a^r \in J$ for some integer $r > 0$. Then $ux_t^r \in J$.*

The proof goes applying Lemma 3.2 by recurrence.

Let $I = \Pi_{q=1}^n \Pi_{j=0}^s ((x_1, \ldots, x_q)^{[p^j]})^{\alpha_{qj}}$ be a principal p-Borel ideal of S, $0 \leq \alpha_{qj} < p$ being some integers. Let $G(I)$ be the set of minimal monomial generators of I. Note that all monomials from $G(I)$ have the same degree.

Lemma 3.4. *Let a, t, q, r be four positive integers such that $a < t < r$, $a < q < r$, $q \neq t$ and $\gamma \in I$ a monomial which is a multiple of $x_q x_r$.Suppose that $x_a\gamma/x_r \in I$ and $x_t\gamma/x_q \in I$. Then either $x_t\gamma/x_r \in I$, or $x_a\gamma/x_q \in I$.*

Proof. Apply induction on s. If $s = 0$ then from $x_t\gamma/x_q \in I$ we get $x_a\gamma/x_q \in I$, since I is strongly stable in this case.

Now suppose $s > 1$ and set $J = \Pi_{q=1}^n ((x_1, \ldots, x_q)^{[p^j]})^{\alpha_{q0}}$, $T = \Pi_{q=1}^n \Pi_{j=1}^{s-1} ((x_1, \ldots, x_q)^{[p^j]})^{\alpha_{q,j+1}}$. Then $I = JT^{[p]}$. We have $\gamma = \beta\alpha^p$ for some monomials $\beta \in J$, $\alpha \in G(T)$.

Suppose from now on that $x_t\gamma/x_r \notin I$. We will show that $x_a\gamma/x_q \in I$. Then x_r does not divide β since otherwise $x_t\gamma/x_r = (x_t\beta/x_r)\alpha^p \in I$, J being strongly stable. Contradiction! If $x_q|\beta$ then $x_a\beta/x_q \in J$ because J is strongly stable and so $x_a\gamma/x_q = (x_a\beta/x_q)\alpha^p \in I$. Remains to study the case when x_q does not divide β. Then $x_q|\alpha$. We have $\alpha/x_q \notin T$ because $\alpha \in G(T)$. From $x_t\gamma/x_q = x_t\beta x_q^{p-1}(\alpha/x_q)^p \in I$ it follows either

(i) there exist $b, 1 \leq b \leq n$ such that $x_b^p|\beta$, $x_b\alpha/x_q \in T$ and $x_t x_q^{p-1}\beta/x_b^p \in T$, or

(ii) $x_t^{p-1}|\beta$, $x_q^{p-1}\beta/x_t^{p-1} \in J$ and $x_t\alpha/x_q \in T$.

Here we use the fact that all minimal monomial generators of $G(T)$ have the same degree and since $\alpha \in G(T)$ we get $v\alpha/x_q \in T$ for v being just one variable. If (i) holds then $x_a\gamma/x_q = (x_a x_q^{p-1}\beta/x_b^p)(x_b\alpha/x_q)^p \in I$ since J is strongly stable.

From now on suppose that (ii) holds. As $x_a\gamma/x_r = x_a x_r^{p-1}\beta(\alpha/x_r)^p \in I$ we get as above the following two cases:

(i') $x_a^{p-1}|\beta$, $x_r^{p-1}\beta/x_a^{p-1} \in J$ and $x_a\alpha/x_r \in T$, or

(ii') there exist $c, 1 \leq c \leq n$ such that $x_c^p|\beta$, $x_c\alpha/x_r \in T$ and $x_a x_r^{p-1}\beta/x_c^p \in T$.

Case (i') holds

It follows $x_r^{p-1}\beta/x_t^{p-1} \in J$ since J is strongly stable. Note that α satisfies over T the condition of γ over I. By induction hypothesis we get either $x_t\alpha/x_r \in T$ or $x_a\alpha/x_q \in T$. If $x_t\alpha/x_r \in T$ then we get $x_t\gamma/x_r = (x_r^{p-1}\beta/x_t^{p-1})(x_t\alpha/x_r)^p \in I$ which is false. Thus we must have $x_a\alpha/x_q \in T$. Then $x_a\gamma/x_q = (x_q^{p-1}\beta/x_a^{p-1})(x_a\alpha/x_q)^p \in I$ since $x_r^{p-1}\beta/x_a^{p-1} \in J$, J being strongly stable.

Case (ii') holds

Suppose first $c \geq t$. From (ii),(ii') we obtain $x_a x_r^{p-1}\beta/x_c^p \in J$ and $x_q^{p-1}\beta/x_t^{p-1} \in J$. Using Lemma 3.3 for $u = x_r^{p-1}\beta/x_c^p$, $v = (x_r x_t)^{p-1}$, $w = (x_c x_q)^{p-1}$ we get $ux_t = x_t x_r^{p-1}\beta/x_c^p \in J$. Then $x_t\gamma/x_r = (x_t x_r^{p-1}\beta/x_c^p)(x_c\alpha/x_r)^p \in I$. Contradiction!

Now suppose $c < t$. By (ii),(ii') we get $x_c\alpha/x_r \in T$, $x_t\alpha/x_q \in T$. Apply the induction hypothesis for the integers c, t, q, r and the ideal T. It follows either $x_c\alpha/x_q \in T$, or $x_t\alpha/x_r \in T$. In the first case we get $x_a\gamma/x_q = (x_a x_q^{p-1}\beta/x_c^p)(x_c\alpha/x_q)^p \in I$ because $x_a x_r^{p-1}\beta/x_c^p \in J$, J being strongly stable.

In the second case we can obtain $x_t\gamma/x_r = (x_r^{p-1}\beta/x_t^{p-1})(x_t\alpha/x_r)^p \in I$, that is a contradiction, providing we show that $x_r^{p-1}\beta/x_t^{p-1} \in J$. But from $x_a x_r^{p-1}\beta/x_c^p \in J$ we get $x_a x_r^{p-1}\beta/x_c x_t^{p-1} \in J$ since J is strongly stable. Moreover if $c \leq a$ then we get even $x_r^{p-1}\beta/x_t^{p-1} \in J$. If $c > a$ then we get the same thing using Lemma 3.3 for $u = x_r^{p-1}\beta/(x_c x_t^{p-1})$, $w = x_c x_t^{p-1}$, $v = x_r^{p-1}$ and c instead t. □

Proposition 3.5. *Let $z = \Sigma_{j=1}^s \gamma_j u_j e_{\sigma_j}$ be a multigraded 3-cycle of $K_i(x; S/I)$, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \dots n\}$, $|\sigma_j| = i$ for all $1 \leq j \leq s$. Suppose that $s > 1$, $m(u_j) \leq m(\sigma_j)$ for all j, $\sigma_1 = \{a, t, r\}$, $a < t < r$, $\sigma_1 = \max_{1 \leq j \leq s} \sigma_j$ and $x_a u_1 \in I$. Then there exists a multigraded 3-cycle y of length ≤ 3 such that $in(z) = in(y)$. Moreover if the length of y is 3 then the homology class of y contains a monomial cycle.*

Proof. By Lemmas 1.2 and 1.3 we have $m(\sigma_j) = r$ and $x_r u_j \in I$ for all $1 \leq j \leq s$. Set $\gamma = x_r u_1 \in I$. Then $x_a\gamma/x_r \in I$ by hypothesis. We may suppose $x_t u_1 \notin I$ because otherwise $y = in(z)$ is a monomial cycle. Then σ_1 has a neighbour σ_j in z for $j > 1$ by Lemma 1.1, let us say $\sigma_j = \{a, q, r\}$. As $\sigma_j < \sigma_1$ we get $t < q$. We have $u_j = x_t u_1/x_q$ because z is multigraded. It follows $x_t\gamma/x_q = x_r u_j \in I$. By Lemma 3.4 we get $x_a\gamma/x_q \in I$ since $x_t\gamma/x_r = x_t u_1 \notin I$. Therefore $\partial((u_1/x_q)e_{atqr}) = y - (x_r u_1/x_q)e_{atq}$ for $y = u_1 e_{\sigma_1} - u_j e_{\sigma_j} + (x_a u_1/x_q)e_{tqr}$. From the above we see that $(\gamma/x_q)e_{atq}$ is a monomial cycle and so y is a cycle. □

Remark 3.6. If $x_a u_1 \notin I$ but $x_t u_1 \in I$ then $(x_r u_1/x_q)e_{atq}$ might be not a monomial cycle as happens in the proof of the above proposition. Note that in the Example 2.2 $(x_r u_1/x_q)e_{atq} = x_3^{p-1}x_4^p e_{123}$ is not a monomial cycle because $x_3(x_3^{p-1}x_4^p) \notin I$. However in this example z is already a binomial cycle, that is a cycle of length ≤ 3.

Lemma 3.7. *Let a, t, q, r be four positive integers such that $a < t < r$, $a < q < r$ and $\gamma \in I$ a monomial which is a multiple of $x_q x_r$. Suppose that $x_t\gamma/x_r \in I$, $x_a\gamma/x_q \in I$ and $x_t\gamma/x_q \notin I$. Then the following statements hold:*

(i) *If $q > t$ then $x_a\gamma/x_r \in I$.*
(ii) *If $q < t$ then $x_t x_a\gamma/x_r x_q \in I$.*

Proof. Apply induction on s. If $s = 0$ then from $x_t\gamma/x_r \in I$ we get $x_a\gamma/x_r \in I$ and also $x_t x_a\gamma/x_r x_q \in I$ since I is strongly stable in this case.

Now suppose $s > 1$ and set J, T and $\gamma = \beta\alpha^p$ for some monomials $\beta \in J$, $\alpha \in G(T)$ as in the proof of Lemma 3.4. Like there we may suppose that x_r does not divide β. From $x_t\gamma/x_r = x_t\beta x_r^{p-1}(\alpha/x_r)^p \in I$ it follows either

(j) there exist $b, 1 \leq b \leq n$, $b \neq t$ such that $x_b^p|\beta$, $x_b\alpha/x_r \in T$ and $x_t x_r^{p-1}\beta/x_b^p \in J$, or

(jj) $x_t^{p-1}|\beta$, $x_r^{p-1}\beta/x_t^{p-1} \in J$ and $x_t\alpha/x_r \in T$.

In case (j) we have

$$x_a\gamma/x_r = ((x_a/x_t)(x_t\beta x_r^{p-1}/x_b^p))(x_b\alpha/x_r)^p \in I,$$

J being strongly stable.

Case (jj) when $x_q|\beta$

If $t < q$ then $x_t\beta/x_q \in J$ because J is strongly stable. Thus $x_t\gamma/x_q \in I$. Contradiction! If $t > q$ then we get

$$x_t x_a\gamma/x_q x_r = ((x_a/x_q)(x_r^{p-1}\beta/x_t^{p-1})(x_t\alpha/x_r)^p \in I.$$

Case (jj) when $x_q \not|\beta$.

Then $x_q|\alpha$. As $\alpha/x_q \notin T$ it follows from $x_a\gamma/x_q \in I$ either

(j') there exist $c, 1 \leq c \leq n$ such that $x_c^p|\beta$, $x_c\alpha/x_q \in T$ and $x_a x_q^{p-1}\beta/x_c^p \in J$, or

(jj') $x_a^{p-1}|\beta$, $x_q^{p-1}\beta/x_a^{p-1} \in J$ and $x_a\alpha/x_q \in T$.

Subcase (j'), $c \geq t$

Let $u = x_q^{p-1}\beta/x_c^p$. We have $x_t\gamma/x_q = (x_t u)(x_c\alpha/x_q)^p \in I$ because by (jj) it follows $x_t u = (x_a x_t^{p-1}/x_c^p)(x_q^{p-1}/x_r^{p-1})(x_r^{p-1}\beta/x_t^{p-1}) \in J$, J being strongly stable. Contradiction!

Subcase (j'), $t > c$, $t < q$.

Apply induction on s for α, T. Since $x_t\alpha/x_r \in T$, $x_c\alpha/x_q \in T$ we get either $x_t\alpha/x_q \in T$ or $x_c\alpha/x_r \in T$. If $x_t\alpha/x_q \in T$ then $x_t\gamma/x_q = (x_q^{p-1}\beta/x_t^{p-1})(x_t\alpha/x_q)^p \in I$ because $x_r^{p-1}\beta/x_t^{p-1} \in J$, J being strongly stable. Contradiction! Note that we did not use the condition $t < q$ in order to get this contradiction from $x_t\alpha/x_q \in T$. Now suppose that $x_c\alpha/x_r \in T$. Then we have $x_a\gamma/x_r = (x_a x_r^{p-1}\beta/x_c^p)(x_c\alpha/x_r)^p \in I$ if we show that $v = x_a x_r^{p-1}\beta/x_c^p \in J$. By (j') we have $x_a x_q^{p-1}\beta/x_c^p \in J$. If $a \geq c$ then we get $x_q^{p-1}\beta/x_c^{p-1} \in J$, J being strongly stable. If $a < c$ we get the same thing applying Lemma 3.3 to $u' = x_q^{p-1}\beta/x_c^p$ because $x_a u' \in J$ and $u'(x_c^p/x_q^{p-1}) = \beta \in J$.

Set $\hat{u} = x_r^{p-1}\beta/x_t^{p-1}x_c^{p-1}$. We have $x_c^{p-1}\hat{u} \in J$ and $\hat{u}(x_q^{p-1}x_t^{p-1}/x_r^{p-1}) \in J$ from above. By Lemma 3.3 we get $x_t^{p-1}\hat{u} \in J$, that is $x_r^{p-1}\beta/x_c^{p-1} \in J$. If $c \geq a$ it follows $v \in J$ since J is strongly stable. If $c < a$ apply Lemma 3.3 for $u" = x_r^{p-1}\beta/x_c^p$ having $x_c u" \in J$ and $u"(x_a x_q^{p-1}/x_r^{p-1}) \in J$. It follows $v = x_a u" \in J$.

Subcase (j'), $t > c$, $t > q$.

Apply induction on s for α, T. Since $x_t\alpha/x_r \in T$, $x_c\alpha/x_q \in T$ we get either $x_t\alpha/x_q \in T$ or $x_t x_c\alpha/(x_q x_r) \in T$. We saw above that $x_t\alpha/x_q \in T$ gives a contradiction. Suppose that $x_t x_c\alpha/(x_q x_r) \in T$. Then we have $x_t x_a\gamma/(x_q x_r) = (x_a x_r^{p-1}x_q^{p-1}\beta/(x_t^{p-1}x_c^p))(x_t x_c\alpha/(x_r x_q))^p \in I$ if we show that $w = x_a x_r^{p-1}x_q^{p-1}\beta/(x_t^{p-1}x_c^p) \in J$. By (j') we have $x_a x_q^{p-1}\beta/x_c^p \in J$. As in the previous case we get $x_q^{p-1}\beta/x_c^{p-1} \in J$ if either $a \geq c$ or $a < c \leq q$. If $a < q < c$ the same tricks bring only that $x_q u' \in J$, that is $x_q^p\beta/x_c^p \in J$.

Suppose $a \geq c$ or $a < c \leq q$. We have $x_c^{p-1}\hat{u} \in J$ and $\hat{u}(x_q^{p-1}x_t^{p-1}/x_r^{p-1}) \in J$ as above. By Lemma 3.3 we get now $x_q^{p-1}\hat{u} \in J$, that is $x_r^{p-1}x_q^{p-1}\beta/(x_t^{p-1}x_c^{p-1}) \in$

J. Therefore $w \in J$ if $a \leq c \leq q$ because J is strongly stable. If $a > c$ then $y = x_r^{p-1}x_q^{p-1}\beta/(x_t^{p-1}x_c^p)$ satisfies $x_c y \in J$ and $yx_t^{p-1}x_a/x_r^{p-1} \in J$. Therefore by Lemma 3.3 we get $w \in J$. If $a < q < c$ then $z = \hat{u}/x_c$ satisfies $x_c^p z \in J$ and $z(x_q^p x_t^{p-1}/x_r^{p-1}) \in J$. By Lemma 3.3 we get as above $x_q^p x_r^{p-1}\beta/(x_t^{p-1}x_c^p) = x_q^p z \in J$ and so $w \in J$ because J is strongly stable.

Subcase (jj'), $t < q$.

As above $x_t\alpha/x_q \notin T$. Apply induction on s for α, T. Since $x_t\alpha/x_r \in T$, $x_c\alpha/x_q \in T$ we get $x_a\alpha/x_r \in T$. It follows that

$$x_a\gamma/x_r = (x_r^{p-1}\beta/x_a^{p-1})(x_a\alpha/x_r)^p \in I$$

if $x_r^{p-1}\beta/x_a^{p-1} \in J$. Set $\bar{u} = x_r^{p-1}\beta/x_a^{p-1}x_t^{p-1}$ and note that $x_a^{p-1}\bar{u} \in J$ by (jj) and $\bar{u}(x_t^{p-1}x_q^{p-1}/x_r^{p-1}) \in J$ by (jj'). By Lemma 3.3 we get $x_t^{p-1}\bar{u} \in J$ which is enough.

Subcase (jj'), $t > q$.

As in the previous case we use induction hypothesis to get $x_t x_a\alpha/(x_r x_q) \in T$. We have

$$x_t x_a\gamma/(x_r x_q) = (x_r^{p-1}x_q^{p-1}\beta/(x_t^{p-1}x_a^{p-1}))(x_t x_a\alpha/(x_r x_q))^p \in I$$

if we show that $v' = x_r^{p-1}x_q^{p-1}\beta/(x_t^{p-1}x_a^{p-1}) \in J$. Apply Lemma 3.3 for $\tilde{u} = x_r^{p-1}\beta/(x_t^{p-1}x_a^{p-1})$ because $x_a^{p-1}\tilde{u} \in J$ and $\tilde{u}(x_q^{p-1}x_t^{p-1}/x_r^{p-1}) \in J$. We obtain $x_q^{p-1}\tilde{u} \in J$ which is enough.

Proposition 3.8. *Let $z = \Sigma_{j=1}^s \gamma_j u_j e_{\sigma_j}$ be a multigraded 3-cycle of $K_i(x; S/I)$, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \ldots n\}$, $|\sigma_j| = i$ for all $1 \leq j \leq s$. Suppose that $s > 1$, $m(u_j) \leq m(\sigma_j)$ for all j, $\sigma_1 = \{a, t, r\}$, $a < t < r$, $\sigma_1 = \max_{1 \leq j \leq s} \sigma_j$ and $x_t u_1 \in I$. Then there exists a multigraded 3-cycle y of length ≤ 3 such that $in(z) = in(y)$. Moreover if the length of y is 3 then the homology class of y contains a monomial cycle.*

Proof. We follow the proof of Proposition 3.5. Set $\gamma = x_r u_1 \in I$. Then $x_t\gamma/x_r \in I$ by hypothesis and we may suppose $x_a u_1 \notin I$. Thus σ_1 has a neighbour σ_j in z for $j > 1$ by Lemma 1.1, let us say $\sigma_j = \{t, q, r\}$. We have $u_j = x_a u_1/x_q$ because z is multigraded. It follows $x_a\gamma/x_q = x_r u_j \in I$. Suppose $t < q$. By Lemma 3.7 we get $x_a\gamma/x_r \in I$ since $x_t\gamma/x_q = x_a u_1 \notin I$. Therefore $\partial((u_1/x_q)e_{atqr}) = y - (x_r u_1/x_q)e_{atq}$ for $y = u_1 e_{\sigma_1} + u_j e_{\sigma_j} - (x_t u_1/x_q)e_{tqr}$. From the above we see that $(\gamma/x_q)e_{atq}$ is a monomial cycle and so y is a cycle. Now suppose $q < t$. Then the same lemma gives that $x_t u_j \in I$, that is $u_1 e_{\sigma_1} - u_j e_{\sigma_j}$ is a binomial cycle. □

Proposition 3.9. *Let $z = \Sigma_{j=1}^s \gamma_j u_j e_{\sigma_j}$ be a multigraded 3-cycle of $K_i(x; S/I)$, $\gamma_j \in K^*$, u_j monomials, $\sigma_j \subset \{1, \ldots n\}$, $|\sigma_j| = i$ for all $1 \leq j \leq s$. Suppose that $s > 1$, $m(u_j) \leq m(\sigma_j)$ for all j, $\sigma_1 = \{a, t, r\}$, $a < t < r$ and $\sigma_1 = \max_{1 \leq j \leq s} \sigma_j$. Then there exists a multigraded 3-cycle y of length ≤ 4 such that $in(z) = in(y)$. Moreover if the length of y is 3 then the homology class of y contains a monomial cycle and if the length is 4 then the homology class of y contains a binomial cycle.*

Proof. We follow the proof of Propositions 3.5 and 3.8. Set $\gamma = x_r u_1 \in I$. Using the quoted propositions we may suppose $x_t u_1 \notin I$ and $x_a u_1 \notin I$. Thus σ_1 has two neighbours σ_j, σ_k in z for $j, k > 1$ by Lemma 1.1, let us say $\sigma_j = \{a, q, r\}$ and

$\sigma_k = \{c,t,r\}$. We have $u_k = x_a u_1/x_c$, $u_j = x_t u_1/x_q$ because z is multigraded. It follows $x_a\gamma/x_c = x_r u_k \in I$ and $x_t\gamma/x_q \in I$ by Lemma 1.3. As $u_1 e_{\sigma_1} = in(z)$ we get $\sigma_j < \sigma_1$ and so $t < q$.

Case $c = q$.

Thus we have $x_a x_r u_1/x_q = x_a\gamma/x_q \in I$. We see that $y' = (x_r u_1/x_q)e_{atq}$ is a monomial cycle and so $y = y' + \partial((u_1/x_q)e_{atqr})$ is a cycle of length ≤ 3 such that $in(z) = in(y)$.

Case $a < t < \min\{c,q\}$

Suppose $c < q$. By Lemma 3.7 we get either $x_a\gamma/x_q \in I$ or $x_t\gamma/x_c \in I$. If $c > q$ then by Lemma 3.4 we get the same thing. Above we already studied the case when $x_a x_r u_1/x_q \in I$. If $x_t x_r u_1/x_c = x_t\gamma/x_c \in I$ then similarly $y" = (x_r u_1/x_c)e_{atc}$ is monomial cycle and $y = y" + \partial((u_1/x_q)e_{atcr})$ is a cycle of length ≤ 3 such that $in(z) = in(y)$.

Case $a < c < t < q$.

Using Lemma 3.7 as above we get either $x_t\gamma/x_c \in I$ (case already studied above) or $x_t x_a\gamma/(x_q x_c) \in I$. In the second case it follows that $\varphi = (x_r u_1/x_q)e_{atq} - (x_a x_r u_1/(x_q x_c))e_{ctq}$ is a binomial cycle. Then

$$y_1 = \varphi + \partial((u_1/x_q)e_{atqr}) = u_1 e_{\sigma_1} - u_j e_{\sigma_j} + (x_a u_1/x_q)e_{tqr} - (x_a x_r u_1/(x_q x_c))e_{ctq}$$

is a cycle and the cycle

$$y = y_1 - \partial((x_a u_1/(x_c x_q))e_{ctqr} = u_1 e_{\sigma_1} - u_j e_{\sigma_j} - u_k e_{\sigma_k} + (x_a x_t u_1/(x_c x_q))e_{cqr}$$

is of length 4. But as in Example 2.5 we may see that

$$y - \partial((u_1/x_q)e_{atqr}) + \partial((x_a u_1/(x_c x_q))e_{ctqr} = -(x_r x_a u_1/(x_c x_q))e_{ctq} + (x_r u_1/x_q)e_{atq}.$$

Theorem 3.10. *$H_3(x; S/I)$ has a basis of binomial cycles.*

For the proof apply Proposition 3.9.

4. Monomial cycle basis on Koszul homology modules of some principal p-Borel ideals.

The principal p-Borel ideals $I \subset S$ such that S/I is Cohen-Macaulay have the form $I = \Pi_{j=0}^{s}(m^{[p^j]})^{\alpha_j}$, $0 \leq \alpha_j < p$. For these ideals is well known the description of a canonical monomial cycle basis of $H_i(x; S/I)$. Fix $2 \leq i \leq n$. Let $0 \leq t \leq s$ be an integer and for $v \in G(m^{\alpha_t})$ denote $v' = v/x_{m(v)}$. Let $B_{it}(I)$ be the following set of elements from $K_i(x; S/I)$

$$\{wv'^{p^t} x_\sigma^{p^t-1} e_\sigma :\ w \in G(\Pi_{j>t}(m[p^j])^{\alpha_j}), v \in G(m^{\alpha_t}), \sigma \subset \bar{n}, |\sigma| = i, m(\sigma) = m(v)\}$$

and $B_i(I) = \cup_{t=0}^{s} B_{it}(I)$.

Theorem 4.1 (Aramova-Herzog [2]). *The elements of $B_i(I)$ are cycles in $K_i(x; S/I)$ and their homology classes form a basis in $H_i(x; S/I)$ for $i \geq 2$.*

Remark 4.2. This result holds independently of the characteristic of K as we had pointed the definition of p-Borel ideals is pure combinatorial. But note that Theorem 4.1 does not hold if $\alpha_j \geq p$ for some j. Indeed, the ideal $I = (x_1, x_2)^4 \subset S = K[x_1, x_2]$ is strongly stable and a monomial basis of $H_2(x; S/I)$ is given by $T = \{x_1^3 e_1 \wedge$

$e_2, x_1^2x_2e_1\wedge e_2, x_1x_2^2e_1\wedge e_2, x_2^3e_1\wedge e_2\}$ by [1] (see also [9]). Since $I=(x_1,x_2)^2(x_1^2,x_2^2)$ one can compute $B_0(I)=T$ and $B_1(I)=\{x_1x_2e_1\wedge e_2\}$ but $x_1x_2e_1\wedge e_2$ is not cycle in $K_2(x;S/I)$. So the condition $\alpha_j<p$ is necessary and this is an obstruction for an extension of Theorem 4.1.

The question appeared in Remark 4.2 perhaps can be solved extending somehow the Theorem 4.1 for the case when α_j are arbitrary. In some special cases a possible tool could be the following lemma.

Lemma 4.3. *Let $I=\Pi_{j=0}^s(m^{[p^j]})^{\alpha_j}$, where $\alpha_j\geq 0$ are arbitrary integers. If $n=2$ then there exist some integers $0\leq j_0<j_1<\ldots<j_k$ and some positive integers $(\gamma_t)_{0\leq t\leq k}$ such that $\gamma_t<p^{j_{t+1}-j_t}$ for $t<k$ and $I=\Pi_{t=0}^k(m^{[p^{j_t}]})^{\gamma_t}$.*

For the proof apply by recurrence the relation $m^{p^t}m^{[p^t]}=(m^{p^t})^2$.

Set $I=\Pi_{t=0}^k(m^{[p^{j_t}]})^{\gamma_t}$ as above but for any n and let $C_{it}(I)$ be the following set of elements from $K_i(x;S/I)$

$$\{wv'^{p^{j_t}}x_\sigma^{p^{j_t}-1}e_\sigma:\ w\in G(\Pi_{r>t}(m^{[p^{j_r}]})^{\gamma_r}), v\in G(m^{\gamma_t}), \sigma\subset\bar{n}, |\sigma|=i, m(\sigma)=m(v)\}$$

and $C_i(I)=\cup_{t=0}^sC_{it}(I)$. A variant of Theorem 4.1 is the following theorem:

Theorem 4.4. *The elements of $C_i(I)$ are cycles in $K_i(x;S/I)$ and their homology classes form a basis in $H_i(x;S/I)$ for $i\geq 2$.*

Since Lemma 4.3 works only in the case $n=2$ this gives almost nothing more than 4.1. Unfortunately the ideals of type $T=m^{p^j}m^{[p^j]}$ could be bad for example when $p=3$ and $n=3$ then $T=m^6\setminus\{x_1^2x_2^2x_3^2\}$.

Let M be a graded S-module and $\beta_{ij}(M)=\dim_K\operatorname{Tor}_S^i(K,M)_j$ the ij-th graded Betti number of M.

Corollary 4.5. *$\beta_{ij}(S/I)$ does not depend on the characteristic of the field K for all i,j.*

For the proof note that $H_i(x;S/I)\cong\operatorname{Tor}_i^S(K,S/I)$ and so $\beta_{ij}(S/I)$ is the sum of some $|C_{it}(I)|$ which has nothing to do with the characteristic of K.

Remark 4.6. Note that $\beta_{ij}(S/I)$ does not depend on the characteristic of K when I is stable by [9]. In [14] it shows that the extremal graded Betti numbers of S/I (see [3]) do not depend on the characteristic of K when I is a Borel type ideal (see [11]). In particular this happens for p-Borel ideals and so we might ask if all $\beta_{ij}(S/I)$ do not depend on the characteristic of K in the case of p-Borel ideals. The Corollary 4.5 is a small hope.

From now on let I be the p-Borel ideal generated by the monomial $x_{n-1}^\gamma x_n^\alpha$ for some integer $\gamma,\alpha\geq 0$, that is $I=\Pi_{j=0}^s((m_{n-1}^{[p^j]})^{\gamma_j}(m^{[p^j]})^{\alpha_j})$, where $m_{n-1}=(x_1,\ldots,x_{n-1})$, and γ_j,α_j are defined by the p-adic expansion of γ, respectively α. The main result of this section is the following:

Theorem 4.7. *Suppose that $\alpha_j+\gamma_j<p$ for all $0\leq j\leq s$. Then*

(1) *$H_i(x;S/I)$ has a monomial cycle basis for all $i\geq 2$, and*

(2) $\beta_{ij}(S/I)$ *does not depend on the characteristic of* K *for all* i, j.

For the proof we need some preparations. Suppose $\alpha > 0$. Let $r = \max\{j : \alpha_j > 0\}$ and set $J = \Pi_{j=0}^{s}(m_{n-1}^{[p^j]})^{\gamma_j}$ and

$$I' = (\Pi_{j=0}^{r-1}(m^{[p^j]})^{\alpha_j})(m^{[p^r]})^{\alpha_r-1}.$$

Then

Lemma 4.8. $(I : x_n^{p^r}) = JI'$.

Proof. Obviously if L, T are some monomial ideals and v is a monomial then $(LT : v) = (L : v)T + L(T : v)$. Applying this fact we get

$$(I : x_n^{p^r}) = J\Sigma\ \Pi_{j=0}^{r}\Pi_{k=1}^{\alpha_j}(m^{[p^j]} : x_n^{c_{jk}}) = J\Sigma\ \Pi_{j=0}^{r}\Pi_{k=1}^{\alpha_j}(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}}),$$

where the sum is taken over all integers $0 \le c_{jk} \le p^j$ such that $\Sigma_{j=0}^{r}\Sigma_{k=1}^{\alpha_j}c_{jk} = p^r$. For each j let $\Lambda_j \subset \{k : 1 \le k \le \alpha_j, c_{jk} > 0\}$ be any subset. Set $u = \Sigma_{j=0}^{r}\Sigma_{k\in\Lambda_j}(p^j - c_{jk})$. We claim that $x_n^u\Pi_{j=0}^{r}(m_{n-1}^{[p^j]})^{\alpha_j-|\Lambda_j|} \subset I'$. Clearly if our claim holds then $(I : x_n^{p^r}) \subset JI'$, the other inclusion being trivial. Note that the claim holds because $u \ge (\Sigma_{j=0}^{r}|\Lambda_j|p^j) - p^r$.

Let a be an integer such that $0 \le a \le \alpha$ and $a = \Sigma_{j=0}^{r}a_jp^j$, $0 \le a_j < p$ the p-adic expansion of a. Set $\alpha_a = \Sigma_{j,\alpha_j\ge a_j}(\alpha_j - a_j)p^j$ and $\alpha_{aj} = \alpha_j - a_j$ if $\alpha_j \ge a_j$ and 0 otherwise. Set

$$I_a = J(\Pi_{j=0}^{r}(m^{[p^j]})^{\alpha_{aj}}),$$

where J is defined above. Let $\pi : S \to \bar{S} = K[x_1, \ldots, x_{n-1}]$ be the $\bar{S}$-morphism given by $x_n \to 0$.

Lemma 4.9. $\pi(I : x_n)$ *is the p-Borel ideal generated by the monomial* $x_{n-1}^{\gamma}x_{n-1}^{\alpha_a}$, *that is* $\pi(I : x_n^a) = \pi(I_a)$.

Proof. It is enough to show the above equality for the case $\gamma = 0$. As in the proof of Lemma 4.8 we have

$$(I : x_n^a) = \Sigma\ \Pi_{j=0}^{r}\Pi_{k=1}^{\alpha_j}(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}}),$$

where the sum is taken over all integers $0 \le c_{jk} \le p^j$ such that $\Sigma_{j=0}^{r}\Sigma_{k=1}^{\alpha_j}c_{jk} = a$. Note that $\pi(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}})$ is $m_{n-1}^{[p^j]}$ if $c_{jk} < p^j$ and $\bar{S}$ otherwise. It follows that $\pi(m_{n-1}^{[p^j]}, x_n^{p^j-c_{jk}}) \subset \pi(I_a)$, the equality holds only when $\min\{a_j, \alpha_j\} = |\{k : c_{jk} = p^j\}|$ for all j, k. Hence $\pi(I : x_n^a) = \pi(I_a)$. □

Let $T \subset S$ be an arbitrary ideal and $\bar{T} = \pi(T)$.

Lemma 4.10. $H_i(x; \bar{S}/\bar{T}) \cong H_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T}) \oplus H_{i-1}(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T})$ *and in particular* $\beta_{ij}^{S}(\bar{S}/\bar{T}) = \beta_{ij}^{\bar{S}}(\bar{S}/\bar{T}) + \beta_{i-1,j}^{\bar{S}}(\bar{S}/\bar{T})$, *where* $\beta_{ij}^{S}(\bar{S}/\bar{T})$ *is the* i, j*-th graded Betti number of* $\bar{S}/\bar{T}$ *over* S.

Proof. By [6, Proposition 1.6.21] we have

$$H_i(x;\bar{S}/\bar{T}) \cong H_i(x;S/(\bar{T},x_n)) \cong H_i(x_1,\dots,x_{n-1};S/(\bar{T}S)) \otimes_S (\wedge^1 S) \cong$$
$$H_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T}) \otimes_S (\wedge^1 S),$$

the last isomorphism follows because S is flat over $\bar{S}$. This is enough because $\beta_{ij}^{\bar{S}}(\bar{S}/\bar{T}) = \dim_k \operatorname{Tor}_i^{\bar{S}}(K,\bar{S}/\bar{T})_j = \dim_k H_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T})_j$.

Because of the above isomorphism we may write

$$H_i(x;\bar{S}/\bar{T}) = H_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T}) \oplus H_{i-1}(x_1,\dots,x_{n-1};\bar{S}/\bar{T}) \wedge e_n.$$

where by abus of notation we write

$$H_{i-1}(x_1,\dots,x_{n-1};\bar{S}/\bar{T}) \wedge e_n$$

for $\{cls(z \wedge e_n) : z\ cycle\ of\ K_{i-1}(x_1,\dots,x_{n-1};\bar{S}/\bar{T})\}$. We have the following multigraded exact sequence

$$(*) \qquad 0 \to S/(T:x_n)(-1) \to S/T \to \bar{S}/\bar{T} \to 0,$$

where first map is given by multiplication with x_n. Applying Koszul homology long exact sequence to (*) we get the following multigraded exact sequence:

$$(**) H_i(x;S/(T:x_n)(-1)) \to H_i(x;S/T) \to H_i(x;\bar{S}/\bar{T}) \to H_{i-1}(x;S/(T:x_n)(-1)),$$

where we denote by δ_i the last map. Next lemma describes how acts δ_i.

Lemma 4.11. *δ_i maps $H_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T})$ in zero and if z is a cycle of $K_{i-1}(x_1,\dots,x_{n-1};\bar{S}/\bar{T})$ then δ_i maps $cls(z \wedge e_n)$ in*

$$(-1)^{i-1} cls(z) \in H_{i-1}(x_1,\dots,x_{n-1};S/(T:x_n)(-1)).$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K_i(x;S/(T:x_n)(-1)) & \to & K_i(x;S/T) & \to & K_i(x;\bar{S}/\bar{T}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_{i-1}(x;S/(T:x_n)(-1)) & \to & K_{i-1}(x;S/T) & \to & K_{i-1}(x;\bar{S}/\bar{T}) & \to & 0 \end{array}$$

Let w be a cycle of $K_i(x;\bar{S}/\bar{T})$. By construction of δ_i we must lift w to an element $v \in K_i(x;S/T)$. Then $\partial(v) = x_n y$ for a cycle $y \in K_{i-1}(x;S/(T:x_n)(-1))$ and we may write $\delta_i(cls(w)) = cls(y)$. Here we may take $v = w \in K_i(x;S/T)$ which is a cycle. Then we have $y = 0$ and so $\delta_i(w) = 0$. Now we take $w = z \wedge e_n$. As in the first case we may take $v = z \wedge e_n$ but this time this is not cycle in $K_i(x;S/T)$. We have $\partial(z \wedge e_n) = \partial(z) \wedge e_n + (-1)^{i-1} x_n z = (-1)^{i-1} x_n z$ since $\partial(z) = 0$. Then $\delta_i(cls(z \wedge e_n)) = (-1)^{i-1} cls(z)$.

Let f_i be the composite map $K_i(x;S/T) \to K_i(x;\bar{S}/\bar{T}) \to H_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T})$ where the second map q_2 is the second projection of the direct sum given by Lemma 4.10. Then f_i has a canonical section ρ_i^T given by $cls(z) \to cls(z) \in H_i(x;S/T)$, z being a cycle of $K_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T})$. Let $\eta_i^T : H_i(x_1,\dots,x_{n-1};\bar{S}/\bar{T}) \to H_i(x;\bar{S}/\pi(T:x_n)$ be the canonical map associated to the surjection $\bar{S}/\bar{T} \to \bar{S}/\pi(T:x_n)$.

Corollary 4.12. *The following statements hold:*

(1) $\delta_{i+1} = (-1)^i \rho_i^{(T:x_n)} \eta_i^T q_2$,
(2) $\operatorname{Ker} \delta_{i+1} \cong H_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T}) \oplus \operatorname{Ker} \eta_i^T$,
(3) $\operatorname{Im} \delta_{i+1} \cong \operatorname{Im} \eta_i^T$.

Lemma 4.13. *Let ue_σ, $u \in S$ monomial with $m(u) < n$. Suppose that ue_σ is a monomial cycle of $K_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T})$ and induces an element of $\operatorname{Ker} \eta_i^T$, that is $ue_\sigma = \partial(z)$ in $K_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(T : x_n))$ for an element $z \in K_{i+1}(x_1, \ldots, x_{n-1}; \bar{S}/\pi(T : x_n))$. Then $ue_\sigma \wedge e_n + (-1)^i x_n z$ is a cycle in $K_{i+1}(x; S/T)$. If ue_σ is zero in $K_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(T : x_n))$ then $ue_\sigma \wedge e_n$ is a cycle also in $K_{i+1}(x; S/T)$.*

Proof. We have $\partial(ue_\sigma \wedge e_n + (-1)^i x_n z) = \partial(ue_\sigma) \wedge e_n + (-1)^{i-1} x_n z + (-1)^i x_n \partial(z) = \partial(ue_\sigma) \wedge e_n = 0$ because ue_σ is a monomial cycle in $K_i(x_1, \ldots, x_{n-1}; \bar{S}/\bar{T})$. The second statement holds because then $z = 0$. □

Now we may return to give the proof of Theorem 4.7.

Proof. Apply induction on $c = \Sigma_{j=0}^s \alpha_j p^j$. If $c = 0$ then we are in the case of Theorem 4.1 and Corollary 4.5. Suppose $c > 0$ and set $r = \max\{j : \alpha_j \neq 0\}$. By Lemma 4.8 $I' = (I : x_n^{p^r})$ is the p-Borel ideal generated by the monomial $x_{n-1}^\gamma x_n^{\alpha - p^r}$ and from induction hypothesis $H_i(x; S/I')$ has monomial cyclic basis and $\beta_{ij}^S(S/I')$ does not depend on the characteristic of K for all i, j.

Let $0 \leq a \leq p^r$ be an integer. By decreasing induction we show that $H_i(x; S/(I : x_n^a)$ has monomial cycle basis and $\beta_{ij}^S(S/(I : x_n^a))$ does not depend on the characteristic of K for all i, j. Above we saw the case $a = p^r$. Suppose $a < p^r$. The exact multigraded sequence (**) given before Lemma 4.11 with Corollary 4.12 give for $T = (I : x_n^a)$ the following exact multigraded sequence

$$0 \to \operatorname{Im} \delta_{i+1} \cong \operatorname{Im} \eta_i^{(I:x_n^a)} \to H_i(x; S/(I : x_n^{a+1})) \to H_i(x; S/(I : x_n^a)) \to \operatorname{Ker} \delta_i \cong$$

$$H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^a)) \oplus \operatorname{Ker} \eta_{i-1}^{(I:x_n^a)} \to 0,$$

where $\eta_i^{(I:x_n^a)} : H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^a)) \to H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^{a+1}))$ is given in Corollary 4.12. By Lemma 4.9 we see that $\pi(I : x_n^a) = \pi(I_a)$ is the p-Borel ideal generated by a power of x_{n-1} and it is subject to Theorem 4.1 and Corollary 4.5 because $\gamma_j + \alpha_{aj} \leq \gamma_j + \alpha_j < p$ for all j. In particular, $B_i(\pi(I : x_n^a)) = B_i(\pi(I_a))$ is in $H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^a))$ a monomial cycle basis.

Using the induction hypothesis on a we see that $H_i(x; S/(I : x_n^{a+1}))$ has a monomial cyclic basis and $\beta_{ij}^S(S/(I : x_n^{a+1}))$ does not depend on the characteristic of K for all i, j. Then the conclusion follows from the above multigraded exact sequence and Lemma 4.13 if we show the following statements:

(1) A monomial cycle basis of $H_i(x_1, \ldots, x_{n-1}; \bar{S}/\pi(I : x_n^a))$ can be lifted to a monomial cycle subset of $H_i(x; S/(I : x_n^a))$.
(2) $\operatorname{Ker} \eta_i^{(I:x_n^a)}$ has a monomial cycle basis which can be lifted to a monomial cycle subset of $H_i(x; S/(I : x_n^a))$.
(3) $\dim_K \operatorname{Ker} \eta_i^{(I:x_n^a)}$ does not depend on the characteristic of K.

Actually (1) was already seen in the proof of Lemma 4.11. For (2) and (3) we study how act $\eta_i^{\pi(I_a)}$ on $B_i(\pi(I_a))$. We have the following cases:

Case $a_0 < p-1$.

Then $a+1 = (\Sigma_{j>0} a_j p^j) + (a_0+1)$ is the p-adic expansion of $a+1$, that is $(a+1)_j = a_j$ for all $j>0$ and $(a+1)_0 = a_0+1$. We have $\alpha_{a+1,j} = \alpha_{aj}$ for $j>0$. If $\alpha_0 \leq a_0$ then $\alpha_0 = \alpha_{a+1,0} = 0$ and $\eta_i^{\pi(I_a)}$ acts identical because $\pi(I_a) = \pi(I_{a+1})$. Thus $\operatorname{Ker} \eta_i^{\pi(I_a)} = 0$. If $\alpha_0 > a_0$ then $\alpha_{a+1,0} = \alpha_{a,0} - 1$. Thus $\eta_i^{\pi(I_a)}$ acts identical on $\cup_{t\geq 1} B_{it}(\pi(I_a))$ and send $B_{io}(\pi(I_a))$ in zero since if $v \in G(m_{n-1}^{\alpha_{a0}})$ then $v' \in G(m_{n-1}^{\alpha_{a0}-1})$. So the monomial cyclic basis of $\operatorname{Ker} \eta_i^{\pi(I_a)}$ is given by $B_{io}(\pi(I_a))$.

Case $a_j = p-1$ for $0 \leq j < t$, $a_t < p-1$.

Then $a+1 = (a_t+1)p^t + \Sigma_{j>t} a_j p^j$ is the p-adic expansion of $a+1$, that is $(a+1)_j = 0$ for $j<t$, $(a+1)_j = a_t+1$ for $j=t$ and $(a+1)_j = a_j$ for $j>t$. We have $\alpha_{a+1,j} = \alpha_{a,j}$ for $j>t$ and so $\eta_i^{\pi(I_a)}$ acts identical on $\cup_{j>t} B_{ij}(\pi(I_a))$. If $\alpha_t \leq a_t$ then $\alpha_{at} = \alpha_{a+1,t}$ and $\eta_i^{\pi(I_a)}$ acts identical on $B_{it}(\pi(I_a))$. If $\alpha_t > a_t$ then $\alpha_{a+1,t} = \alpha_{at} - 1$ and $\eta_i^{\pi(I_a)}$ send $B_{it}(\pi(I_a))$ to zero. Suppose $j<t$. Then $\alpha_{a,j} = \alpha_{a+1,j}$ and $\eta_i^{\pi(I_a)}$ acts identical on $B_{ij}(\pi(I_a))$. Otherwise we have $\alpha_{a,j} > \alpha_{a+1,j}$ and $\eta_i^{\pi(I_a)}$ send $B_{ij}(\pi(I_a))$ to zero.

Consequently, given $j \geq 0$ in both cases $\eta_i^{\pi(I_a)}$ either acts identical on $B_{ij}(\pi(I_a))$ or send it to zero. It follows that $\operatorname{Ker} \eta_i^{\pi(I_a)}$ has a monomial cyclic basis which can be lifted to $H_i(x; S/(I : x_n^a))$ by Lemma 4.13. It consists of some $B_{ij}(\pi(I_a))$ whose cardinal does not depend on the characteristic of K. This ends our decreasing induction. Thus the ideal $(I : x_n^a)$ satisfy the conditions (1), (2) from the Theorem 4.7 for all $0 \leq a \leq p^r$. In particular this holds for $a=0$.

Remark 4.14. Note that the above proof shows also that some non-principal p-Borel ideals of the form $(I : x_n^a)$ have monomial cyclic bases.

We end this section with an example illustrating the proof of Theorem 4.7.

Example 4.15. Let $n=3$, $p=2$, $S = K[x_1,x_2,x_3]$, $m = (x_1,x_2,x_3)$, $I = m^{[2]}m$. Using Theorem 4.1 a cyclic basis of $H_2(x; S/I)$ is given by $B_{21}(I) = \{x_1x_2e_1 \wedge e_2, x_1x_3e_1 \wedge e_3, x_2x_3e_2 \wedge e_3\}$ and $B_{20}(I) = \{x_i^2e_1 \wedge e_2, x_i^2e_1 \wedge e_3, x_i^2e_2 \wedge e_3 : 1 \leq i \leq 3\}$. We will show this independently using the procedure from the proof of Theorem 4.7. Let $\pi : S \to \bar{S} = K[x_1,x_2]$ be the $\bar{S}$-morphism given by $x_3 \to 0$. Then $\bar{I} = \pi(I) = m_2^{[2]}m_2$, where $m_2 = (x_1,x_2)$ and $\pi(I : x_3) = m_2^{[2]}$, $(I : x_3^2) = m$. Note that monomial cyclic basis of $H_2(x_1,x_2; \bar{S}/m_2^{[2]}m_2)$, $H_2(x_1,x_2; \bar{S}/m_2^{[2]})$ are given by $B_{21}(\bar{I}) = \{x_1x_2e_1 \wedge e_2\}$ and $B_{20}(\bar{I}) = \{x_1^2e_1 \wedge e_2, x_2^2e_1 \wedge e_2\}$ respectively $\{x_1x_2e_1 \wedge e_2\}$. The map η_2^I maps $B_{20}(\bar{I})$ in zero and it is identity on $B_{21}(\bar{I})$. The maps η_1^I, $\eta_i^{(I:x_3)}$, $i=1,2$ are zero maps.

We have the following multigraded exact sequence

$$\operatorname{Im} \eta_2^{(I:x_3)} = 0 \to H_2(x; S/m) \to H_2(x; S/(I : x_3)) \to$$

$$H_2(x_1,x_2; \bar{S}/m_2^{[2]}) \oplus H_1(x_1,x_2; \bar{S}/m_2^{[2]}) \wedge e_3 \to 0.$$

As the monomial cyclic basis of $H_2(x;S/m)$, $H_2(x_1,x_2;\bar{S}/m_2^{[2]})$, $H_1(x_1,x_2;\bar{S}/m_2^{[2]})$ are $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$, respectively $\{x_1x_2e_1\wedge e_2\}$, respectively $\{x_1e_1, x_2e_2\}$ we see that a monomial cycle basis in $H_2(x;S/(I:x_3))$ is given by

$$T_2(I:x_3)=\{x_3e_1\wedge e_2, x_3e_1\wedge e_3, x_3e_2\wedge e_3, x_1x_2e_1\wedge e_2, x_1e_1\wedge e_3, x_2e_2\wedge e_3\}.$$

Now consider the multigraded exact sequence

$$0\to \operatorname{Im}\eta_2^I\to H_2(x;S/(I:x_3))\to H_2(x;S/I)\to$$
$$H_2(x_1,x_2;\bar{S}/m_2^{[2]}m_2)\oplus H_1(x_1,x_2;\bar{S}/m_2^{[2]}m_2)\wedge e_3\to 0.$$

As the monomial cyclic basis of $\operatorname{Im}\eta_2^I$, $H_2(x_1,x_2;\bar{S}/m_2^{[2]}m_2)$, $H_1(x_1,x_2;\bar{S}/m_2^{[2]}m_2)$ are $\{x_1x_2e_1\wedge e_2\}$, respectively $T_2(m_2^{[2]})=\{x_1x_2e_1\wedge e_2, x_1^2e_1\wedge e_2, x_2^2e_1\wedge e_2\}$, respectively $T_1(m_2^{[2]})=\{x_1^2e_1\wedge e_3, x_2^2e_1\wedge e_3, x_1^2e_2\wedge e_3, x_2^2e_2\wedge e_3\}$ we see that a monomial cycle basis in $H_2(x;S/I)$ is given by

$$x_3[T_2(I:x_3)\setminus\{x_1x_2e_1\wedge e_2\}]\cup T_2(m_2^{[2]})\cup T_1(m_2^{[2]})=B_{20}(I)\cup B_{21}(I)=B_2(I).$$

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