

# ON THE RADICAL OF A MONOMIAL IDEAL

JÜRGEN HERZOG, YUKIHIKE TAKAYAMA AND NAOKI TERAJ

ABSTRACT. Algebraic and combinatorial properties of a monomial ideal and its radical are compared.

## 1. INTRODUCTION

There are simple examples of Cohen-Macaulay ideals whose radical is not Cohen-Macaulay. The first such example is probably due to Hartshorne [5], who proved that in positive characteristic the toric ring  $K[s^4, s^3t, st^3, t^4]$  is a set theoretic complete intersection. With CoCoA or other computer algebra systems many other examples, also in characteristic zero, can be constructed. The following example due Conca was computed with CoCoA: let  $S = K[x_1, x_2, x_3, x_4, x_5]$  and  $J = (x_2^2 - x_4x_5, x_1x_3 - x_3x_4, x_3x_4 - x_1x_5) \subset S$ . Then  $S/J$  is a 2-dimensional Cohen-Macaulay ring,  $\sqrt{J} = (x_1x_3 - x_1x_5, x_3x_4 - x_1x_5, x_2^2 - x_4x_5, x_1^2x_2 - x_1x_2x_4, x_2x_3^2 - x_2x_3x_5)$  and  $S/\sqrt{J}$  is *not* Cohen-Macaulay. Indeed, the depth of  $S/\sqrt{J}$  equals 1. On the other hand it is well-known that the Cohen-Macaulay property of a monomial ideal is inherited by its radical. The reason is that the radical of a monomial ideal is essentially obtained by polarization and localization. This observation, was communicated to the third author by David Eisenbud. Both operations, polarization and localization, preserve the Cohen-Macaulay property. An explicit proof of this fact can be found in [11]. The purpose of this paper is to exploit this idea and to show that many other nice properties are inherited by the radical of a monomial ideal.

## 2. THE COMPARISON

For the proof of the main result of this paper we need some preparation. We begin with the following extension [10, Theorem 1.1] of Hochster's formula [1, Theorem 5.3.8] describing the local cohomology of a monomial ideal.

Let  $K$  be a field,  $S = K[x_1, \dots, x_n]$  the polynomial ring and  $I \subset S$  a monomial ideal. The unique minimal monomial system of generators of  $I$  is denoted by  $G(I)$ . For  $i = 1, \dots, n$  we set

$$t_i = \max\{\nu_i(u) : u \in G(I)\},$$

where for a monomial  $u \in S$ ,  $u = x_1^{a_1} \cdots x_n^{a_n}$  we set  $\nu_i(u) = a_i$  for  $i = 1, \dots, n$ .

For  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ , we set

$$G_a = \{i : 1 \leq i \leq n, a_i < 0\},$$

---

1991 *Mathematics Subject Classification.* 13D02, 13P10, 13D40, 13A02.

and define the simplicial complex  $\Delta_a(I)$  whose faces are the sets  $L \setminus G_a$  with  $G_a \subset L$ , and such that  $L$  satisfies the following condition: for all  $u \in G(I)$  there exists  $i \notin L$  such that  $\nu_i(u) > a_i \geq 0$ .

Notice that the inequality  $a_i \geq 0$  in the definition of  $\Delta_a(I)$  follows from the condition  $i \notin L \supset G_a$ . It is included only for the reader's convenience.

With the notation introduced one has

**Theorem 2.1** (Takayama [10]). *Let  $I \subset S$  be a monomial ideal. Then the Hilbert series of the local cohomology modules of  $S/I$  with respect to the  $\mathbb{Z}^n$ -grading is given by*

$$\text{Hilb}(H_{\mathfrak{m}}^i(S/I), \mathbf{t}) = \sum_{F \in \Delta} \sum_a \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(I); K) \mathbf{t}^a$$

where  $\Delta$  is the simplicial complex corresponding to the Stanley-Reisner ideal  $\sqrt{I}$ , and the second sum is taken over all  $a \in \mathbb{Z}^n$  such that  $a_i \leq t_i - 1$  for all  $i$ , and  $G_a = F$ .

As a first application of this theorem we have

**Corollary 2.2.** *Let  $I \subset S$  be a monomial ideal. Then*

$$a(S/I) \leq \sum_{i=1}^n t_i - n,$$

where  $a(S/I)$  is the  $a$ -invariant of  $S/I$ .

*Proof.* By Theorem 2.1, we know that  $H_{\mathfrak{m}}^i(R)_a = 0$  for all  $i$  and for all  $a \in \mathbb{Z}^n$  such that  $a_i > t_i - 1$  for some  $i$ . Thus in particular, if  $d = \dim R$ , then  $H_{\mathfrak{m}}^d(R)_j = 0$  for  $j > \sum_{i=1}^n t_i - n$ .  $\square$

We say that  $S/I$  has *maximal  $a$ -invariant* if the upper bound in Corollary 2.2 is attained, that is, if  $a(S/I) = \sum_{i=1}^n t_i - n$ .

For our main theorem the next corollary is important.

**Corollary 2.3.** *Let  $I \subset S$  be a monomial ideal. Then we have the following isomorphisms of  $K$ -vector spaces*

$$H_{\mathfrak{m}}^i(S/I)_a \cong H_{\mathfrak{m}}^i(S/\sqrt{I})_a$$

for all  $a \in \mathbb{Z}^n$  with  $a_i \leq 0$  for  $1 \leq i \leq n$ .

*Proof.* Consider the multigraded Hilbert series of  $H_{\mathfrak{m}}^i(S/I)$  and  $H_{\mathfrak{m}}^i(S/\sqrt{I})$ . Let  $a \in \mathbb{Z}^n$  be such that  $a_i \leq 0$  for all  $1 \leq i \leq n$ . Then by Theorem 2.1, we have

$$\begin{aligned} \dim_K H_{\mathfrak{m}}^i(S/I)_a &= \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(I); K), \quad \text{and} \\ \dim_K H_{\mathfrak{m}}^i(S/\sqrt{I})_a &= \dim_K \tilde{H}_{i-|F|-1}(\Delta_a(\sqrt{I}); K), \end{aligned}$$

For a monomial  $u$  we set  $\text{supp}(u) = \{i: x_i \text{ divides } u\}$ . Now since for every  $u \in G(I)$  there exists  $v \in G(\sqrt{I})$  such that  $\text{supp}(u) \supset \text{supp}(v)$ , and since for every  $v \in G(\sqrt{I})$  there exists  $u \in G(I)$  such that  $\text{supp}(v) = \text{supp}(u)$ , it follows that  $\Delta_a(I) = \Delta_a(\sqrt{I})$ . Thus we have  $\dim_K H_{\mathfrak{m}}^i(S/I)_a = \dim_K H_{\mathfrak{m}}^i(S/\sqrt{I})_a$ .  $\square$

Let  $M$  be a graded  $S$ -module. For the convenience of the reader we recall the following two concepts which generalize the Cohen-Macaulay property and non-pure shellability of simplicial complexes.

The following definition is due to Stanley [9, Section II, 3.9]:

**Definition 2.4.** Let  $M$  be a finitely generated graded  $S$ -module. The module  $M$  is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

of  $M$  by graded submodules of  $M$  such that each quotient  $M_i/M_{i-1}$  is CM, and  $\dim M_1/M_0 < \dim M_2/M_1 < \dots < \dim M_r/M_{r-1}$ .

It is known (see for example [6, Corollary 1.7]) that if  $M$  is sequentially Cohen-Macaulay, then the filtration given in the definition is uniquely determined. We call it the *attached filtration* of the sequentially Cohen-Macaulay module  $M$ .

The uniqueness of the filtration is seen as follows: suppose  $\text{depth } M = t$ , then  $M_1$  is the image of the natural map  $\text{Ext}_S^{n-t}(\text{Ext}_S^{n-t}(M, \omega_S), \omega_S) \rightarrow M$ . Here  $\omega_S = S(-n)$  is the canonical module of  $S$ . Then one notices that  $M/M_1$  is again sequentially Cohen-Macaulay and uses induction on the length of the attached sequence.

In case  $M$  is a cyclic module, say,  $M = S/I$ , with attached filtration  $0 = M_0 \subset M_1 \subset \dots$ , each of the the modules  $M_i$  is an ideal in  $S/I$ , and hence is of the form  $I_i/I$  for certain (uniquely determined) ideals  $I_i \subset S$ . Thus  $S/I$  is sequentially Cohen-Macaulay, if and only if there exists a chain of graded ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_r = S$$

such that each factor module  $I_{i+1}/I_i$  is Cohen-Macaulay with

$$\dim I_{i+1}/I_i < \dim I_{i+2}/I_{i+1}$$

for  $i = 0, \dots, r-2$ . Moreover if this property is satisfied, then this chain of ideals is uniquely determined.

In the particular case that  $I$  is a monomial ideal, the natural map

$$\text{Ext}_S^{n-t}(\text{Ext}_S^{n-t}(S/I, \omega_S), \omega_S) \rightarrow S/I$$

is a homomorphism of multigraded  $S$ -modules. This implies that the attached chain of ideals of the sequentially Cohen-Macaulay module  $S/I$  is a chain of monomial ideals.

Now let us briefly describe the other concept which was introduced by Dress [4]:

**Definition 2.5.** Let  $M$  be a finitely generated graded  $S$ -module. A filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

of  $M$  by graded submodules of  $M$  is called *clean* if for all  $i = 1, \dots, r$  there exists a minimal prime ideal  $P_i$  of  $M$  such that  $M_i/M_{i-1} \cong S/P_i$ . The module  $M$  is called *clean* if it has a clean filtration.

Again, if  $M = S/I$  is cyclic, then  $S/I$  is clean if there exists a chain of ideals  $I = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_{r-1} \subset I_r = S$  such that  $I_{i+1}/I_i \cong S/P_i$  with  $P_i$  a minimal prime ideal of  $I$ . In other words, for all  $i = 0, \dots, r-1$  there exists  $f_{i+1} \in I_{i+1}$  such

that  $I_{i+1} = (I_i, f_{i+1})$  and  $P_i = I_i : f_{i+1}$ . In case  $I$  is a monomial ideal we require that all  $f_i$  are monomials.

Dress [4] shows that a Stanley-Reisner ideal  $I_\Delta$  is clean if and only the simplicial complex  $\Delta$  is non-pure shellable in the sense of Björner and Wachs [3].

In the proof of our main theorem we use polarization, as indicated in the introduction. Let  $I = (u_1, \dots, u_m)$  with  $u_i = x_1^{a_{i1}} \cdots x_n^{a_{in}}$ . We fix some number  $i$  with  $1 \leq i \leq n$ , introduce a new variable  $y$ , and set  $v_k = x_1^{a_{k1}} \cdots x_i^{a_{ki}-1} y \cdots x_n^{a_{kn}}$  if  $a_{ki} > 1$ , and  $v_k = u_k$  otherwise. We call  $J = (v_1, \dots, v_m)$  the *1-step polarization of  $I$  with respect to the variable  $x_i$* . The element  $y - x_i$  is regular on  $S[y]/J$  and  $(S[y]/J)/(y - x_i)(S[y]/J) \cong S/I$ , see [1, Lemma 4.2.16].

Let as above  $t_i = \max\{\nu_i(u_j) : j = 1, \dots, m\}$ , and set  $t = \sum_{i=1}^n t_i - n$ . Then it is clear that if we apply  $t$  suitable 1-step polarizations, we end up with a squarefree monomial ideal  $I^p$ , which is called the *complete polarization of  $I$* .

Now we are ready to present the main result of this section.

**Theorem 2.6.** *Let  $K$  be a field,  $S = K[x_1, \dots, x_n]$  the polynomial ring over  $K$ , and  $I \subset S$  a monomial ideal. Suppose that  $S/I$  satisfies one of the following properties:  $S/I$  is (i) Cohen-Macaulay, (ii) Gorenstein, (iii) sequentially Cohen-Macaulay, (iv) generalized Cohen-Macaulay, (v) Buchsbaum, (vi) clean, or (vii) level and has maximal  $a$ -invariant. Then  $S/\sqrt{I}$  satisfies the corresponding property.*

*Proof.* We first use the trick, mentioned in the introduction, to show that the Betti-numbers  $\beta_i(I)$  of  $I$  do not increase when passing to  $\sqrt{I}$ .

We denote by  $I^p$  the complete polarization of  $I$ . Let  $T$  be the polynomial ring in the variables that are needed to polarize  $I$ . Then  $I^p$  is a squarefree monomial ideal in  $T$  with  $\beta_i(I^p) = \beta_i(I)$  for all  $i$ . It is easy to see that if we localize at the multiplicative set  $N$  generated by the *new* variables which are needed to polarize  $I$ , one obtains  $I^p T_N = (\sqrt{I}) T_N$ . Since localization is an exact functor, the localized free resolution will be a possibly non-minimal free resolution of  $(\sqrt{I}) T_N$ . Since the extension  $S \rightarrow T_N$  is flat, the desired inequality follows.

Proof of (i) and (ii): The inequality  $\beta_i(\sqrt{I}) \leq \beta_i(I)$  implies that  $\text{depth } S/\sqrt{I} \geq \text{depth } S/I$ . On the other hand,  $\dim S/I = \dim S/\sqrt{I}$ . This implies that  $S/\sqrt{I}$  is Cohen-Macaulay, if  $S/I$  is so.

Suppose now that  $S/I$  is Gorenstein. Then  $\beta_q(S/I) = 1$  where  $q$  is the codimension of  $I$ , see [1, Theorem 3.3.7 and Corollary 3.3.9]. Therefore,  $\beta_q(S/\sqrt{I}) \leq 1$ . Since  $I$  and  $\sqrt{I}$  have the same codimension, we see that  $\beta_q(S/\sqrt{I}) > 0$ , and hence  $\beta_q(S/\sqrt{I}) = 1$ . Again using [1, Theorem 3.3.7 and Corollary 3.3.9] we conclude that  $S/\sqrt{I}$  is Gorenstein. This fact follows also from [2, Corollary 3.4].

Proof of (iii): Since  $S/I$  is sequentially Cohen-Macaulay there exists a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k = S$$

such that  $I_{j+1}/I_j$  is Cohen-Macaulay for all  $j = 0, \dots, k-1$  and such that  $\dim I_1/I_0 < \dim I_2/I_1 < \cdots < \dim I_k/I_{k-1}$ .

Suppose  $x_1^a$  with  $a > 1$  divides a generator of  $I$ . Then we apply a 1-step polarization for  $x_1$  to all the ideals  $I_i$ , and obtain a chain of ideals  $J = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_k = \tilde{S}$  where  $\tilde{S} = S[y]$ . It follows that  $y - x_1$  is  $\tilde{S}/J_i$ -regular and  $(\tilde{S}/J_i)/(y - x_1)(\tilde{S}/J_i) \cong S/I_i$  for all  $i$ . Therefore  $y - x_1$  is  $J_{i+1}/J_i$ -regular, and  $(J_{i+1}/J_i)/(y - x_1)(J_{i+1}/J_i) \cong I_{i+1}/I_i$ . Thus  $J$  is sequentially Cohen-Macaulay.

Since the complete polarization  $I_i^p$  of the ideals  $I_i$  for  $i = 1, \dots, k$ , is obtained by a sequence of 1-step polarizations, it follows that  $I^p$  is sequentially Cohen-Macaulay. As  $I_i^p/I_{i+1}^p$  is Cohen-Macaulay, we conclude as in the proof of (i) that  $\sqrt{I_{i+1}^p}/\sqrt{I_i^p}$  is Cohen-Macaulay of the same dimension as  $I_{i+1}/I_i$ . This shows that  $\sqrt{I}$  is sequentially Cohen-Macaulay.

Proof of (iv) and (v): Assuming that  $S/I$  is generalized Cohen-Macaulay or Buchsbaum, one has that  $S/I$  is equidimensional and that  $H_m^i(S/I)_j = 0$  for all  $i < \dim S/I$ , and all but finitely many  $j$ . Since  $I$  and  $\sqrt{I}$  have the same minimal prime ideals, it follows that  $\sqrt{I}$  is again equidimensional.

Let  $\mathbb{Z}_-^n$  be the set of all  $a \in \mathbb{Z}^n$  such that  $a_i \leq 0$  for  $i = 1, \dots, n$ . By Corollary 2.3,  $H_m^i(S/I)_a = H_m^i(S/\sqrt{I})_a$  for all  $a \in \mathbb{Z}_-^n$ . Moreover, by Hochster's formula,  $H_m^i(S/\sqrt{I})_a = 0$  for all  $a \notin \mathbb{Z}_-^n$ . Therefore,  $\dim_K H_m^i(S/\sqrt{I})_j \leq \dim_K H_m^i(S/I)_j$  for all  $j \leq 0$  and  $H_m^i(S/J)_j = 0$  for  $j > 0$ . It is known [8] that a squarefree monomial ideal is Buchsbaum if and only if it is generalized Cohen-Macaulay. Thus (iv) and (v) follow.

Proof of (vi): Assuming that  $S/I$  is clean, there exists a chain of monomial ideals  $I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_{r-1} \subset I_r = S$  such that  $I_{i+1}/I_i \cong S/P_i$  with  $P_i$  a minimal prime ideal of  $I$ . We claim that  $\sqrt{I_{i+1}}/\sqrt{I_i} = S/P_i$ , if  $\sqrt{I_{i+1}} \neq \sqrt{I_i}$ . This then implies that  $S/\sqrt{I}$  is clean, since the prime ideals  $P_i$  are also minimal prime ideals of  $\sqrt{I}$ .

In order to prove this claim we introduce some notation: let  $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$  and  $v = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$  be two monomials. Then we set

$$u : v = \prod_{i=1}^n x_i^{\max\{a_i - b_i, 0\}}, \quad \text{and} \quad u_{red} = \prod_{\substack{i \\ a_i > 0}} x_i.$$

We then have

$$(1) \quad (u : v)_{red} = (u_{red} : v_{red}) \prod_{\substack{i \\ a_i > b_i > 0}} x_i.$$

Note that if  $I$  is a monomial ideal with monomial generators  $u_1, \dots, u_m$ , then

$$\sqrt{I} = ((u_1)_{red}, \dots, (u_m)_{red}) \quad \text{and} \quad I : v = (u_1 : v, \dots, u_m : v).$$

Back to the proof of our claim, our assumption implies that for all  $i = 0, \dots, r - 1$  there exists a monomial  $v_{i+1} \in I_{i+1}$  such that  $I_{i+1} = (I_i, v_{i+1})$  and  $P_i = I_i : v_{i+1}$ . Suppose  $P_i = (x_{i_1}, \dots, x_{i_s})$ . Then  $P_i = I_i : v_{i+1}$  if and only if

- (a) for all  $j = 1, \dots, s$  there exists  $u \in I_i$  such that  $u : v_{i+1} = x_{i_j}$ , and

(b) for all monomial generators  $w \in I_i$  there exists an integer  $j$  with  $1 \leq j \leq s$  such that  $x_{i_j} | (w : v_{i+1})$ .

We need to show that  $P_i = \sqrt{I_i} : (v_{i+1})_{red}$ , if  $(v_{i+1})_{red} \notin \sqrt{I_i}$ , and prove this by checking (a) and (b) for the pair  $\sqrt{I_i}$  and  $(v_{i+1})_{red}$ .

Let  $j$  be an integer with  $1 \leq j \leq s$ . Then there exists  $u \in I_i$  such that  $u : v_{i+1} = x_{i_j}$ . Suppose  $u = \prod_{k=1}^n x_k^{a_k}$  and  $v_{i+1} = \prod_{k=1}^n x_k^{b_k}$ , then (1) implies that  $x_{i_j} = (u : v_{i+1})_{red} = (u_{red} : (v_{i+1})_{red})w$  where  $w = \prod_{k, a_k > b_k > 0} x_k$ . Suppose  $x_{i_j}$  divides  $w$ , then  $u_{red} : (v_{i+1})_{red} = 1$ . This implies that  $(v_{i+1})_{red} \in \sqrt{I_i}$ , a contradiction. Therefore  $u_{red} : (v_{i+1})_{red} = x_{i_j}$ , and this proves (a). The argument also shows that  $b_{i_j} = 0$  for  $j = 1, \dots, s$ .

For the proof of (b), let  $w \in I_i$  be a monomial generator. Then there exists an integer  $j$  with  $1 \leq j \leq s$  such that  $x_{i_j} | (w : v_{i+1})$ . It follows that  $x_{i_j}$  divides  $(w : v_{i+1})_{red}$ . Let  $w = \prod_{k=1}^n x_k^{c_k}$ . Then (1) implies that  $x_{i_j}$  divides  $(w_{red} : (v_{i+1})_{red}) \prod_{k, c_k > b_k > 0} x_k$ . However,  $b_{i_j} = 0$ , as we have seen in the proof of (a). Therefore,  $x_{i_j}$  divides  $(w_{red} : (v_{i+1})_{red})$ . Since  $\sqrt{I_i}$  is generated by the monomials  $w_{red}$  where the monomials  $w$  are the generators of  $I_i$ , condition (b) follows.

Proof of (vii): By assumption  $S/I$  is level. This means that  $S/I$  is Cohen-Macaulay and that all generators of the canonical module  $\omega_{S/I}$  of  $S/I$  have the same degree, say  $g$ . In this situation the  $a$ -invariant  $a(S/I)$  of  $S/I$  is just  $-g$ , see [1, Section 3.6]. Suppose  $d = \dim S/I$ ; then  $I$  has a graded minimal free resolution  $\mathbb{F}$  of length  $q = n - d - 1$  with  $F_q = S^b(-c)$ . Since  $\omega_{S/I}$  may be represented as the cokernel of  $F_{q-1}^* \rightarrow F_q^*$ , which is dual of the map  $F_q \rightarrow F_{q-1}$  with respect to  $S(-n)$ , it follows that  $a(S/I) = c - n$ .

For  $i = 1, \dots, n$  we set again

$$t_i = \max\{\nu_i(u) : u \in G(I)\}.$$

By Corollary 2.2, one has the upper bound  $a(S/I) \leq \sum_{i=1}^n t_i - n$ . Since we assume that  $S/I$  has maximal  $a$ -invariant, the upper bound is reached. Let  $I^p \subset T$  the complete polarization of  $I$ . This polarization requires precisely  $t = \sum_{i=1}^n t_i - n$  1-step polarizations. It follows that  $S/I$  is obtained from  $T/I^p$  as a residue class ring modulo a regular sequence of linear forms of length  $t$ . From the above description of the  $a$ -invariant we now conclude that  $a(T/I^p) = a(S/I) - t = 0$ . Let  $\mathbb{G}$  be the multigraded minimal free resolution of the squarefree monomial ideal  $I^p$ . Since  $\text{proj dim } I^p = \text{proj dim } I = q$ , and since  $a(T/I^p) = 0$ , we see that  $G_q = T(-m)^b$ , where  $m = n + t = \dim T$ . This implies that  $G_q$  as a multigraded module is isomorphic to  $T(-e)^b$  where  $e = (1, 1, \dots, 1)$ .

For  $i = 1, \dots, m$  let  $e_i$  be the  $i$ th canonical basis vector of  $\mathbb{Z}^m$ . Then  $e = \sum_{i=1}^m e_i$ , and we may assume that  $\deg x_i = e_i$  for  $i = 1, \dots, n$ , while the new variables have the multidegrees  $e_i$  with  $i = n + 1, \dots, m$ . We define a new multigrading on  $T$  and  $T/I^p$ : for an element  $f$  of multidegree  $a$  we set  $\deg' f = \pi(a)$ , where  $\pi : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$  is the projection onto the first  $n$  components of  $\mathbb{Z}^m$ .

As above, let  $N$  be the multiplicative set generated by the  $t$  new variables which are needed to polarize  $I$ . Then  $I^p T_N = \sqrt{I} T_N$ , and localization with respect to  $N$  preserves the new multigrading since  $\deg' f = 0$  for all  $f \in N$ . Therefore  $\mathbb{G}_N$  is,

with respect to the new grading, a multigraded free  $T_N$ -resolution of  $\sqrt{I}T_N$  with  $(G_q)_N = T_N(-1, \dots, -1)^b$  and  $(-1, \dots, -1) \in \mathbb{Z}^n$ .

Let  $\mathbb{H}$  be the multigraded minimal free  $S$ -resolution of  $\sqrt{I}$ . Then  $\mathbb{H}T_N$  is the minimal multigraded free  $T_N$ -resolution of  $\sqrt{I}T_N$ . A comparison with the (possibly non-minimal) graded free  $T_N$ -resolution  $\mathbb{G}_N$  shows that  $H_q$  is a direct summand of copies of  $S(-1, \dots, -1)$ . Since  $S/I$  and  $S/\sqrt{I}$  are Cohen-Macaulay of the same dimension, we see that  $q = \text{proj dim } I = \text{proj dim } \sqrt{I}$ . Therefore all summands in the last step of the resolution  $\mathbb{H}$  of  $S/\sqrt{I}$  have the same shift. This shows that  $S/\sqrt{I}$  is level.  $\square$

**Remark 2.7.** In Theorem 2.6(i) (or (iv)), it suffices to require that  $I$  is an arbitrary homogeneous (generalized) Cohen-Macaulay ideal whose radical  $\sqrt{I}$  is a monomial ideal, i.e. we do not need to require that  $I$  itself is a monomial ideal.

Indeed it is enough to prove that there is a surjective homomorphism  $H_{\mathfrak{m}}^i(S/I) \rightarrow H_{\mathfrak{m}}^i(S/\sqrt{I})$  for all  $i$ . The natural surjective map  $S/I \rightarrow S/\sqrt{I}$  induce for all  $i$  commutative diagrams

$$\begin{array}{ccc} \text{Ext}^i(S/\sqrt{I}, S) & \longrightarrow & \text{Ext}^i(S/I, S) \\ \downarrow & & \downarrow \\ H_{\sqrt{I}}^i(S) & \longrightarrow & H_I^i(S). \end{array}$$

Since  $H_{\sqrt{I}}^i(S) \cong H_I^i(S)$  and since  $\text{Ext}^i(S/\sqrt{I}, S) \rightarrow H_{\sqrt{I}}^i(S)$  is an essential extension (see [12]), it follows that  $\text{Ext}_S^i(S/\sqrt{I}, S) \rightarrow \text{Ext}_S^i(S/I, S)$  is injective for all  $i$ . Hence the desired conclusion follows by local duality.

On the other hand, as for the Gorenstein property, we must assume that  $I$  is a monomial ideal. For example,  $I = (xy + yz, xz)$  is a complete intersection, hence, a Gorenstein ideal, while  $\sqrt{I} = (xy, yz, xz)$  is not Gorenstein.

### 3. THE INVERSE PROBLEM

The results of the previous section indicate the following question: for a subset  $F \subset [n]$ , let  $P_F$  be the prime ideal generated by the  $x_i$  with  $i \in F$ . The minimal prime ideals of a squarefree  $I$  are all of this form, and since  $I$  is a radical ideal it is the intersection of its minimal prime ideals, say,  $I = \bigcap_{i=1}^r P_{F_i}$  with  $F_i \subset [n]$ .

Suppose  $I$  is Cohen-Macaulay. For which exponents  $a_{ij}$  is the ideal

$$J = \bigcap_{i=1}^r (x_j^{a_{ij}} : j \in F_i)$$

again Cohen-Macaulay?

Of course if we raise the  $x_i$  uniformly to some power, say  $x_i$  is replaced by  $x_i^{a_i}$  everywhere in the intersection, then the resulting ideal  $J$  is the image of the flat map  $S \rightarrow S$  with  $x_i \mapsto x_i^{a_i}$  for all  $i$ . Thus in this case  $J$  will be Cohen-Macaulay, if  $I$  is so. On the other hand, if we allow arbitrary exponents, the question seems to be quite delicate, and we do not know a general answer. However, if we require that for *all* choices of exponents the resulting ideal is again Cohen-Macaulay, a complete answer is possible.

We need a definition to state the next result. Let  $L$  be a monomial ideal. Lyubeznik [7] defines the *size* of  $L$  as follows: let  $L = \bigcap_{j=1}^r Q_j$  be an irredundant primary decomposition of  $L$ , where the  $Q_i$  are monomial ideals. Let  $h$  be the height of  $\sum_{j=1}^r Q_j$ , and denote by  $v$  the minimum number  $t$  such that there exist  $j_1, \dots, j_t$  with  $\sqrt{\sum_{i=1}^t Q_{j_i}} = \sqrt{\sum_{j=1}^r Q_j}$ . Then  $\text{size } L = v + (n - h) - 1$ .

Since for monomial ideals the operations of forming sums and taking radicals can be exchanged, the numbers  $v$  and  $h$ , and hence the size of  $L$  depends only on the associated prime ideals of  $L$ .

We shall need the following result of Lyubeznik [7, Proposition 2]:

**Lemma 3.1.** *Let  $L$  be a monomial ideal in  $S$ . Then  $\text{depth } S/L \geq \text{size } L$ .*

Now we can state the main result of this section.

**Theorem 3.2.** *Let  $I \subset S = K[x_1, \dots, x_n]$  be a Cohen-Macaulay squarefree monomial ideal, and write*

$$I = \bigcap_{i=1}^r P_{F_i},$$

where the sets  $F_i \subset [n]$  are pairwise distinct, and all have the same cardinality  $c$ .

For  $i = 1, \dots, r$  and  $j = 1, \dots, c$  we choose integers  $a_{ij} \geq 1$ , and set

$$Q_{F_i} = (x_j^{a_{ij}} : j \in F_i) \quad \text{for } i = 1, \dots, r.$$

Then the following conditions are equivalent:

(a) for all choices of the integers  $a_{ij}$  the ideal

$$J = \bigcap_{i=1}^r Q_{F_i}$$

is Cohen-Macaulay;

(b) for each subset  $A \subset [r]$ , the ideal  $I_A = \bigcap_{i \in A} P_{F_i}$  is Cohen-Macaulay;

(c)  $\text{height } P_{F_i} + P_{F_j} = c + 1$  for all  $i \neq j$ ;

(d) for  $r \geq 2$  either  $|\bigcup_{i=1}^r F_i| = c + 1$ , or  $|\bigcap_{i=1}^r F_i| = c - 1$ ;

(e) after a suitable permutation of the elements of  $[n]$  we either have

$$F_i = \{1, \dots, i - 1, i + 1, \dots, c, c + 1\} \quad \text{for } i = 1, \dots, r,$$

or

$$F_i = \{1, \dots, c - 1, c - 1 + i\} \quad \text{for } i = 1, \dots, r;$$

(f)  $\text{size } I = \dim S/I$ ;

(g)  $S/L$  is Cohen-Macaulay for any monomial ideal  $L$  such that  $\text{Ass } L = \text{Ass } I$ .

*Proof.* (a)  $\Rightarrow$  (b): Let  $Q_{F_i} = (x_j^2 : j \in F_i)$  if  $i \in A$ , and  $Q_{F_i} = P_{F_i}$  if  $i \notin A$ . By assumption,  $J = \bigcap_{i=1}^r Q_{F_i}$  is Cohen-Macaulay. Hence the complete polarization  $J^p$  of  $J$  is again Cohen-Macaulay. We have  $J^p = \bigcap_{i=1}^r Q_{F_i}^p$  with  $Q_{F_i}^p = (x_j y_j : j \in F_i)$  if

$i \in A$ , and  $Q_{F_i}^p = P_{F_i}$  if  $i \notin A$ . Let  $N$  be the multiplicative set generated by all the variables  $x_i$ . Then  $J_N^p$  is Cohen-Macaulay, and hence

$$J_N^p = \bigcap_{i \in A} (y_j : j \in F_i).$$

This shows that  $I_A = \bigcap_{i \in A} P_{F_i}$  is Cohen-Macaulay.

(b)  $\Rightarrow$  (c): Consider the exact sequence

$$0 \longrightarrow S/(P_{F_i} \cap P_{F_j}) \longrightarrow S/P_{F_i} \oplus S/P_{F_j} \longrightarrow S/(P_{F_i} + P_{F_j}) \longrightarrow 0.$$

The rings  $S/P_i$  and  $S/P_j$  are Cohen-Macaulay of dimension  $n - c$ , while  $S/(P_{F_i} + P_{F_j})$  is Cohen-Macaulay of dimension  $n - d$  where  $d$  is the height of  $P_{F_i} + P_{F_j}$ . The exact sequence yields that  $S/(P_{F_i} \cap P_{F_j})$  is Cohen-Macaulay if and only if  $d = c + 1$ .

Since by assumption  $S/P_{F_i} \cap P_{F_j}$  is Cohen-Macaulay for all  $i \neq j$ , the assertion follows.

(c)  $\Rightarrow$  (d): We must show: given a collection of subsets  $F_1, \dots, F_r \subset [n]$  with

- (i)  $|F_i| = c$  for all  $i$ ;
- (ii)  $|F_i \cup F_j| = c + 1$  for all  $i \neq j$ .

Then either  $|\bigcup_{i=1}^r F_i| = c + 1$ , or  $|\bigcap_{i=1}^r F_i| = c - 1$ .

Suppose this is not the case. Then, since  $|F_1 \cap F_2| = c - 1$  and  $|F_1 \cup F_2| = c + 1$ , there exist integers  $i$  and  $j$  such that  $F_1 \cap F_2 \not\subset F_i$ , and  $F_j \not\subset F_1 \cup F_2$ . The conditions (i) and (ii) then imply that there exists an element  $x \in F_1 \cap F_2$  such that  $F_1 \cup F_2 \setminus \{x\} = F_i$ , and an element  $y \in F_j \setminus (F_1 \cup F_2)$  such that  $F_j = \{y\} \cup (F_1 \cap F_2)$ . It follows that  $F_i \cup F_j = (F_1 \cup F_2) \cup \{y\}$ . This contradicts (ii).

(d)  $\Rightarrow$  (e): Assume that  $|\bigcup_{i=1}^r F_i| = c + 1$ . After a suitable permutation of the elements of  $[n]$  we may assume that  $\bigcup_{i=1}^r F_i = \{1, \dots, c + 1\}$ . Since  $|F_i| = c$ , there exists  $j_i \in \{1, \dots, c + 1\}$  such that  $F_i = \{1, \dots, c + 1\} \setminus \{j_i\}$ . Since the sets  $F_i$  are pairwise distinct it follows that  $j_i \neq j_k$  for  $i \neq k$ . Thus after applying again suitable permutation we may assume that  $j_i = i$  for  $i = 1, \dots, r$ .

The second statement follows similarly.

(e)  $\Rightarrow$  (f): In the first case,  $v = 2$  and  $h = (c + 1)$ , while in the second case,  $v = r$  and  $h = c - 1 + r$ . Thus in both cases  $\text{size } I = n - c = \dim S/I$ .

(f)  $\Rightarrow$  (g): By Lemma 3.1 and the remark preceding the lemma, we have

$$\text{depth } S/L \geq \text{size } L = \text{size } I = \dim S/I = \dim S/L.$$

Hence  $S/L$  is Cohen-Macaulay.

Finally the implication (g)  $\Rightarrow$  (a) is trivial.  $\square$

**Corollary 3.3.** *With notation as above, the following conditions are equivalent:*

- (a)  $J$  is a Gorenstein ideal for all choices of the integers  $a_{ij}$ ;
- (b)  $r = 1$  or  $c = 1$ .

*Proof.* If  $r = 1$  or  $c = 1$ , then  $J$  is complete intersection for all choices of the integers  $a_{ij}$ . Thus (b) implies (a).

Conversely suppose condition (b) is not satisfied. We assume that  $c > 1$ , and have to show that  $r = 1$ . By Theorem 3.2 we have  $|\bigcap_{i=1}^r F_i| = c - 1$  or  $|\bigcup_{i=1}^r F_i| = c + 1$ .

In the first case we may assume that  $F_i = \{1, \dots, c-1, i+c-1\}$  for  $i = 1, \dots, r$ . Assume  $r > 1$ , and let  $Q_{F_1} = (x_1^2, x_2, \dots, x_c)$  and  $Q_{F_i} = P_{F_i}$  for  $i \geq 2$ . Then  $J = \bigcap_{i=1}^r Q_{F_i} = (x_1^2, x_1x_2, \prod_{i=0}^{r-1} x_{c+i})$  is not Gorenstein, a contradiction.

In the second case suppose that  $r \geq 3$ . With the same argument as in the proof of Theorem 3.2 it follows that  $I_A = \bigcap_{i \in A} P_{F_i}$  is a Gorenstein ideal for all subsets  $A \subset [r]$ . Therefore  $P_{F_1} \cap P_{F_2} \cap P_{F_3}$  is Gorenstein. We may assume that  $F_1 = \{1, 2, \dots, c\}$ ,  $F_2 = \{2, 3, \dots, c+1\}$  and  $F_3 = \{1, 3, 4, \dots, c+1\}$ . Then  $P_{F_1} \cap P_{F_2} \cap P_{F_3} = (x_1x_2, x_1x_{c+1}, x_2x_{c+1}, x_3, \dots, x_c)$  is not Gorenstein, a contradiction.

On the other hand, if  $r = 2$ , then  $|\bigcap_{i=1}^r F_i| = c-1$ , and we are again in the first case. Thus we must have that  $r = 1$ .  $\square$

**Remark 3.4.** From a view point of Stanley-Reisner rings, the ideal  $I$  in the first case of condition (e) in Theorem 3.2 corresponds to an iterated cone of a 0-dimensional simplicial complex. In this case it is known that  $S/I$  itself is Gorenstein if the corresponding 0-dimensional simplicial complex consists of at most 2 points, see [9, Theorem 5.1(e)]. The corollary also follows from this fact.

## REFERENCES

- [1] W. Bruns and J. Herzog, “Cohen-Macaulay rings” (Revised edition), Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998.
- [2] W. Bruns and J. Herzog, On multigraded resolutions, Math. Proc. Camb. Phil. Soc. **118** (1995), 245–257.
- [3] A. Björner and M.L. Wachs, Shellable non-pure complexes and posets II, Trans. AMS **349** (1997) 3945–3975.
- [4] A. Dress, A new algebraic criterion for shellability, Beiträge zur Algebra und Geometrie **34** (1993), 45–55.
- [5] R. Hartshorne, Complete intersections in characteristic  $p > 0$ , Amer. J. Math. **101** (1979), 380–383.
- [6] J. Herzog and E. Sbarra, Sequentially Cohen-Macaulay modules and local cohomology, “Algebra, arithmetic and geometry, Part I, II” (Mumbai, 2000), 327–340, Tata Inst. Fund. Res. Stud. Math., **16**, Tata Inst. Fund. Res., Bombay, 2002.
- [7] G. Lyubeznik, On the arithmetic rank of monomial ideals, J. Alg. **112** (1988), 86–89.
- [8] P. Schenzel, On the number of faces of simplicial complexes and the purity of Frobenius, Math. Z. **178** (1981), 125–142.
- [9] R.P. Stanley, “Combinatorics and commutative algebra”, Birkhäuser, second edition, 1996.
- [10] Y. Takayama, A generalized Hochster’s formula for local cohomologies of monomial ideals, preprint 2004.
- [11] A. Taylor, The inverse Groebner basis problem in codimension two, J. Symb. Comp. **33** (2002), 221–238.
- [12] N. Terai, Local cohomology modules with respect to monomial ideals, preprint 1998.

JÜRGEN HERZOG, FACHBEREICH MATHEMATIK UND INFORMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY

*E-mail address:* `juergen.herzog@uni-essen.de`

YUKIHIDE TAKAYAMA, DEPARTMENT OF MATHEMATICAL SCIENCES, RITSUMEIKAN UNIVERSITY, 1-1-1 NOJIHIGASHI, KUSATSU, SHIGA 525-8577, JAPAN

*E-mail address:* `takayama@se.ritsumei.ac.jp`

NAOKI TERAII, DEPARTMENT OF MATHEMATICS, FACULTY OF CULTURE AND EDUCATION, SAGA UNIVERSITY, SAGA 840-8502, JAPAN

*E-mail address:* `terai@cc.saga-u.ac.jp`