# THE STRONG LEFSCHETZ PROPERTY AND CERTAIN COMPLETE INTERSECTION EXTENSIONS

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Abstract. Harima and Watanabe [3] showed that simple extensions of a standard graded Gorenstein Artinian $K$ algebras with the strong Lefschetz property have again strong Lefschetz property. We study the strong Lefschetz property for certain complete intersection extensions and give another proof in the simple extension case.

## Introduction

Let $K$ be a field, $A$ be a standard graded Artinian $K$-algebra and $a \in A$ a homogeneous form of degree $k$. The element $a$ is called a *Lefschetz element* if for all integers $i$ the $K$-linear map $a\colon A_i \to A_{i+k}$ (induced by multiplication with $a$) has maximal rank. One says that $A$ has the *weak Lefschetz property* if there exists a Lefschetz element $a \in A$ of degree 1. An element $a \in A_1$ for which all powers $a^r$ are Lefschetz is called a *strong Lefschetz element*, and $A$ is said to have the *strong Lefschetz property* if $A$ admits a strong Lefschetz element. Note that the set of Lefschetz elements $a \in A_1$ form a Zariski open subset of $A_1$. The same holds true for the set of strong Lefschetz elements.

Assuming that the characteristic of $K$ is zero and the defining ideal of $A$ is generated by generic forms, it is conjectured that $A$ has the strong Lefschetz property. Thus in particular, $A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ should have the strong Lefschetz property for generic forms $f_1, \ldots, f_n$. Note that such an algebra is an Artinian complete intersection. It is expected that *any* standard graded Artinian complete intersection over a base field of characteristic 0 has the strong Lefschetz property. Stanley [6] and later J. Watanabe [7] proved this in case $A$ is a monomial complete intersection. Stanley used the Hard Lefschetz Theorem to prove this result, while Watanabe used the representation theory of the Lie algebra $sl(2)$.

Our paper studies how preserve the strong Lefschetz property on certain complete intersection extensions using the following:

**Theorem 0.1** (Harima-Watanabe [3]). *Let $K$ be a field of characteristic $0$, $(A, m)$ be a standard graded Artinian Gorenstein $K$-algebra having the strong Lefschetz property, and let $B$ be a finite free graded $K$-algebra such that the algebra map*

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*$A \to B$ preserve the grading. Suppose that $B/mB$ is Gorenstein and has the strong Lefschetz property. Then $B$ has the strong Lefschetz property too.*

**Corollary 0.2** (Harima-Watanabe [3])**.** *Let $K$ be a field of characteristic $0$, $A$ be a standard graded Artinian Gorenstein $K$-algebra having the strong Lefschetz property, and let $f \in A[x]$ be a monic homogeneous polynomial. Then the algebra $B = A[x]/(f)$ has the strong Lefschetz property.*

The proof only uses techniques from linear algebra. The result implies in particular Stanley's theorem. Together with J. Herzog we gave in [4] another proof of the above corollary using linear algebra. Later we noticed that our main results are contained in [3]. Our direct proof deserves to be published since from our approach we also get some new results (see the Propositions 2.3, 2.4). In Section 1 I also include some consequences of Harima-Watanabe Theorem 0.1 concerning certain complete intersection extensions of standard graded Artinian Gorenstein $K$-algebras having the strong Lefschetz property.

## 1. Complete intersection extensions

Let $K$ be a field of characteristic zero and $A$ a standard graded Artinian Gorenstein $K$-algebra having the strong Lefschetz property. Let $f = (f_1, \ldots, f_r)$ be a regular sequence of $A[y]$, $y = (y_1, \ldots, y_r)$.

**Theorem 1.1.** *If $K \otimes_A (A[y]/(f))$ has strong Lefschetz property then $A[y]/(f)$ has strong Lefschetz property too.*

*Proof.* It is enough to show that $B = A[y]/(f)$ is a finite free $A$-agebra because then we may apply Harima-Watanabe Theorem 0.1. Let $S = K[x_1, \ldots, x_n]$, $A = S/a$ for a graded ideal $a \subset S$ and $g = (g_1, \ldots, g_r) \in S[y]^r$ a lifting of $f$ to $S[y]$. We claim that $g$ is also a regular sequence in $S[y]$. Indeed, let $d$ be a positive integer such that $(x_1^d, \ldots, x_r^d) \subset a$. Then the kernel of the canonical surjective map $C = S[y]/(x_1^d, \ldots, x_r^d, g) \to B$ is nilpotent and so $C$ is Artinian like $B$. Thus $\{x_1^d, \ldots, x_r^d, g_1, \ldots, g_r\}$ is a regular sequence in $S[y]$ because $S[y]$ is Cohen-Macaulay. In particular $g$ is a regular sequence in $S[y]$.

Now note that $D = S[y]/(g)$ is finite free over $S$ because $S$ is regular and $D$ is a maximal Cohen-Macaulay $S$-module. By base change $A \otimes_S D \cong B$ is finite free over $A$, which is enough.

As an immediate consequence of the above theorem we obtain

**Corollary 1.2.** *Let $\{f_1, f_2\}$ be a regular sequence of $A[y_1, y_2]$. Then $A[y_1, y_2]/(f_1, f_2)$ has strong Lefschetz property.*

*Proof.* It is enough to apply Theorem 1.1 since $K \otimes_A (A[y_1, y_2]/(f_1, f_2))$ has strong Lefschetz property by [2]. □

Using the above corollary by induction it follows:

**Corollary 1.3.** *Let $t$ be a positive integer , and $(f_1, \dots, f_{2t})$ a system of polynomials from $A[y_1, \dots, y_{2t}]$ such that for $i = 1, \dots, t$ the polynomials $\{f_{2i-1}, f_{2i}\}$ induce a regular sequence in $K[y_{2i-1}, y_{2i}]$. Then the $K$-algebra*

$$A[y_1, \dots, y_{2t}]/(f_1, \dots, f_{2t}]$$

*has the strong Lefschetz property.*

Another consequence of Corollary 1.2 as well of Corollary 0.2 is the following:

**Corollary 1.4.** *For $i = 1, \dots, n$ let $f_i \in A[x_1, \dots, x_i]$ be a homogeneous polynomial which is monic in $x_i$. Then the $K$-algebra*

$$A[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

*has the strong Lefschetz property.*

The result implies in particular that $K[x_1, \dots, x_n]/(f_1, \dots, f_n)$ has the strong Lefschetz property, if for $i = 1, \dots, n$, $f_i \in K[x_1, \dots, x_i]$ is a homogeneous and monic polynomial in $x_i$. In the special case that $f_i = x_i^{a_i}$ for $i = 1, \dots, n$, we obtain the theorem of Stanley [6]. The slightly more general result with the $f_i$ as described before, can also be deduced directly from Stanley's theorem using the following result of Wiebe [8, Proposition 2.9]: let $I \subset K[x_1, \dots, x_n]$ be a graded ideal, and assume that $K[x_1, \dots, x_n]/\operatorname{in}(I)$ has the strong Lefschetz property, where $\operatorname{in}(I)$ is the initial ideal with respect to some term order. Then $K[x_1, \dots, x_n]/I$ has the strong Lefschetz property.

In the above situation we have $\operatorname{in}(f_i) = x_i^{\deg f_i}$ for $i = 1, \dots, n$, if we choose the lexicographical order induced by $x_n > x_{n-1} > \dots > x_1$. Since the initial terms of the generators form a regular sequence it follows that

$$\operatorname{in}(I) = (\operatorname{in}(f_1), \dots, \operatorname{in}(f_n)) = (x_1^{a_1}, \dots, x_n^{a_n}).$$

One cannot expect to find a proof of Corollary 0.2 as well as of the results above when $A$ is not Gorenstein (see example bellow). For weak Lefschetz property something can be done even when $A$ is not Gorenstein as we will see in 2.3.

**Example 1.5.** Let $S = K[x_1, ..., x_3]$, $I \subset S$ be the monomial ideal generated by $\{x_1, .., x_3\}^2 \setminus \{x_3^2\}$, and $A/(I, x_3^5)$. The Hilbert series of $A$ is

$$\operatorname{Hilb}_A(t) = 1 + 3t + t^2 + t^3 + t^4$$

and by [3, Example 5] $A$ has strong Lefschetz property but $A[z]/(z^2)$ has not.

We conclude this section with the following

**Example 1.6.** Let $A = K[x_1, ..., x_5]/(x_1^4, ..., x_4^4, x_5^2)$, $f = x_1 + x_2 + .... + x_5$ and $B = A/(f^8)$. B has the interesting property that it is not strong Lefschetz but nevertheless has the maximal rank property. The Hilbert series of $B$ is given by

$$\operatorname{Hilb}_B(t) = 1 + 5t + 14t^2 + 30t^3 + 51t^4 + 71t^5 + 84t^6 + 84t^7 + 70t^8 + 46t^9 + 16t^{10}.$$

Let $b \in B$ be a generic linear form, and set $C = B/(b^9)$. We use the "Randomized" command of CoCoA to produce generic forms. Then

$$\operatorname{Hilb}_C(t) = 1 + 5t + 14t^2 + 30t^3 + 51t^4 + 71t^5 + 84t^6 + 84t^7 + 70t^8 + 45t^9 + 12t^{10}.$$

It follows that the map $B_1 \xrightarrow{b^9} B_{10}$ is not surjective but also not injective because $\dim_K B_1 + \dim_K C_{10} = 5 + 12 > 16 = \dim_K B_{10}$. Thus $B$ does not have the strong Lefschetz property.

On the other hand it can be checked that $B$ has the maximal rank property, that is, any generic form in $B$ has maximal rank. Such an example seems to be new, see [5].

## 2. Simple extensions

Let $A$ be a standard graded $K$-algebra and $I \subset A$ a graded ideal. For convenience we will say that $a \in A$ is Lefschetz for $A/I$ if the residue class $a + I$ is a Lefschetz element of $A/I$. In this section we give an independent proof to the Corollary 0.2.

In the proof of Corollary 0.2 we shall use the following two lemmata.

**Lemma 2.1.** *Let $A$ be a standard graded $K$-algebra, $f, g \in A$ homogeneous elements which are nonzero divisors on $A$. Then $f$ is Lefschetz for $A/(g)$ if and only if $g$ is Lefschetz for $A/(f)$.*

*Proof.* Consider the long exact sequence for Koszul homology (see [1, Corollary 1.6.13])

$$\cdots \to H_1(g; A) \to H_1(f, g; A) \to H_0(g; A) \xrightarrow{\quad f \quad} H_0(g; A) \to H_0(f, g; A) \to 0.$$

Since $g$ is a non-zerodivisor on $A$ this yields the exact sequence

$$0 \longrightarrow H_1(f, g; A) \longrightarrow A/(g) \xrightarrow{\quad f \quad} A/(g) \to H_0(f, g; A) \to 0.$$

Similarly we obtain an exact sequence

$$0 \longrightarrow H_1(f, g; A) \longrightarrow A/(f) \xrightarrow{\quad g \quad} A/(f) \to H_0(f, g; A) \to 0.$$

Comparing this two exact sequences, the assertion follows. □

**Lemma 2.2.** *Let $K$ be field of characteristic 0, $A$ a standard graded Artinian $K$-algebra with strong Lefschetz property and $f \in A[y]$ a monic homogeneous polynomial. Then for any strong Lefschetz element $a \in A_1$ there exists a non-zero element $c \in K$ such that $f(a/c)$ is a Lefschetz element of $A$.*

*Proof.* Let $f = y^d + a_1 y^{d-1} + \cdots + a_d$, and $s = \max\{i\colon A_i \neq 0\}$. We may assume that $d \leq s$ because otherwise the statement is trivial. For $c \in K$ we set $f_c = y^d + \sum_{i=1}^{d} c^i a_i y^{d-i}$. Let $a \in A_1$ be a strong Lefschetz element. Then $a^d$ is a Lefschetz element, that is, for all $i$ the multiplication map $f_0(a)\colon A_i \to A_{i+d}$ has maximal rank.

Fix $i \leq s - d$ and $K$-bases of the nonzero $K$-vector spaces $A_i$ and $A_{i+d}$, and let $D_c$ be the matrix describing the $K$-linear map $f_c(a)\colon A_i \to A_{i+d}$. Note that the entries of $D_c$ are polynomial expressions in $c$ with coefficients in $K$. Now $P_c(a)$ has maximal rank if and only if one maximal minor $M_j(D_c)$ of $D_c$ does not vanish. In particular, $M_{j_0}(D_0) \neq 0$ for some $j_0$. Since $M_{j_0}(D_c)$ is a (non-zero) polynomial expression in $c$ with coefficients in $K$, there exist only finitely many $c \in K$ such that $M_{j_0}(D_c) = 0$. Thus, since $K$ is infinite, we have $M_{j_0}(D_c) \neq 0$ for infinitely many $c \in K$, and so $f_c(a)\colon A_i \to A_{i+d}$ has maximal rank for infinitely many $c \in K$. Since $A$ has only finitely many non-zero components, we can therefore find $c \in K$, $c \neq 0$ such that $f_c(a)$ has maximal rank for all $i$. Then $a/c \in A_1$ has the desired property, since $f(a/c) = f_c(a)/c^d$. $\square$

Now we are ready to begin with the new proof of Corollary 0.2. Let $A$ be a standard graded Artinian Gorenstein $K$-algebra having the strong Lefschetz property.

In a first step we will prove: suppose $C = A[x]/(x^r)$ has the strong Lefschetz property for all $r > 1$, then $B = A[x]/(f)$ has the strong Lefschetz property for any monic homogeneous polynomial $f \in A[x]$.

Let $U_r \subset B_1$ be the Zariski open set of elements $b \in B_1$ for which $b^r$ is a Lefschetz element. If $U_r \neq \emptyset$ for all $r \geq 1$, then the finite intersection $U = \bigcap_r U_r$ is non-empty, as well, and any $b \in U$ is then a strong Lefschetz element. Thus it suffices to show that for each $r \geq 1$ there exists an element $b_r \in B_1$ such that $b_r^r$ is a Lefschetz element.

By Lemma 2.2 we may choose an element $a \in A_1$ such that $f(a)$ is a Lefschetz element of $A$. It follows that $f(x)$ is Lefschetz for $A[x]/(a - x)$. Thus by Lemma 2.1, the element $b_1 = a - x$ is Lefschetz for $B$.

In case $r > 1$, we may view $f(y)$ as a polynomial in $C[y]$ where $C = A[x]/(x^r)$. By our assumption $C$ has a strong Lefschetz element. Now Lemma 2.2 implies that we can find a strong Lefschetz element $c \in C_1$ such that $f(c)$ is a Lefschetz element of $C$. Since the strong Lefschetz elements form a nonempty Zariski open set in $C_1$, we may assume that $c = a + \lambda x$ with $a \in A_1$ and $\lambda \in K$, $\lambda \neq 0$. Applying the substitution $x \mapsto b_r = \lambda^{-1}(x - a)$ it follows that $f(x)$ is Lefschetz for $A[x]/b_r^r$. Thus by Lemma 2.2, the element $b_r^r$ is Lefschetz for $A[x]/(f)$.

In order to complete the proof of the Corollary 0.2 it remains to be shown that if $A$ is a standard graded Artinian Gorenstein $K$-algebra having the strong Lefschetz property, then $A[x]/(x^q)$ has the strong Lefschetz property. We use Lemma 2.1 and show instead that if $a \in A_1$ is a strong Lefschetz element, then for all $k$ the element $x^q$ is Lefschetz for $B = A[x]/(a + x)^k$.

In $B$ we have

$$x^k = -\sum_{j=0}^{k-1} \binom{k}{j} a^{k-j} x^j.$$

By induction on $r$ it follows that

$$x^r = (-1)^{r-k-1} \sum_{j=0}^{k-1} \binom{r-j-1}{r-k} \binom{r}{j} a^{r-j} x^j \quad \text{for} \quad r \geq k.$$

Note that

$$\binom{r-j-1}{r-k} \binom{r}{j} = \frac{k}{r-j} \binom{r}{k} \binom{k-1}{j},$$

so that

$$x^r = (-1)^{r-k-1} \sum_{j=0}^{k-1} \binom{r}{k} \binom{k-1}{j} \frac{k}{r-j} a^{r-j} x^j$$

for all $r \geq k$. Thus for all $r \geq 0$ we have

$$x^r = \sum_{j=0}^{k-1} c_{rj} a^{r-j} x^j$$

with

$$c_{rj} = \begin{cases} \delta_{rj}, & \text{if} \quad r \leq k-1 \\ (-1)^{r-k-1}\binom{r}{k}\binom{k-1}{j}\frac{k}{r-j}, & \text{if} \quad r \geq k, \end{cases}$$

where $\delta_{rj}$ denotes the Kronecker symbol.

Now we show that the map $\beta_t^q : B_t \to B_{t+q}$ given by multiplication with $x^q$ has maximal rank.

We denote by $\alpha_i^j \colon A_i \to A_{i+j}$ the $K$-linear map given by multiplication with $a^j$. For each element $ux^i \in A_{t-i}x^i$ we have

$$x^q(ux^i) = \sum_{j=0}^{k-1} c_{q+i,j} a^{q+i-j} ux^j = \sum_{j=0}^{k-1} c_{q+i,j} \alpha_{t-i}^{q+i-j}(u) x^j.$$

Since for each $j$ the $K$-vectorspace $B_j$ has the direct sum decomposition

$$B_j = \bigoplus_{i=0}^{k-1} A_{t-i}x^i,$$

the linear map $\beta_t^q$ can be described by the following block matrix

$$M = \begin{pmatrix} c_{q,0}\alpha_t^q & c_{q+1,0}\alpha_{t-1}^{q+1} & \cdots & c_{q+k-1,0}\alpha_{t-k+1}^{q+k-1} \\ c_{q,1}\alpha_t^{q-1} & c_{q+1,1}\alpha_{t-1}^{q} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{q,k-1}\alpha_t^{q-k+1} & c_{q+1,k-1}\alpha_{t-1}^{q-k+2} & \cdots & c_{q+k-1,k-1}\alpha_{t-k+1}^{q} \end{pmatrix}.$$

Our aim is to show that $M$ has maximal rank. Assume first that $q < k$, then

$$M = \begin{pmatrix} 0 & N \\ \text{id} & * \end{pmatrix},$$

where $N = (c_{q+j,i}\alpha_{t-j}^{q+j-i})_{\substack{i=0,\dots,q-1 \\ j=k-q,\dots,k-1}}$. It follows that $M$ has maximal rank if and only if $N$ has maximal rank. Thus the general case is treated if we can prove that for all $q$ the matrix

$$N = (c_{q+j,i}\alpha_{t-j}^{q+j-i})_{\substack{i=0,\dots,s-1 \\ j=r-q,\dots,k-1}} \quad \text{with} \quad s = \min\{q,k\} \quad \text{and} \quad r = \max\{q,k\}$$

has maximal rank.

We show this by applying certain block row and block column operations in order to simplify the matrix without changing its rank. The kind of operations we will apply are the following:

(i) multiplication of a block row or a block column of $N$ with a non-zero rational number;

(ii) for $d \in \mathbb{Q}$ , $d \neq 0$ and $j < i$ compose $d\alpha_{t-i}^{i-j}$ with each block $c_{jl}\alpha_{t-j}^{j-l}$ of the $j$th block column of $N$ to obtain a $j$th block column whose blocks are $dc_{jl}\alpha_{t-j}^{j-l} \circ \alpha_{t-i}^{i-j} = dc_{jl}\alpha_{t-i}^{i-l}$, and subtract this new block column from the $i$th block column to obtain the new $i$th block column whose blocks are

$$(c_{il} - dc_{jl})\alpha_{t-i}^{i-l}, \quad l = 0, \cdots, s-1.$$

These operations only change the coefficients $c_{ji}$ of the block entries, that is, the matrix $N = (c_{q+j,i}\alpha_{t-j}^{q+j-i})_{\substack{i=0,\ldots,s-1\\ j=r-q,\ldots,k-1}}$ will be transformed into a matrix of the form $N' = (c'_{q+j,i}\alpha_{t-j}^{q+j-i})_{\substack{i=0,\ldots,s-1\\ j=r-q,\ldots,k-1}}$ with certain new coefficients $c'_{q+j,i} \in \mathbb{Q}$.

Consider the "coefficient matrix" $L = (c_{q+j,i})_{\substack{i=0,\ldots,s-1\\ j=r-q,\ldots,k-1}}$ of $N$. Then the coefficient matrix $L'$ of $N'$ is obtained from $L$ by the following row and column operations:

(i) multiplication or division of a row or a column with a non-zero rational number;

(ii) subtraction of a multiple of the $j$th column from the $i$th column where $j < i$.

Next we intend to show that by these operations $L$ can be transformed into a matrix $L'$ such that all entries of $L'$ on the anti-diagonal are non-zero, while the entries below the anti-diagonal are all zero.

We first simplify $L$ by dividing each $j$th column by $(-1)^{j-k-1}\binom{j}{k}k$ and each $i$th row by $\binom{k-1}{i}$. The result of these operations is the matrix

$$\begin{pmatrix} 1/r & 1/(r+1) & \cdots & 1/(r+s-1) \\ 1/(r-1) & 1/r & \cdots & 1/(r+s-2) \\ \vdots & \vdots & \ddots & \vdots \\ 1/(r-s+1) & 1/(r-s+2) & \cdots & 1/r \end{pmatrix},$$

which we again denote by $L$.

We will use the following simple fact from linear algebra: suppose $F = (f_{ij})_{i,j=1,\ldots,n}$ is an $n \times n$-matrix with coefficients in a field $K$. Then the following conditions are equivalent:

(a) the matrix $F$ can be transformed by operations of type (ii) into a matrix $F'$ with $f'_{ij} \neq 0$ for $i + j = n + 1$, and $f'_{ij} = 0$ for $i + j > n + 1$;

(b) $\det(F_i) \neq 0$ for $i = 1, \ldots, n$ where $F_i = (f_{kl})_{\substack{k=i,\ldots,n\\ l=1,\ldots i}}$.

Indeed, it is clear that (a) $\Rightarrow$ (b). Conversely, assuming (b) we have $\det(F_n) = f_{n1} \neq 0$. Thus by subtracting suitable multiples of the first column from the other columns we obtain a matrix $G = (g_{ij})$ with $g_{ni} = 0$ for $i = 2, \ldots, n$, and such that $\det(F_i) = \det(G_i)$ for $i = 1, \ldots, n$, where $G_i = (g_{kl})_{\substack{k=i,\ldots,n\\ l=1,\ldots i}}$. Applying the induction hypothesis to the matrix $G' = (g_{kl})_{\substack{k=1,\ldots,n-1\\ l=2,\ldots,n}}$, the assertion follows.

Applying this result from linear algebra, we see that $L$ can be transformed by operations of type (ii) into the matrix $L'$ of the desired form if for all integers $0 \leq t < s$ the matrices of the shape

$$S = (1/(r - i + j))_{i,j=0,\ldots,t}$$

are non-singular. It is an easy exercise in linear algebra to show that this is indeed the case.

After all these operations our matrix $N$ is transformed into the matrix $N'$ whose anti-diagonal has the block entries

$$c'_1\alpha^{r-s+1}_{t+q-r}, c'_2\alpha^{r-s+3}_{t+q-r-1}, \ldots, c'_{s-1}\alpha^{q+k-1}_{t-k+1}$$

with non-zero rational coefficients $c'_i$, and whose block entries below the anti-diagonal are all zero.

We will show that for $i = 0, \ldots, s-1$ either all $\alpha^{r-s+2i+1}_{t+q-r-i}$ are injective maps, or else all $\alpha^{r-s+2i+1}_{t+q-r-i}$ are surjective maps. Then clearly $N'$ has maximal rank, and consequently $N$ has maximal rank.

For all integers $i$ and $j$ with $0 \leq i < j$ the maps

$$\alpha^{j-i}_i \colon A_i \longrightarrow A_j$$

have maximal rank, by assumption. In particular, $\alpha^{j-i}_i$ is injective if $\dim A_i \leq \dim A_j$ and surjective if $\dim A_i \geq \dim A_j$.

Let $\sigma = \max\{i \colon A_i \neq 0\}$. Then, since $A$ is Gorenstein, the Hilbert function of $A$ is symmetric (see e.g. [1, Corollary 4.4.6, Remark 4.4.7]), that is,

$$\dim A_i = \dim A_{\sigma-i} \quad \text{for all} \quad i,$$

and since $A$ has the weak Lefschetz property ($A$ even has the strong Lefschetz property), the Hilbert function of $A$ is unimodal (see e.g. [2, Remark 3.3]). It then follows that

$$\dim A_i \leq \dim A_j \quad \text{if and only if} \quad i \leq \sigma - j.$$

Thus we conclude that

$$\alpha^{j-i}_i \quad \text{is} \quad \begin{cases} \text{injective} & \text{if} \quad i \leq \sigma - j, \\ \text{surjective} & \text{if} \quad i \geq \sigma - j. \end{cases}$$

Thus in case of the maps $\alpha^{r-s+2i+1}_{t+q-r-i}$, we have to compare the size of the numbers $t+q-r-i$ and $\sigma - [(r-s+2i+1)+(t+q-r-i)] = \sigma - t - q + s - i - 1$. Since it does not depend on $i$ which of the two numbers is less than or equal to other, it follows $\alpha^{r-s+2i+1}_{t+q-r-i}$ is injective for all $i$, or $\alpha^{r-s+2i+1}_{t+q-r-i}$ is surjective for all $i$, as desired.

In case the $K$-algebra $A$ is not Gorenstein, our new proof of Corollary 0.2 yields the following weaker result.

**Proposition 2.3.** *Let $K$ be a field, $A$ a standard graded Artinian $K$-algebra having the strong Lefschetz property, and let $f \in A[x]$ be a monic homogeneous polynomial. Then the algebra $B = A[x]/(f)$ has the weak Lefschetz property.*

*Proof.* Recall the following step in the above proof: by Lemma 2.2 we may choose an element $a \in A_1$ such that $f(a)$ is a Lefschetz element of $A$. It follows that $f(x)$ is Lefschetz for $A[x]/(a-x)$. Thus by Lemma 2.1, $b = a - x$ is Lefschetz for $B$. $\square$

Analyzing the arguments in the above proof of Corollary 0.2 we see that all results remain valid if the characteristic of the base field is large enough. More precisely we have

**Proposition 2.4.** *Let $K$ be a field and $A$ an Artinian Gorenstein $K$-algebra having the strong Lefschetz property with socle degree $\sigma = \max\{t\colon A_t \neq 0\}$ and multiplicity $e(A) = \sum_{t=0}^{\sigma} \dim_K A_t$. Let $f \in A[x]$ be a homogeneous monic polynomial of degree $q$. Then $B = A[x]/(f)$ has the strong Lefschetz property if*

$$\operatorname{char} K \geq \begin{cases} 2q+\sigma-1, & \textit{and } f = x^q, \\ \max\{e(A), 2q+\sigma-1\}, & \textit{otherwise.} \end{cases}$$

*Proof.* In case $f = x^q$ we must make sure that all the binomials in the expression $x^r = (-1)^{r-k-1} \sum_{j=0}^{k-1} \binom{r-j-1}{r-k} \binom{r}{j} a^{r-j} x^j$ are units in the field $K$, and this must be satisfied for all $r = q + k$ where are less than or equal the socle degree of $A[x]/(x^q)$. Since the socle degree of $A[x]/(x^q))$ is equal to $q + \sigma - 1$, we therefore need that $\operatorname{char} K$ does not divide any prime number $\leq 2q + \sigma - 1$.

In the general case we had to apply Lemma 2.2. For the proof of this lemma it was necessary that the field $K$ has enough elements, so that for all the polynomials in $c$ defined by the maximal minors considered in the proof we find a common element $c \in K$ for which these polynomials do not vanish. This is possible if $\operatorname{char} K > e(A)$. □

## References


[1] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised Edition, Cambridge, 1996.

[2] T. Harima, J. Migliore, U. Nagel, J. Watanabe, The weak and strong Lefschetz properties for artinian $K$-algebras, arXiv:math.AC/0208201.

[3] T. Harima, J. Watanabe, The finite free extension of $K$-algebras with the strong Lefschetz property, Rend. Sem. Mat. Univ. Padova, **110** (2003), 119-146.

[4] J. Herzog, D. Popescu, The strong Lefschetz property and simple extensions, arXive:math.AC/0506537.

[5] J. Migliore, R.M. Miro-Roig, Ideals of generic forms and the ubiquity of the weak Lefschetz property, arXiv:math.AC/0205133.

[6] R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods 1 (1980), 168-184.

[7] J. Watanabe, The Dilworth number of Artinian rings and finite posets with rank function, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math.,Vol **11**, Kinokuniya Co. North Holland, Amsterdam, (1987), 303-312.

[8] A. Wiebe, The Lefschetz property for componentwise linear ideals and Gotzmann ideals, arXiv:math.AC/0307223.



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