

# ASYMPTOTIC COHOMOLOGICAL FUNCTIONS ON PROJECTIVE VARIETIES

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## 1. INTRODUCTION

Our purpose here is to consider certain cohomological invariants associated to complete linear systems on projective varieties. These invariants — called asymptotic cohomological functions — are higher degree analogues of the volume of a divisor. We establish the continuity of asymptotic cohomological functions on the real Néron–Severi space and describe several interesting connections which link them to classical phenomena, for example Zariski decompositions of divisors, or Mumford’s index theorem for the cohomology of line bundles on abelian varieties.

Our concepts have their origins in the Riemann–Roch problem. The classical version asking how  $h^0(X, \mathcal{O}_X(mD))$  changes as a function of  $m$  (where  $X$  is an irreducible complex projective variety, and  $D$  is a Cartier divisor on  $X$ ), has only been answered in dimensions up to two, by Riemann and Roch for curves, and by Zariski [24], and Cutkosky and Srinivas [5] for surfaces. The lack of a satisfactory answer in higher dimensions makes it important to look at the question from an asymptotic point of view. For ample divisors, the by now classical asymptotic Riemann–Roch theorem of Kleiman [17] and Snapper [23] settles the issue. For arbitrary divisors, however, the question has only surfaced recently in the form of the volume of a divisor, ie. the asymptotic rate of growth of the number of global sections of its multiples.

The notion of the volume first arose implicitly in Cutkosky’s work [4], where he used asymptotic computations to establish the non-existence of rational Zariski decompositions on a certain threefold. It was then studied subsequently by Demailly, Ein, Fujita, Lazarsfeld, and others, while pioneering efforts regarding other asymptotic invariants of linear systems were made Nakayama [21] and Tsuji. In this process the properties of the volume were more fully explored, and instead of thinking of the volume as an invariant linked to a single divisor, one started to consider it as a function on the Néron–Severi space, thus as an intrinsic invariant of the underlying variety  $X$ .

More precisely, the volume of a Cartier divisor  $D$  on an irreducible projective variety  $X$  of dimension  $n$  is defined to be

$$\mathrm{vol}_X(D) = \limsup_m \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

For ample divisors,  $\mathrm{vol}_X(D) = (D^n)$ , and considered as a function of the divisor  $D$ , it descends to a degree  $n$  homogeneous continuous functions on the real Néron–Severi space. The point of view that the volume should be defined on numerical equivalence classes originates in [19], although it was also realized independently in [2].

The volume function is log-concave, which — according to the influential paper [22] of Okounkov, is an indication that it is a ‘good’ notion of multiplicity. In a different direction, Demailly, Ein and Lazarsfeld in [8] show that the volume of a divisor is the normalized limsup

of the moving self-intersection numbers of its multiples. The analogous notion on compact Kähler manifolds has been studied by Boucksom [2].

Asymptotic cohomological functions of divisors are direct generalizations of the volume function. Let  $X$  be an irreducible projective variety of dimension  $n$ ,  $D$  an integral Cartier divisor on  $X$ . Then for every  $0 \leq i \leq n$ , the  $i$ th *asymptotic cohomological function* associated to  $X$  is defined to be

$$\widehat{h}^i(X, D) \stackrel{\text{def}}{=} \limsup_m \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

Note that by definition,  $\widehat{h}^0(X, D) = \text{vol}_X(D)$ . There is a pronounced difference between the case of the volume function and higher asymptotic cohomological functions: for a non-big divisor, the volume is zero, while this is not so in general for the higher asymptotic cohomological functions. Therefore, asymptotic cohomological functions of higher degree potentially carry information about non-effective divisors as well.

Probably the most striking property of the volume of divisors was that it defines a continuous function on  $N^1(X)_{\mathbb{R}}$ . Our main focus here is to prove the corresponding statement for asymptotic cohomological functions. More concretely, in Theorem 5.1, we establish the following (for its  $\widehat{h}^0$  predecessor see [19, Theorem 2.2.44].)

**Theorem** (Continuity of asymptotic cohomological functions). *Let  $X$  be an irreducible projective variety of dimension  $n$  over  $\mathbb{C}$ . Then for all  $0 \leq i \leq n$ , the functions  $\widehat{h}^i$  are invariant with respect to numerical equivalence of divisors, and*

$$\widehat{h}^i : N^1(X)_{\mathbb{Q}} \rightarrow \mathbb{R}^{\geq 0}$$

*defines a continuous function on  $N^1(X)_{\mathbb{Q}}$ . These functions are homogeneous of degree  $n$ , and satisfy the following Lipschitz-type estimate: there exists a constant  $C$  such that for all pairs  $\xi, \eta \in N^1(X)_{\mathbb{Q}}$ , one has*

$$|\widehat{h}^i(X, \xi) - \widehat{h}^i(X, \eta)| \leq C \cdot \sum_{k=1}^n (\max\{\|\xi\|, \|\eta\|\})^{n-k} \cdot \|\xi - \eta\|^k$$

*for some fixed norm  $\|\cdot\|$ .*

**Corollary.** *With notation as in the Theorem, the asymptotic cohomological functions  $\widehat{h}^i$  extend uniquely to continuous functions*

$$\widehat{h}^i : N^1(X)_{\mathbb{R}} \rightarrow \mathbb{R}^{\geq 0},$$

*which are homogeneous of degree  $n$ .*

The main ingredients of the proof are boundedness of numerically trivial divisors, and cohomological estimates coming from a Mayer–Vietoris-type exact sequence of sheaves.

On certain classes of varieties with additional structure, notable examples being complex abelian varieties and generalized flag varieties, the cohomology of line bundles exhibits an interesting chamber structure, in that  $\text{Pic}(X)_{\mathbb{R}}$  is divided into a set of open cones, and for the integral points in each such cone there is a single non-vanishing cohomology group. This behaviour manifests itself in the form of Mumford’s index theorem for complex abelian varieties, and the Borel–Weil–Bott theorem on generalized flag varieties.

These phenomena are suggestively similar to the behaviour of the volume function on smooth projective surfaces. As described in [1], the volume on a smooth projective surface is piecewise polynomial with respect to a locally finite, locally rational polyhedral chamber decomposition on the cone of big divisors.

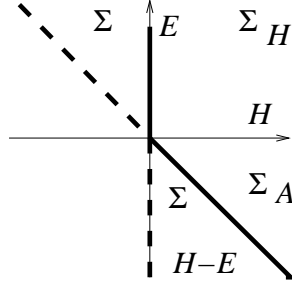
Surprisingly enough, there exists some sort of a generalization of these phenomena to arbitrary irreducible projective varieties, which rests upon the notion of asymptotic cohomological functions. These are meaningful on any irreducible projective variety, and in a certain sense they give back the classical decompositions of  $\text{Pic}(X)_{\mathbb{R}}$  on abelian varieties, and generalized flag varieties.

In many cases — e.g. smooth projective surfaces, toric varieties (see [13]) — these invariants lead to a chamber structure similar to that seen in the case of abelian varieties or generalized flag varieties, by considering the maximal regions in the Néron–Severi space where each asymptotic cohomological function is given by a single polynomial.

It is informative to have a look at a concrete example before exploring asymptotic cohomological functions in more detail.

**Example.** Consider the projective plane  $\mathbb{P}^2$  blown up at a point, which we denote by  $X$ . The Néron–Severi space of  $X$  is generated by the exceptional divisor  $E$  of the blow-up and  $H$ , the pull-back of the class of a line in  $\mathbb{P}^2$ .

The effective cone is spanned by the rays of  $E$  and  $H - E$ . A class  $\alpha = xH - yE \in N^1(X)_{\mathbb{R}}$  is nef in and only if  $\alpha \cdot E \geq 0$  and  $\alpha \cdot (H - E) \geq 0$ , and the nef cone is generated by the rays spanned by  $H$  and  $H - E$ .



With the notation of the picture,  $\Sigma_A$  is the nef cone, the effective cone is generated by the rays of  $E$  and  $H - E$ , thus it is the union of the cones  $\Sigma_A$  and  $\Sigma_H$ .

We will describe the asymptotic cohomological functions in some of the regions. A direct computation shows that for  $D = xH - yE$ , both  $x, y > 0$  (ie. when  $D$  is nef),

$$\widehat{h}^i(X, \alpha) = \begin{cases} \alpha^2 = x^2 - y^2 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

The asymptotic cohomological functions are more interesting in the cone  $\Sigma_H$ , ie. in the part of  $\text{Big}(X)$ , which consists of non-nef divisors. Consider  $D = xH + yE$ , where  $x, y > 0$ . Then the asymptotic cohomological functions on  $\Sigma_L$  are

$$\widehat{h}^i(X, xH + yE) = \begin{cases} x^2 & \text{if } i = 0 \\ y^2 & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

For the part of  $N^1(X)_{\mathbb{R}}$  between the lines of  $E$  and  $E - H$ , the asymptotic cohomological functions are given by

$$\widehat{h}^i(X, \alpha) = \begin{cases} 0 & \text{if } i = 0 \\ y^2 - x^2 & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

Results on ordinary coherent cohomology have natural implications for asymptotic cohomological functions. Among these consequences, one obtains an asymptotic version of Serre duality: with notation as above, let  $D$  be an arbitrary  $\mathbb{Q}$ -divisor. Then one has

$$\widehat{h}^i(X, D) = \widehat{h}^{n-i}(X, -D) .$$

The reason for which we obtain an asymptotic duality statement in this generality, is that asymptotic cohomological functions are invariant under pullbacks with respect to birational maps. More generally, if  $f : Y \rightarrow X$  is a finite surjective map of  $n$ -dimensional irreducible projective varieties,  $D$  and integral Cartier divisor on  $X$ , then

$$\widehat{h}^i(Y, f^*D) = d \cdot \widehat{h}^i(X, D) ,$$

where  $d$  is the degree of  $f$ .

Asymptotic cohomological functions yield a vanishing result, which generalizes the asymptotic version of Serre vanishing for big divisors. In particular, a multiplier ideal argument (Proposition 2.15) shows that

**Proposition.** *If  $X$  is a smooth projective variety over the complex numbers,  $D$  a big  $\mathbb{Q}$ -divisor on  $X$  with  $d$ -dimensional stable base locus, then*

$$\widehat{h}^i(X, D) = 0$$

for all  $i > d$ .

Our investigation would not be complete without examples. More than just illustrations of the theory, they were the driving force behind many of the developments. We treat abelian varieties, homogeneous spaces, and smooth surfaces in detail. In the first two cases, we draw on earlier work of Mumford, and Borel–Weil–Bott, respectively.

In the case of surfaces, the computation of asymptotic cohomological functions does not rely on actual computations of ordinary cohomology groups, rather, it makes use of the approach in [1], where the authors use Zariski decomposition to determine the volume of divisors.

About the organization of this paper: the basic properties of asymptotic cohomological functions are given in Section 2. We illustrate these functions in a few concrete examples in Section 3. Section 4 hosts the technical basis for this work, it contains estimates on differences of dimensions of cohomology groups of Cartier divisors. Based on this, we prove our main result, the continuity of asymptotic cohomological functions in Section 5. The last section contains technical material, which is used in other parts of the paper, but does not essentially contribute to the understanding of our results.

**Acknowledgments.** I would like to thank Tommaso De Fernex, Katalin Friedl, Bill Fulton, Mel Hochster, Leonardo Mihăilescu, Endre Szabó and Alex Wolfe for the many helpful discussions and comments on previous versions of this work. The observations of Lawrence Ein and Mihnea Popa were particularly useful, and influenced this paper considerably. I am indebted to Brian Conrad for sharing his technical expertise, which led to significant improvements. Finally, I would like to express my gratitude towards my former thesis advisor, Rob Lazarsfeld, for his invaluable guidance during my graduate years.

## 2. BASIC PROPERTIES

This section develops the basic theory of asymptotic cohomological functions on projective varieties.

First let us fix notation. In what follows,  $X$  will be an irreducible projective variety over the complex numbers, unless otherwise mentioned. We will use line bundle and divisor

notation interchangeably, depending on the context. As we will be dealing with  $\mathbb{Q}$ - and  $\mathbb{R}$ -divisors frequently, there will be a preference for divisor language. If it does not cause confusion, we will use the shorthand notation  $H^i(X, D)$  for  $H^i(X, \mathcal{O}_X(D))$ , where  $D$  is a Cartier divisor, as it will often simplify the appearance of our formulas.

The Néron–Severi space  $N^1(X)$  is the group of Cartier divisors modulo numerical equivalence. The rational Néron–Severi space is denoted by  $N^1(X)_{\mathbb{Q}}$ , and similarly for divisor classes with real coefficients. The notations  $\text{Nef}(X)$  and  $\text{Big}(X)$  stand for the convex cones in  $N^1(X)_{\mathbb{R}}$  generated by the classes of nef, and big divisors, respectively.

Once past the definition of asymptotic cohomological functions, we will present the asymptotic counterparts of some important properties of coherent cohomology (e.g. Serre vanishing, Serre duality, Künneth formula), and describe the behaviour of asymptotic cohomological functions with respect to pullbacks. The section ends with a generalization of asymptotic Serre vanishing to big divisors.

**Definition 2.1** (Asymptotic cohomological functions). Let  $X$  be an irreducible projective variety of dimension  $n$ ,  $D$  a Cartier divisor on  $X$ . The value of the  $i$ th asymptotic cohomological function associated to  $X$  at  $D$  is defined to be

$$\widehat{h}^i(X, \mathcal{O}_X(D)) \stackrel{\text{def}}{=} \limsup_m \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

The case  $i = 0$  is the volume of the divisor  $D$ , and has been studied in detail in [1],[8], [19, Section 2.2.C], Note that it is established in [19, Section 11.4.A], that

$$\widehat{h}^0(X, \mathcal{O}_X(D)) = \lim_m \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

It is not known if the same holds for  $i \geq 1$ , except in special cases.

**Remark 2.2.** Let  $X$  be as above,  $\mathcal{F}$  a coherent sheaf,  $D$  a Cartier divisor on  $X$ . Then there exists a constant  $C$  depending on  $X, D$  and  $\mathcal{F}$  only, such that  $h^i(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) \leq Cm^n$ . If in addition  $D$  is nef then  $h^i(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) \leq Cm^{n-1}$  for all  $i \geq 1$ .

Consequently, on the one hand, the value of asymptotic cohomological functions is always finite. On the other hand, if  $D$  is nef, then

$$\widehat{h}^i(X, \mathcal{O}_X(D)) = 0$$

for all  $i \geq 1$ . This latter statement can be considered as an asymptotic version of Serre vanishing. In this case, the asymptotic Riemann–Roch theorem implies that for a nef divisor  $D$ ,

$$\widehat{h}^0(X, \mathcal{O}_X(D)) = (D^n).$$

(For proofs of the cited statements see [19, Section 1.2.B.], or [18]).

**Example 2.3** (Asymptotic cohomological functions on curves). As a first example, consider the case of curves. Let  $C$  be an irreducible projective curve,  $L$  a line bundle on  $C$ . By the asymptotic Riemann–Roch theorem

$$h^0(X, L^{\otimes m}) - h^1(X, L^{\otimes m}) = m \cdot \deg_C L + O(1).$$

As  $L$  is ample if and only if  $\deg_C(L) > 0$ , we obtain that

$$\widehat{h}^0(X, L) = \begin{cases} \deg_C(L) & \text{if } \deg_C(L) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$\widehat{h}^1(X, L) = \begin{cases} 0 & \text{if } \deg_C(L) \geq 0 \\ \deg_C(L) & \text{otherwise.} \end{cases}$$

**Example 2.4** (Weak asymptotic holomorphic Morse inequalities). In [6, 7], Demailly established a set of inequalities for the dimensions of the cohomology groups of the difference of two nef divisors on smooth varieties. These so-called holomorphic Morse inequalities can be quickly extended to arbitrary projective varieties by appealing to resolution of singularities.

With notation as so far, let  $D = F - G$  be a difference of two nef Cartier divisors. Then Demailly's weak holomorphic Morse inequality states that

$$h^i(X, \mathcal{O}_X(mD)) \leq m^n \frac{F^{n-i} \cdot G^i}{(n-i)! i!} + o(m^n),$$

hence

$$\widehat{h}^i(X, \mathcal{O}_X(D)) \leq \binom{n}{i} F^{n-i} \cdot G^i.$$

Our next aim is to see how asymptotic cohomological functions behave under 'infinitesimal' perturbations. Fix an arbitrary Cartier divisor  $D$  and a coherent sheaf  $\mathcal{F}$  on  $X$ . Instead of considering the sequence  $h^i(X, \mathcal{O}_X(mD))$  for  $m \geq 1$ , we consider  $h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})$ , and ask what its properly normalized upper limit is.

**Proposition 2.5** (Invariance under infinitesimal perturbations). *Let  $X$  be an  $n$ -dimensional irreducible projective variety,  $\mathcal{F}$  a coherent sheaf,  $D$  a divisor on  $X$ . Then*

$$\limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})}{m^n/n!} = \text{rank } \mathcal{F} \cdot \widehat{h}^i(X, \mathcal{O}_X(D)).$$

*Proof.* We start with the case when  $\mathcal{F} = \mathcal{O}_X(N)$  is an invertible sheaf associated to the Cartier divisor  $N$ . Let us write  $N$  as the difference of two effective divisors  $N = E - F$ , which do not contain  $D$ . This way, we obtain the following short exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(mD + (E - F)) \rightarrow \mathcal{O}_X(mD + E) \rightarrow \mathcal{O}_F(mD + E) \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD + E) \rightarrow \mathcal{O}_E(mD + E) \rightarrow 0. \end{aligned}$$

By looking at the corresponding long exact sequences, we see that

$$|h^i(X, mD + (E - F)) - h^i(X, mD + E)| \leq \max\{h^{i-1}(F, mD + E), h^i(F, mD + E)\}$$

and

$$|h^i(X, mD + E) - h^i(X, mD)| \leq \max\{h^{i-1}(E, mD + E), h^i(E, mD + E)\}.$$

Observe that by Proposition 2.2 the right-hand side terms in both inequalities are at most  $C' \cdot m^{n-1}$  for some positive constant  $C'$  independent of  $m$ . By the triangle inequality we obtain

$$|h^i(X, mD + N) - h^i(X, mD)| \leq C \cdot m^{n-1}$$

for some constant  $C$ . After dividing both sides by  $m^n/n!$  and taking limsups, we obtain by Lemma 6.1 that

$$\limsup_m \frac{h^i(X, mD + N)}{m^n/n!} = \widehat{h}^i(X, D).$$

Next, consider a direct sum of invertible sheaves  $\mathcal{F} = L_1 \oplus \dots \oplus L_r$ . Then

$$h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F}) = \sum_{j=1}^r h^i(X, \mathcal{O}_X(mD) \otimes L_j),$$

therefore,

$$\begin{aligned} & \left| h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F}) - r \cdot h^i(X, \mathcal{O}_X(mD)) \right| \\ & \leq \sum_{j=1}^r \left| h^i(X, \mathcal{O}_X(mD) \otimes L_j) - h^i(X, \mathcal{O}_X(mD)) \right|, \end{aligned}$$

and consequently,

$$\begin{aligned} & \left| \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})}{m^n/n!} - r \cdot \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!} \right| \\ & \leq \sum_{j=1}^r \left| \frac{h^i(X, \mathcal{O}_X(mD) \otimes L_j)}{m^n/n!} - \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!} \right|. \end{aligned}$$

By the case of line bundles, the right-hand side of the previous inequality converges to zero, hence by (3) of Lemma 6.1 we can conclude that

$$\limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})}{m^n/n!} = \limsup_m r \cdot \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!} = \text{rank } \mathcal{F} \cdot \widehat{h}^i(X, \mathcal{O}_X(D)).$$

For the general case, observe that for every coherent sheaf  $\mathcal{F}$  of rank  $r$  on  $X$  there exists a map  $\varphi$ , line bundles  $L_1, \dots, L_r$ , and an exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{K} \rightarrow \bigoplus_{j=1}^r L_j \xrightarrow{\varphi} \mathcal{F} \rightarrow \mathcal{C} \rightarrow 0,$$

where  $\dim \text{supp } \mathcal{K} \leq n-1$  and  $\dim \text{supp } \mathcal{C} \leq n-1$ . Split up this sequence into short exact sequences:

$$0 \rightarrow \mathcal{K} \rightarrow \bigoplus_{j=1}^r L_j \rightarrow \mathcal{G} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{C} \rightarrow 0,$$

where  $\mathcal{G} = \text{Ker}(\mathcal{F} \rightarrow \mathcal{C})$ . It follows that the sequences

$$0 \rightarrow \mathcal{O}_X(mD) \otimes \mathcal{K} \rightarrow \bigoplus_{j=1}^r \mathcal{O}_X(mD) \otimes L_j \rightarrow \mathcal{O}_X(mD) \otimes \mathcal{G} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_X(mD) \otimes \mathcal{G} \rightarrow \mathcal{O}_X(mD) \otimes \mathcal{F} \rightarrow \mathcal{O}_X(mD) \otimes \mathcal{C} \rightarrow 0$$

are then exact. As  $\dim \text{supp } \mathcal{K} \leq n-1$  and  $\dim \text{supp } \mathcal{C} \leq n-1$ , we can apply Lemma 6.3 to both sequences. Hence

$$\limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{G})}{m^n/n!} = \limsup_m \frac{h^i(X, \bigoplus_{j=1}^r \mathcal{O}_X(mD) \otimes L_j)}{m^n/n!},$$

and

$$\limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})}{m^n/n!} = \limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{G})}{m^n/n!}.$$

Consequently,

$$\begin{aligned} \limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})}{m^n/n!} &= \limsup_m \frac{h^i(X, \bigoplus_{j=1}^r \mathcal{O}_X(mD) \otimes L_j)}{m^n/n!} \\ &= \text{rank } \mathcal{F} \cdot \widehat{h}^i(X, \mathcal{O}_X(D)). \end{aligned}$$

□

**Corollary 2.6.** *With notation as above, if  $\dim \operatorname{supp} \mathcal{F} < \dim X$ , then*

$$\limsup_m \frac{h^i(X, \mathcal{O}_X(mD) \otimes \mathcal{F})}{m^n/n!} = 0 .$$

**Proposition 2.7** (Homogeneity of asymptotic cohomological functions). *Let  $X$  be an  $n$ -dimensional irreducible projective variety,  $D$  a divisor on  $X$ ,  $a > 0$  arbitrary integer. Then*

$$\widehat{h}^i(X, aD) = a^n \cdot \widehat{h}^i(X, D)$$

for all  $i \geq 0$ .

*Proof.* Homogeneity is a consequence of the following statement: if

$$\alpha_r^{(i)} \stackrel{\text{def}}{=} \limsup_k \frac{h^i(X, (ak+r)D)}{(ak+r)^n/n!} ,$$

then

$$\alpha_0^{(i)} = \dots = \alpha_{a-1}^{(i)} .$$

Grant this for the moment. Then

$$\widehat{h}^i(X, aD) = \limsup_k \frac{h^i(X, akD)}{k^n/n!} = a^n \limsup_k \frac{h^i(X, akD)}{(ak)^n/n!} = a^n \widehat{h}^i(X, D) ,$$

as we wanted to show.

We are left with proving that  $\alpha_0^{(i)} = \dots = \alpha_{a-1}^{(i)}$ . But this follows immediately from Proposition 2.5 applied with  $N = 2D, \dots, (a-1)D$ .  $\square$

**Remark 2.8.** Homogeneity of asymptotic cohomological functions allows us to extend them to  $\mathbb{Q}$ -divisors. For an arbitrary  $\mathbb{Q}$ -divisor  $D$ , set

$$\widehat{h}^i(X, D) \stackrel{\text{def}}{=} \frac{1}{a^n} \widehat{h}^i(X, aD) ,$$

where  $a$  is a positive integer with  $aD$  integral. It follows from Proposition 2.7 that the right-hand side is independent of the choice of  $a$ .

Our next goal is to describe how asymptotic cohomological functions behave with respect to pull-backs.

**Proposition 2.9** (Asymptotic cohomological functions of pullbacks). *Let  $f : Y \rightarrow X$  be a proper surjective map of irreducible projective varieties with  $\dim X = n$ ,  $D$  a divisor on  $X$ .*

(1) *If  $f$  is generically finite with  $\deg f = d$ , then*

$$\widehat{h}^i(Y, f^* \mathcal{O}_X(D)) = d \cdot \widehat{h}^i(X, \mathcal{O}_X(D)) .$$

(2) *If  $\dim Y > \dim X = n$ , then*

$$\widehat{h}^i(Y, f^* \mathcal{O}_X(D)) = 0$$

for all  $i$ 's.

*Proof.* Both cases follow from an analysis of the Leray spectral sequence

$$E_2^{pq}(m) = H^p(X, R^q f_* (f^* \mathcal{O}_X(mD))) \Rightarrow H^{p+q}(Y, f^* \mathcal{O}_X(mD)) .$$

For (1), we show that

$$h^i(Y, f^* \mathcal{O}_X(mD)) = h^i(X, (f_* \mathcal{O}_Y) \otimes \mathcal{O}_X(mD)) + C \cdot m^{n-1}$$

for a constant  $C$  depending on  $X, Y, f$  and  $D$ . The projection formula for direct images implies that

$$E_2^{pq}(m) = H^p(X, R^q f_*(f^* \mathcal{O}_X(mD))) \simeq H^p(X, R^q f_* \mathcal{O}_Y \otimes \mathcal{O}_X(mD))$$

for all  $p, q \geq 0$ . As  $f$  is generically finite, the higher direct image sheaves  $R^q f_* \mathcal{O}_Y$  are supported on proper subschemes of  $X$ . Hence by Corollary 2.6

$$h^p(X, R^q f_* \mathcal{O}_Y \otimes \mathcal{O}_X(mD)) \leq C \cdot m^{n-1}$$

for some constant  $C$  (not depending on  $m$ ) for all  $q$ 's except  $q = 0$ . Therefore, for every  $r \geq 2$  and every diagonal  $p + q = i$  all terms  $E_r^{pq}(m)$  — except possibly one — will grow in terms of  $m$  at most as  $C \cdot m^{n-1}$ , and the only possible exception will be of the form

$$h^i(X, f_* \mathcal{O}_Y \otimes \mathcal{O}_X(mD)) + O(m^{n-1}).$$

Hence

$$h^i(Y, f^* \mathcal{O}_X(mD)) = h^i(X, f_* \mathcal{O}_Y \otimes \mathcal{O}_X(mD)) + O(m^{n-1})$$

as we wanted.

As  $f$  is generically finite of degree  $d$ ,  $\text{rank } f_* \mathcal{O}_Y = d$ , therefore Proposition 2.5 implies

$$\widehat{h}^i(Y, f^* \mathcal{O}_X(D)) = \limsup_m \frac{h^i(X, f_* \mathcal{O}_Y \otimes \mathcal{O}_X(mD))}{m^n/n!} = d \cdot \widehat{h}^i(X, \mathcal{O}_X(D)).$$

For the case (2), consider again the Leray spectral sequence. We can see by induction on  $r$  that  $\dim E_r^{pq,(m)} \leq C_r \cdot m^n$  for constants  $C_r$  independent of  $m$ . Therefore,

$$h^i(Y, f^* \mathcal{O}_X(mD)) \leq n \cdot C_n \cdot m^n,$$

from which the proposition follows as  $\dim Y > \dim X = n$ .  $\square$

**Corollary 2.10** (Birational invariance of asymptotic cohomological functions). *Let  $f : Y \rightarrow X$  be a proper surjective birational map of irreducible projective varieties of dimension  $n$ , let  $D$  be a divisor on  $X$ . Then*

$$\widehat{h}^i(Y, f^* \mathcal{O}_X(D)) = \widehat{h}^i(X, \mathcal{O}_X(D))$$

for all  $i \geq 0$ .

**Corollary 2.11** (Asymptotic Serre duality). *Let  $X$  be an irreducible projective variety of dimension  $n$ ,  $D$  a divisor on  $X$ . Then for every  $0 \leq i \leq n$*

$$\widehat{h}^i(X, D) = \widehat{h}^{n-i}(X, -D).$$

*Proof.* Let  $f : Y \rightarrow X$  a resolution of singularities of  $X$ . Then by Corollary 2.10 we have

$$\widehat{h}^i(Y, f^* \mathcal{O}_X(D)) = \widehat{h}^i(X, \mathcal{O}_X(D)),$$

and

$$\widehat{h}^{n-i}(Y, f^* \mathcal{O}_X(-D)) = \widehat{h}^{n-i}(X, f^* \mathcal{O}_X(-D)).$$

Serre duality on the smooth variety  $Y$  gives

$$h^i(Y, f^* \mathcal{O}_X(mD)) = h^{n-i}(Y, K_Y \otimes f^* \mathcal{O}_X(-mD))$$

for every  $m \geq 1$ . By Proposition 2.5

$$\limsup_m \frac{h^{n-i}(Y, K_Y \otimes f^* \mathcal{O}_X(-mD))}{m^n/n!} = \widehat{h}^{n-i}(Y, f^* \mathcal{O}_X(-D)),$$

therefore,

$$\widehat{h}^i(Y, f^* \mathcal{O}_X(D)) = \widehat{h}^{n-i}(Y, f^* \mathcal{O}_X(-D)).$$

This implies

$$\widehat{h}^i(X, \mathcal{O}_X(D)) = \widehat{h}^{n-i}(X, \mathcal{O}_X(-D)) .$$

□

**Remark 2.12.** By homogeneity, the previous version of Serre duality remains valid for  $\mathbb{Q}$ -divisors. Also, once having proven the continuity of asymptotic cohomological functions on  $N^1(X)_{\mathbb{R}}$  in Corollary 5.3, we obtain

$$\widehat{h}^i(X, \xi) = \widehat{h}^{n-i}(X, -\xi)$$

for every  $\xi \in N^1(X)_{\mathbb{R}}$  and every  $0 \leq i \leq n$ .

**Corollary 2.13.** *With notation as above, let  $D$  be a big  $\mathbb{Q}$ -divisor on  $X$ . Then*

$$\widehat{h}^n(X, D) = 0 .$$

*Proof.* As  $D$  is big,  $-D$  is not, therefore  $\widehat{h}^0(X, -D) = 0$ . Then the corollary follows by the asymptotic version of Serre duality. □

**Remark 2.14** (Künneth formulas for asymptotic cohomological functions). Let  $X_1, X_2$  be irreducible projective varieties of dimensions  $n_1$  and  $n_2$ ,  $D_1, D_2$  Cartier divisors on  $X_1$  and  $X_2$ , respectively. Then

$$h^i(X_1 \times X_2, \pi_1^* D_1 \otimes \pi_2^* D_2) \leq \binom{n_1 + n_2}{n_1} \sum_{i=j+k} \widehat{h}^j(X_1, D_1) \cdot \widehat{h}^k(X_2, D_2) .$$

for all  $i$ 's, where  $\pi_l$  denotes the projection map to  $X_l$  ( $l = 1, 2$ ). Furthermore, we have equality in the case  $i = 0$ , which follows from the observation that the lim sup in the definition of  $\widehat{h}^0$  is a limit ([19, Section 11.4.A.]).

In the remainder of this section we discuss a connection between asymptotic cohomological functions and stable base loci of divisors. As a motivating example, consider a smooth projective surface  $X$  and a big divisor  $D$  on  $X$ .

In Section 3, we will prove the following fact: if  $D$  is a big divisor on a smooth surface  $X$  with Zariski decomposition  $D = P_D + N_D$ , then

$$\widehat{h}^i(X, D) = \begin{cases} (P_D^2) & \text{if } i = 0 \\ -(N_D^2) & \text{if } i = 1 \\ 0 & \text{if } i = 2 . \end{cases}$$

As the intersection matrix of  $N_D$  is negative definite,  $(N_D^2) = 0$  if and only if  $N_D = 0$ . On the other hand, as pointed out in [9],  $\text{supp } N \subseteq \text{B}(D)$ . Therefore, if  $i > \dim \text{B}(D)$ , the dimension of the stable base locus of  $D$ , then  $\widehat{h}^i(X, D) = 0$ . It turns out, that this phenomenon persists on varieties of higher dimension as well.

**Proposition 2.15** (Stable base loci and vanishing of asymptotic cohomological functions). *Let  $D$  be a big divisor on a smooth variety  $X$ , and assume that the dimension of the stable base locus of  $D$  is  $d$ . Then*

$$\widehat{h}^i(X, \mathcal{O}_X(D)) = 0$$

for all  $i > d$ .

*Proof.* Let  $Z_m \subseteq X$  denote the subscheme defined by the asymptotic multiplier ideal sheaf  $\mathcal{J}(\|mD\|)$ . Then  $\dim Z_m \leq d$  for  $m$  large. Consider the exact sequence

$$0 \rightarrow \mathcal{J}(\|mD\|) \otimes \mathcal{O}_X(K_X + mD) \rightarrow \mathcal{O}_X(K_X + mD) \rightarrow \mathcal{O}_{Z_m}(K_X + mD) \rightarrow 0 .$$

A form of Nadel vanishing for asymptotic multiplier ideals ([19], Section 11.2.B) says that for  $i > 0$

$$H^i(X, \mathcal{J}(\|mD\|) \otimes \mathcal{O}_X(K_X + mD)) = 0 .$$

Therefore, if  $i > 0$  then

$$H^i(X, \mathcal{O}_X(K_X + mD)) = H^i(X, \mathcal{O}_{Z_m}(K_X + mD)) .$$

But  $H^i(X, \mathcal{O}_{Z_m}(K_X + mD)) = 0$  for  $m \gg 0$ , as  $Z_m$  is a  $d$ -dimensional scheme in this case. Hence

$$h^i(X, \mathcal{O}_X(K_X + mD)) = 0$$

for  $m \gg 0$ . By Proposition 2.5, this implies

$$\widehat{h}^i(X, \mathcal{O}_X(D)) = \limsup_m \frac{h^i(X, \mathcal{O}_X(K_X + mD))}{m^n/n!} = 0 ,$$

as required.  $\square$

### 3. EXAMPLES

We aim to provide a pool of examples where asymptotic cohomological functions are worked out in detail. The ones presented here give evidence of interesting structures in the Néron–Severi space arising from the functions  $\widehat{h}^i$ . Some of our examples — abelian varieties and generalized flag varieties — are classical in the sense that cohomology groups of line bundles on them have been described many years ago.

In the case of our other source of examples, smooth surfaces, the computation of asymptotic cohomological functions does not rely on information about individual cohomology groups of line bundles, but geometric data in the form of Zariski decompositions. The exposition here draws heavily on [1].

Although we do not discuss it here, toric varieties form another class where asymptotic cohomological functions can be computed explicitly. In addition, they provide interesting combinatorial information (see [13]).

**Remark 3.1.** Although the examples we cover here might suggest that the asymptotic cohomological functions are piecewise polynomial, this is not the case in general. For a counterexample, we refer the reader to [1].

**3.1. Abelian varieties.** Let  $X$  be a  $g$ -dimensional complex abelian variety, expressed as a quotient of a  $g$ -dimensional complex vector space  $V$  by a lattice  $L \subseteq V$ . Line bundles on  $X$  are given in terms of Appel–Humbert data, that is, pairs  $(\alpha, H)$ , where  $H$  is a Hermitian form on  $V$  such that its imaginary part  $E$  is integral on  $L \times L$ , and

$$\begin{aligned} \alpha & : L \rightarrow U(1) \text{ is a function for which} \\ \alpha(l_1 + l_2) & = \alpha(l_1) \cdot \alpha(l_2) \cdot (-1)^{E(l_1, l_2)} . \end{aligned}$$

By the Appel–Humbert theorem any pair  $(\alpha, H)$  determines a unique line bundle  $\mathcal{L}(\alpha, H)$  on  $X$ , and every line bundle on  $X$  is isomorphic to one of the form  $\mathcal{L}(\alpha, H)$  for some  $(\alpha, H)$ . The Hermitian form  $H$  on  $V$  is the invariant two-form associated to  $c_1(\mathcal{L}(\alpha, H))$ .

Fix a line bundle  $\mathcal{L} = \mathcal{L}(\alpha, H)$ . It is called *nondegenerate*, if 0 is not an eigenvalue of  $H$ . For a nondegenerate line bundle  $\mathcal{L}$ , the number of negative eigenvalues of  $H$  is referred

to as the *index* of  $\mathcal{L}$ , which we denote by  $\text{ind } \mathcal{L}$ . The case of a positive definite matrix  $H$  corresponds to the line bundle  $\mathcal{L}(\alpha, H)$  being ample.

**Theorem 3.2** (Mumford's index theorem). [16] *With notation as above, let  $\mathcal{L}$  be a nondegenerate line bundle on  $X$ . Then*

$$h^i(X, \mathcal{L}) = \begin{cases} (-1)^i \chi(\mathcal{L}) = \sqrt{\det_L E} & \text{if } i = \text{ind}(\mathcal{L}) \\ 0 & \text{otherwise.} \end{cases}$$

The first Chern class of line bundles is additive, therefore  $\text{ind}(\mathcal{L}^{\otimes m}) = \text{ind}(\mathcal{L})$  for all  $m \geq 1$ .

Consider  $\text{NonDeg}(X) \subseteq \text{Pic}(X)_{\mathbb{R}}$ , the cone generated by all nondegenerate line bundles. Then  $\text{NonDeg}(X)$  is an open cone in  $\text{Pic}(X)_{\mathbb{R}}$ , and its complement has Lebesgue measure zero. For every  $1 \leq j \leq g$ , define  $\mathcal{C}_j$  to be the cone spanned by nondegenerate line bundles of index  $j$ . Then (apart from the origin)  $\text{NonDeg}(X)$  is the disjoint union of  $\mathcal{C}_1, \dots, \mathcal{C}_g$ . On each  $\mathcal{C}_j$ , we have

$$\widehat{h}^i(X, \xi) = \begin{cases} (-1)^i (\xi^n) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This way, we obtain a finite decomposition of  $N^1(X)_{\mathbb{R}}$  into a set of cones, such that on each cone, the asymptotic cohomological functions are homogeneous polynomials of degree  $g$ .

**Example 3.3.** We will consider the following example in more detail. Consider  $X \stackrel{\text{def}}{=} E \times E$ , the product of an elliptic curve with itself. Fix a point  $P \in E$ . Then the three classes

$$e_1 = [\{P\} \times E], \quad e_2 = [E \times \{P\}], \quad \delta = [\Delta]$$

in  $N^1(X)_{\mathbb{R}}$  are independent ( $\Delta \subseteq E \times E$  is the diagonal) and generate  $N^1(X)_{\mathbb{R}}$ . The various intersection numbers among them are as follows:

$$\delta \cdot e_1 = \delta \cdot e_2 = e_1 \cdot e_2 = 1 \quad \text{and} \quad e_1^2 = e_2^2 = \delta^2 = 0.$$

Any effective curve on  $X$  is nef,  $\overline{\text{NE}}(X) = \text{Nef}(X)$ , furthermore, a class  $\alpha \in N^1(X)_{\mathbb{R}}$  is nef if and only if  $\alpha^2 \geq 0$  and  $\alpha \cdot h \geq 0$  for some (any) ample class  $h$ .

In particular, if  $\alpha = x \cdot e_1 + y \cdot e_2 + z \cdot \delta$  then  $\alpha$  is nef if and only if

$$xy + xz + yz \geq 0 \quad \text{and} \quad x + y + z \geq 0.$$

As a reference for these statements see [19], Section 1.5.B. One can see that the  $\text{Nef}(X)$  is a circular cone inside  $N^1(X)_{\mathbb{R}} \simeq \mathbb{R}^3$ . By continuity, define the index of a real divisor class  $\alpha \in \text{NonDeg}(X)$  to be 0 if it is ample, 2 if  $-\alpha$  is ample, and 1 otherwise. Then

$$\widehat{h}^i(X, \alpha) = \begin{cases} (-1)^{\text{ind}(\alpha)} (\alpha^2) = (-1)^{\text{ind}(\alpha)} (xy + xz + yz) & \text{if } i = \text{ind}(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

**3.2. Smooth surfaces.** For a smooth projective surface  $X$  over  $\mathbb{C}$ , we exhibit a locally finite decomposition of  $\text{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$  into locally polyhedral cones, such that on each chamber of the decomposition, the functions  $\widehat{h}^i$  are given by homogeneous quadratic polynomials coming from intersection numbers of divisors on  $X$ .

The discussion is based on [1], where the authors work out theory of the volume function for surfaces, however, the issue of asymptotic cohomological functions was not raised. Here we treat the general case building on their results. Also, we will make use of the continuity of asymptotic cohomological functions, which we prove in Section 5.

First recall some important facts about our main tool, Zariski decompositions.

**Theorem 3.4** (Existence and uniqueness of Zariski decompositions for  $\mathbb{R}$ -divisors, [15], Theorem 7.3.1). *Let  $D$  be a pseudo-effective  $\mathbb{R}$ -divisor on a smooth projective surface. Then there exists a unique effective  $\mathbb{R}$ -divisor*

$$N_D = \sum_{i=1}^m a_i N_i$$

such that

- (i)  $P_D = D - N_D$  is nef,
- (ii)  $N_D$  is either zero or its intersection matrix  $(N_i \cdot N_j)$  is negative definite,
- (iii)  $P_D \cdot N_i = 0$  for  $i = 1, \dots, m$ .

Furthermore,  $N_D$  is uniquely determined by the numerical equivalence class of  $D$ , and if  $D$  is a  $\mathbb{Q}$ -divisor, then so are  $P_D$  and  $N_D$ . The decomposition

$$D = P_D + N_D$$

is called the Zariski decomposition of  $D$ .

The connection between Zariski decompositions and asymptotic cohomological functions comes from the following result.

**Proposition 3.5** (Section 2.3.C., [19]). *Let  $D$  be a big integral divisor,  $D = P_D + N_D$  the Zariski decomposition of  $D$ . Then*

- (i)  $H^0(X, kD) = H^0(X, kP_D)$  for all  $k \geq 1$  such that  $kP_D$  is integral, and
- (ii)  $\text{vol}(D) = \text{vol}(P_D) = (P_D^2)$ .

By homogeneity and continuity of the volume we obtain that for an arbitrary big  $\mathbb{R}$ -divisor  $D$  with Zariski decomposition  $D = P_D + N_D$  we have  $\text{vol}(D) = (P_D^2) = (D - N_D)^2$ .

Let  $D$  be an  $\mathbb{R}$ -divisor on  $X$ . In determining the asymptotic cohomological functions on  $X$ , we distinguish three cases, according to whether  $D$  is pseudo-effective,  $-D$  is pseudo-effective or none.

**Proposition 3.6.** *With notation as above, if  $D$  is pseudo-effective then*

$$\widehat{h}^i(X, D) = \begin{cases} (P_D^2) & \text{if } i = 0 \\ -(N_D^2) & \text{if } i = 1 \\ 0 & \text{if } i = 2. \end{cases}$$

*Proof.* If  $D = P_D + N_D$  is the Zariski decomposition of the pseudo-effective divisor  $D$ , then  $\widehat{h}^0(X, D) = (P_D^2)$ . Furthermore, if  $D$  is pseudo-effective then by Corollary 2.13 and the continuity of  $\widehat{h}^2$ ,  $\widehat{h}^2(X, D) = 0$ . In order to compute  $\widehat{h}^1$ , consider the equality

$$h^1(X, mD) = h^0(X, mD) + h^2(X, mD) - \chi(X, mD).$$

This implies that

$$\widehat{h}^1(X, D) = \limsup_m \left( \frac{h^0(X, mD)}{m^2/2} + \frac{h^2(X, mD)}{m^2/2} - \frac{\chi(X, mD)}{m^2/2} \right).$$

All three sequences on the right-hand side are convergent. The  $h^0$  sequence by the fact that the volume function is in general a limit. The  $h^2$  sequence converges by  $\widehat{h}^2(X, D) = 0$ . Finally, the convergence of the sequence of Euler characteristics follows from the Asymptotic Riemann–Roch theorem. Therefore the lim sup on the right-hand side is a limit, and  $\widehat{h}^1(X, D) = -(N_D^2)$ .  $\square$

**Corollary 3.7.** *If  $-D$  is pseudo-effective with Zariski decomposition  $-D = P_D + N_D$  then*

$$\widehat{h}^i(X, D) = \begin{cases} 0 & \text{if } i = 0 \\ -(N_{-D}^2) & \text{if } i = 1 \\ (P_{-D}^2) & \text{if } i = 2 . \end{cases}$$

*When neither  $D$  nor  $-D$  are pseudo-effective, one has*

$$\widehat{h}^i(X, D) = \begin{cases} 0 & \text{if } i = 0 \\ -(D^2) & \text{if } i = 1 \\ 0 & \text{if } i = 2 . \end{cases}$$

As in [1], with a careful examination of the variation of Zariski decompositions, one can give a geometric description of the volume, and hence all asymptotic cohomological functions on the big cone of a smooth surface. The main result is the following.

**Theorem 3.8.** *With notation as above, there exists a locally finite decomposition of  $\text{Big}(X)$  into rational locally polyhedral subcones such that on each of those the asymptotic cohomological functions are given by a single homogeneous quadratic polynomial.*

*Proof.* Follows from our description of asymptotic cohomological functions on smooth projective surfaces, and the main theorem of [1].  $\square$

In some cases, the locally finite chamber structure will turn out to be finite polyhedral.

**Proposition 3.9.** *Let  $X$  be a del Pezzo surface. Then there exists a finite decomposition of  $N^1(X)_{\mathbb{R}}$  into rational polyhedral cones such that on each of these cones all asymptotic cohomological functions are given by homogeneous quadratic polynomials.*

*Proof.* The statement is proved by considering the effective cone, its negative and the remaining part separately. Of these three the first two are convex rational polyhedral cones, the third one is not convex, nevertheless its finitely many boundary components are still rational polyhedral. By Proposition 3.6 and [1, Proposition 3.4.], the statement of the corollary holds for all asymptotic cohomological functions on the effective cone. Analogously, Corollary 3.7 implies the same on the negative of the effective cone, and for all classes  $\alpha$  where neither  $\alpha$  nor  $-\alpha$  is effective.  $\square$

**3.3. Generalized flag varieties.** Let  $G$  denote a simply-connected semisimple complex Lie group,  $B \subseteq G$  a Borel subgroup,  $\Delta, \Delta_+$  the set of roots and positive roots, respectively. The factor space  $X = G/B$  is equipped with the structure of an irreducible projective variety over  $\mathbb{C}$ , and there is a natural isomorphism

$$\Lambda_W \simeq \text{Pic}(G/B)$$

given by  $\lambda \mapsto L_\lambda = G \times_B \mathbb{C}_\lambda$ , where  $\Lambda_W$  is the associated weight lattice (cf. [10], Section 23.3.). Set

$$\rho \stackrel{\text{def}}{=} \frac{1}{2} \sum_{v \in \Delta_+} v .$$

The computation of the cohomology groups of the line bundles  $L_\lambda$  is a celebrated result of Borel–Weil and Bott.

**Theorem 3.10** (Borel–Weil–Bott, [3]). *For a given weight  $\lambda$  the following (mutually exclusive) situations can happen.*

(1)  $\lambda + \rho$  is on the boundary of a fundamental chamber of  $W$  in  $\Lambda_W$ . Then

$$H^i(G/B, L_\lambda) = 0$$

for all  $0 \leq i \leq \dim G/B$ .

(2)  $\lambda + \rho$  is in the interior of a chamber. Then there is a unique element  $w \in W$  such that  $w(\lambda + \rho)$  is in the positive chamber. In this case

$$H^i(G/B, L_\lambda) = \begin{cases} H^0(G/B, L_{w(\lambda+\rho)}) & \text{if } i = \text{ind}(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Here  $\text{ind}(\lambda)$  is the number of positive roots  $v$  such that  $(\lambda + \rho, v) < 0$  with respect to the Killing form. This gives rise to a decomposition of  $N^1(G/B)_\mathbb{R}$  into finitely many open polyhedral chambers, on each of which every  $\widehat{h}^i(G/B, L)$  is given by a single homogeneous polynomial. Let us describe this chamber structure in more detail.

To every  $v \in \Delta_+$  we attach the the half-spaces

$$H_{v,\rho}^\pm = \{\lambda \in \Lambda_W | (\lambda + \rho, v) \gtrless 0\} \text{ and } H_v^\pm = \{\alpha \in N^1(G/B)_\mathbb{R} | (\alpha, v) \gtrless 0\} .$$

For every choice of signs, the intersection  $H_{v_1,\rho}^\pm \cap \dots \cap H_{v_n,\rho}^\pm$  — where  $v_1, \dots, v_n$  are *all* the positive roots — is either empty or an open polyhedral cone, we will also refer to them as cohomology chambers.

On such a chamber  $\text{ind}(\lambda)$  is constant, hence by the Borel–Weil–Bott theorem

$$h^i(G/B, L_\lambda) = \begin{cases} h^0(G/B, L_{w(\lambda+\rho)}) & \text{if } i = \text{ind}(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

If  $I \subseteq \Delta_+$  is an arbitrary subset of positive roots, define  $C_I$  to be the set of weights  $\lambda \in \Lambda_W$  for which the sequences  $(m\lambda + \rho, v)$  are eventually positive (for  $m \gg 0$ ) if and only if  $v \in I$ . Then

$$C_I = \Lambda_W \cap \bigcap_{v \in I} H_v^+ \cap \bigcap_{v \notin I} H_v^- .$$

**Corollary 3.11.** *With notation as above*

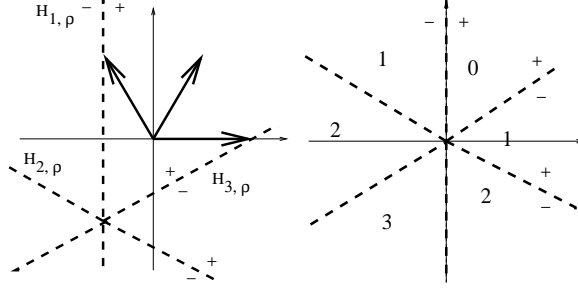
$$\widehat{h}^i(G/B, \alpha) = \begin{cases} (-1)^i (\alpha^n) & \text{if } i = \text{ind}(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

where  $\alpha \in N^1(X)_\mathbb{R}$  and  $\text{ind}(\alpha)$  is defined to be the number of positive roots  $v$  for which  $(\alpha, v) > 0$  holds.

The hyperplanes  $H_v$  (where  $v$  runs through  $\Delta_+$ ) determine a finite rational polyhedral decomposition of  $N^1(X)_\mathbb{R}$ , such that on each piece, the asymptotic cohomological functions are given by homogeneous polynomials. The self-intersection  $(\alpha)^n$  can be determined from the Weyl dimension formula (see [10], Section 24.1).

Let us illustrate the previous discussion on a concrete example.

**Example 3.12.** Let  $G = SL(3, \mathbb{C})$ . Then the upper triangular matrices in  $G$  form a Borel subgroup  $B$ . Consider the root system  $A_2$  attached to  $SL(3, \mathbb{C})$ , let us denote the three positive roots by  $v_1, v_2, v_3$ . Assuming that the root vectors have unit length, they will be  $v_1 = (1, 0), v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), v_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ . The cohomology chambers and the asymptotic cohomology chambers (in that order) look as follows.



For every chamber for the asymptotic cohomological functions, the number inscribed in it denotes the index of real divisor classes in the chamber in question.

#### 4. COHOMOLOGICAL ESTIMATES

The content of this section is to establish the main technical tool in the proof of the continuity of asymptotic cohomological functions, the cohomological estimates based on the Mayer–Vietoris-type exact sequence of sheaves

$$(1) \quad 0 \rightarrow \mathcal{O}_X \left( D - \sum_{j=1}^m A_j \right) \rightarrow \mathcal{O}_X(D) \rightarrow \bigoplus_{1 \leq i \leq m} \mathcal{O}_{A_i}(D) \rightarrow \\ \bigoplus_{1 \leq i_1 < i_2 \leq m} \mathcal{O}_{A_{i_1} \cap A_{i_2}}(D) \rightarrow \cdots \rightarrow \bigoplus_{1 \leq i_1 < \cdots < i_n \leq m} \mathcal{O}_{A_{i_1} \cap \cdots \cap A_{i_n}}(D) \rightarrow 0,$$

where  $D$  is an arbitrary integral divisor,  $A, A_1, \dots, A_m$  are general very ample divisors. The exactness of this sequence is established in Corollary 4.2. The cohomological estimates obtained via this sequence will have a predominant role in the proof of the continuity of asymptotic cohomological functions.

We start out by establishing a local version of the sequence (1) under suitable general position hypotheses.

**Lemma 4.1.** *Let  $R$  be a noetherian local ring,  $n$  a nonnegative integer,  $f_1, \dots, f_m \in R$  elements such that*

- (1) *any  $n$  element subset of  $f_1, \dots, f_m$  forms a regular sequence in  $R$ ,*
- (2) *any  $n + 1$  elements from  $f_1, \dots, f_m$  generate  $R$ .*

*If  $m < n$  then the complex*

$$0 \rightarrow (f_1 \cdots f_m) \rightarrow R \rightarrow \bigoplus_{1 \leq i \leq m} R/(f_i) \rightarrow \cdots \rightarrow R/(f_1, \dots, f_m) \rightarrow 0$$

*is exact. If  $m \geq n$  then*

$$0 \rightarrow (f_1 \cdots f_m) \rightarrow R \rightarrow \bigoplus_{1 \leq i \leq m} R/(f_i) \rightarrow \cdots \rightarrow \bigoplus_{1 \leq i_1 < \cdots < i_n \leq m} R/(f_{i_1}, \dots, f_{i_n}) \rightarrow 0$$

*is exact.*

As this lemma is a dual version of Theorem 16.5 in [20], we only give an indication of the proof.

*Proof.* For any  $1 \leq i \leq m$  let  $M(i)$  be the complex

$$R \rightarrow R/(f_i) \rightarrow 0$$

with  $R$  in degree 0 and  $R/(f_i)$  in degree 1. Define  $M^{(m)}$  as  $M^{(m)} \stackrel{\text{def}}{=} M(1) \otimes \cdots \otimes M(m)$ . In the case  $m \leq n$ ,  $M^{(m)}$  is equal to

$$R \rightarrow \bigoplus_{1 \leq i \leq m} R/(f_i) \rightarrow \cdots \rightarrow R/(f_1, \dots, f_m) \rightarrow 0,$$

while if  $m > n$  then  $M^{(m)}$  is

$$R \rightarrow \bigoplus_{1 \leq i \leq m} R/(f_i) \rightarrow \cdots \rightarrow \bigoplus_{1 \leq i_1 < \cdots < i_n \leq m} R/(f_{i_1}, \dots, f_{i_n}) \rightarrow 0,$$

since all the quotients by ideals generated by at least  $n+1$  of the  $f_i$ 's are zero by assumption. By induction on  $m$  and  $\dim R$  one then proves that

$$H^i(M^{(m)}) = \begin{cases} (f_1 \cdots f_m) & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

□

**Corollary 4.2.** *Let  $X$  be a pure  $n$ -dimensional scheme of finite type over a field  $k$ ,  $D$  an arbitrary Cartier divisor on  $X$ ,  $A_1, \dots, A_m$  effective Cartier divisors on  $X$ , such that*

- (1) *in the local rings of any point in  $X$ , the local equations of any collection of at most  $n$  elements form a regular sequence*
- (2) *the intersection of any  $n+1$  of the  $A_j$ 's is empty.*

If  $m \leq n$  then the sequence

$$0 \rightarrow \mathcal{O}_X \left( D - \sum_{j=1}^m A_j \right) \rightarrow \mathcal{O}_X(D) \rightarrow \bigoplus_{1 \leq i \leq m} \mathcal{O}_{A_i}(D) \rightarrow \cdots \rightarrow \mathcal{O}_{A_1 \cap \cdots \cap A_m}(D) \rightarrow 0$$

is exact, while if  $m > n$  then

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X \left( D - \sum_{j=1}^m A_j \right) \rightarrow \mathcal{O}_X(D) \rightarrow \bigoplus_{1 \leq i \leq m} \mathcal{O}_{A_i}(D) \rightarrow \\ \rightarrow \bigoplus_{1 \leq i_1 < i_2 \leq m} \mathcal{O}_{A_{i_1} \cap A_{i_2}}(D) \rightarrow \cdots \rightarrow \bigoplus_{1 \leq i_1 < \cdots < i_n \leq m} \mathcal{O}_{A_{i_1} \cap \cdots \cap A_{i_n}}(D) \rightarrow 0 \end{aligned}$$

is exact. Moreover, in (1), it suffices to work with local rings at closed points.

Our goal is to provide a cohomology estimate for certain differences of divisors which will serve as the basis of all further results.

**Lemma 4.3.** *Let  $C. = C_{-1} \rightarrow C_0 \rightarrow C_1 \rightarrow \dots$  be an exact sequence of coherent sheaves on a proper scheme  $X$  over a field. Then*

$$|h^i(X, C_{-1}) - h^i(X, C_0)| \leq 2 \cdot \sum_{k=1}^{i+1} h^{i+1-k}(X, C_k).$$

*Proof.* Break up the exact sequence  $C.$  into a set of short exact sequences

$$0 \rightarrow K_i \rightarrow C_i \rightarrow K_{i+1} \rightarrow 0,$$

where  $0 \leq i$  and  $K_0 = C_{-1}$ . Observe that for every  $r \geq 0$ , a simple induction on  $r$  shows that

$$h^r(X, K_s) \leq \sum_{k=0}^r h^{r-k}(X, C_{s+k}).$$

The statement of the lemma then follows via

$$\begin{aligned}
|h^i(X, C_{-1}) - h^i(X, C_0)| &\leq h^i(X, K_1) + h^{i-1}(X, K_1) \\
&\leq \sum_{k=1}^{i+1} h^{i+1-k}(X, C_k) + \sum_{k=1}^i h^{i-k}(X, C_k) \\
&\leq 2 \cdot \sum_{k=1}^{i+1} h^{i+1-k}(X, C_k) .
\end{aligned}$$

□

**Corollary 4.4** (Basic estimate). *With notation as in Corollary 4.2, for any  $i \geq 0$  and  $m \geq n$*

$$\left| h^i \left( X, D - \sum_{j=1}^m A_j \right) - h^i(X, D) \right| \leq 2 \cdot \sum_{k=1}^{i+1} h^{i+1-k} \left( X, \bigoplus_{1 \leq i_1 < \dots < i_k \leq m} \mathcal{O}_{A_{i_1} \cap \dots \cap A_{i_k}}(D) \right) .$$

*Proof.* Apply Lemma 4.3 to the long exact sequence in Corollary 4.2. □

## 5. CONTINUITY OF ASYMPTOTIC COHOMOLOGICAL FUNCTIONS

The aim of this section is to prove the continuity of asymptotic cohomological functions. We will establish continuity in the following form, which generalizes the result obtained for the volume function in [19], Section 2.2.C. Our proof loosely follows the one given there.

**Theorem 5.1** (Continuity of asymptotic cohomological functions). *Let  $X$  be an irreducible projective variety of dimension  $n$ . Then for all  $0 \leq i \leq n$ ,*

$$\widehat{h}^i : N^1(X)_{\mathbb{Q}} \rightarrow \mathbb{R}^{\geq 0}$$

*defines a continuous function on  $N^1(X)_{\mathbb{Q}}$  which is homogeneous of degree  $n$ , and satisfies the following Lipschitz-type estimate: there exists a constant  $C$  such that for all pairs  $\xi, \eta \in N^1(X)_{\mathbb{Q}}$ , one has*

$$|\widehat{h}^i(X, \xi) - \widehat{h}^i(X, \eta)| \leq C \cdot \sum_{k=1}^n (\max\{\|\xi\|, \|\eta\|\})^{n-k} \cdot \|\xi - \eta\|^k$$

*for some fixed norm  $\|\cdot\|$ .*

**Remark 5.2.** As any two norms on a finite-dimensional real vector space are equivalent, it is indifferent which one we choose. However, the constant  $C$  will certainly depend on it.

**Corollary 5.3.** *With notation as in the Theorem, the asymptotic cohomological functions  $\widehat{h}^i$  extend uniquely to continuous functions*

$$\widehat{h}^i : N^1(X)_{\mathbb{R}} \rightarrow \mathbb{R}^{\geq 0} ,$$

*which are homogeneous of degree  $n$ . Moreover, they satisfy the same Lipschitz-type estimates as on rational classes in the Theorem.*

**Corollary 5.4.** *With notation as in the Theorem, there exists a positive constant  $C$ , such that for any real divisor class  $\xi \in N^1(X)_{\mathbb{R}}$  on  $X$ ,*

$$\widehat{h}^i(X, \xi) \leq C \cdot \|\xi\|^n$$

*for all  $i$ .*

The proof of Theorem 5.1 will come as the conclusion of a series of results, many of which are of somewhat technical nature. Although some of them are probably well-known to experts, we will give proofs in cases when we were not able to find a suitable reference. Some statements which we need in the course of the proofs, but shed no further light to our original problem will be relegated to Section 6.

The cohomological machinery developed in the previous section is only able to deal with pairs  $\xi, \eta$  where

$$\xi = \eta + \alpha,$$

for some ample class  $\alpha \in N^1(X)_{\mathbb{Q}}$ , therefore we need some additional effort to pass to the general case. This will rest upon an observation about normed vector spaces (Proposition 6.4).

We establish the following statements, which, when put together, will convey a proof of Theorem 5.1. As usual,  $X$  will denote an irreducible complex projective variety of dimension  $n$ , all divisors are  $\mathbb{Q}$ -Cartier unless explicitly mentioned otherwise.

**Claim A** (Invariance with respect to numerical equivalence) *Let  $D$  be an arbitrary, and  $P$  a numerically trivial divisor. Then for all  $i$ , one has*

$$\widehat{h}^i(X, D + P) = \widehat{h}^i(X, D) .$$

**Claim B** (Local uniform continuity in ample directions) *Assume that the continuity of asymptotic cohomological functions holds for varieties of dimension less than  $\dim(X)$ . Let  $D$  be an arbitrary,  $A$  an ample divisor on  $X$ . Fix an arbitrary norm on  $N^1(X)_{\mathbb{Q}}$ . Then there exists a constant  $C$  independent of  $D$  for which*

$$\left| \widehat{h}^i(X, D - bA) - \widehat{h}^i(X, D) \right| \leq C \cdot \sum_{k=1}^n \|D\|^{n-k} \cdot b^k \|A\|^k .$$

for every  $b \geq 1$ .

**Claim C** (Formal extension) *Let  $V$  be an  $r$ -dimensional normed rational vector space,  $A_1, \dots, A_r$  a basis for  $V$ ,  $f : V \rightarrow \mathbb{R}^{\geq 0}$  a homogeneous function on  $V$ . Assume furthermore, that for every  $1 \leq i \leq r$  there exists a constant  $C_i$  such that for all  $D \in V$ , and all natural numbers  $b \geq 1$ ,*

$$|f(D - bA_i) - f(D)| \leq C_i \cdot \sum_{k=1}^n \|D\|^{n-k} \cdot b^k .$$

Then there exists a constant  $C > 0$ , such that for every  $D, D' \in V$  one has

$$|f(D) - f(D')| \leq C \cdot \sum_{k=1}^n (\max \{ \|D\|, \|D'\| \})^{n-k} \cdot \|D - D'\|^k .$$

These claims will appear below as Corollary 5.10, Proposition 5.15, Proposition 5.16, and Proposition 6.4, respectively. We now prove Theorem 5.1 based on these statements.

*Proof.* According to Proposition 5.15 (Claim B), asymptotic cohomological functions are invariant with respect to numerical equivalence of divisors, therefore they are well-defined on  $N^1(X)$ . By their homogeneity property — established in Proposition 2.7 — they are also well-defined on  $N^1(X)_{\mathbb{Q}}$ . Also, asymptotic cohomological functions are homogeneous functions of degree  $n = \dim X$  on  $N^1(X)_{\mathbb{Q}}$ .

Observe that asymptotic cohomological functions are continuous on irreducible projective varieties of dimension 0. By induction on the dimension of  $X$ , Proposition 5.16 (Claim C) then implies that, for any  $\mathbb{Q}$ -divisor  $D$  and any ample  $\mathbb{Q}$ -divisor  $A$ ,

$$\left| \widehat{h}^i(X, D - bA) - \widehat{h}^i(X, D) \right| \leq C_A \cdot \sum_{k=1}^n \|D\|^{n-k} \cdot b^k \cdot \|A\|^k$$

for some fixed constant  $C_A$ , which is independent of  $D$  (but possibly depends on  $A$ , and certainly depends on the chosen norm) for all natural numbers  $b \geq 1$ . Let  $A_1, \dots, A_r$  be a basis for  $N^1(X)_{\mathbb{Q}}$  consisting of ample divisors. Then the functions  $\widehat{h}^i$  on the rational vector space  $N^1(X)_{\mathbb{Q}}$  satisfy the assumptions in Proposition 6.4 (Claim D), and the theorem follows.  $\square$

**5.1. Rational continuity of asymptotic cohomological functions.** The first step along the way is to establish a weak version of the rational continuity property of the volume.

**Proposition 5.5.** *Let  $X$  be a reduced projective scheme of pure dimension  $n$ ,  $L$  an arbitrary line bundle,  $A_1, \dots, A_k$  very ample divisors on  $X$ , where  $k \leq \dim X$ . Then for every pair of ordered  $k$ -tuples  $(E_1, \dots, E_k)$ ,  $(E'_1, \dots, E'_k)$  with  $E_j, E'_j \in |A_j|$  ( $1 \leq j \leq k$ ) in a dense open subset of  $|A_1| \times \dots \times |A_k|$ , one has*

$$h^i(E_1 \cap \dots \cap E_k, L|_{E_1 \cap \dots \cap E_k}) = h^i(E'_1 \cap \dots \cap E'_k, L|_{E'_1 \cap \dots \cap E'_k}),$$

and the intersections  $E_1 \cap \dots \cap E_k$ ,  $E'_1 \cap \dots \cap E'_k$  are reduced.

*Proof.* For every  $1 \leq j \leq k$  let  $V_j = \mathbb{P}(H^0(X, A_j)^\vee)$  be the projective space parameterizing elements in  $|A_j|$ , let  $T_j = \{(x, D) \mid x \in D\} \subseteq X \times V_j$  denote the total space of the flat family

$$\begin{array}{ccc} T_j & \subseteq & X \times V_j \\ \downarrow \phi_j & & \\ V_j & & \end{array}$$

with  $\phi_j^{-1}(v) \cap (X \times \{v\})$  being the divisor in  $X$  corresponding to  $v \in V_j$ . Consider the family

$$\begin{array}{ccc} T_1 \times_X \dots \times_X T_k & \subseteq & X \times (V_1 \times \dots \times V_k) \\ \downarrow \phi_1 \times \dots \times \phi_k = \psi & & \\ V_1 \times \dots \times V_k & & \end{array}$$

which parametrizes ordered  $k$ -tuples of divisors  $(E_1, \dots, E_k)$  with  $E_j \in |A_j|$  for  $1 \leq j \leq k$  together with a specified point  $x \in \bigcap_{j=1}^k E_j$ . As  $k \leq \dim X$ , the map  $\psi = \phi_1 \times \dots \times \phi_k$  is surjective. Since  $V_1 \times \dots \times V_k$  is integral, by generic flatness there exists a dense open set

$$U \subseteq V_1 \times \dots \times V_k$$

such that

$$\psi|_{\psi^{-1}(U)} : \psi^{-1}(U) \rightarrow U$$

is flat. The map  $\psi$  over  $U$  is a flat family whose fibres are closed subschemes of the form  $E_1 \cap \dots \cap E_k$ ,  $E_j \in |A_j|$  for  $1 \leq j \leq k$ .

By possibly shrinking  $U$  one can arrange via a Bertini-type argument that all the subschemes  $E_1 \cap \dots \cap E_k \subseteq X$  that are the fibres of  $\psi$  over  $U$  are actually (geometrically) reduced (this amounts to showing that the set of points in the base over which the fibres are geometrically reduced is constructible, dense, and has maximal dimension).

Consider the line bundle

$$\mathcal{L} = i^* p_1^* L ,$$

where

$$i : T_1 \times_X \dots \times_X T_k \rightarrow X \times (V_1 \times \dots \times V_k)$$

is the inclusion map and

$$p_1 : X \times (V_1 \times \dots \times V_k) \rightarrow X$$

is the first projection. For the line bundle  $\mathcal{L}$  one has

$$\mathcal{L}|_{\psi^{-1}(u)} \simeq L|_{E_1 \cap \dots \cap E_k} ,$$

where  $\psi^{-1}(u) \cap X \times \{u\} = E_1 \cap \dots \cap E_k$  inside  $X$ . Then the statement of the lemma follows from the semi continuity theorem [12], III.12. applied to  $\mathcal{L}$  over the integral base  $U$ .  $\square$

**Corollary 5.6.** *With notation as above, one has*

$$\widehat{h}^i(E_1 \cap \dots \cap E_k, L|_{E_1 \cap \dots \cap E_k}) = \widehat{h}^i(E'_1 \cap \dots \cap E'_k, L|_{E'_1 \cap \dots \cap E'_k})$$

for all  $i$ ,  $1 \leq k \leq n$  and for every pair of ordered  $k$ -tuples  $(E_1, \dots, E_k)$ ,  $(E'_1, \dots, E'_k)$  with  $E_j, E'_j \in |A_j|$  ( $1 \leq j \leq k$ ) in a dense open subset of  $|A_1| \times \dots \times |A_k|$ , for all  $1 \leq j \leq n$ .

In order to be able to use Corollary 4.4 to prove our rational continuity result, we need the following Bertini-type result on intersections of divisors in a very ample linear system.

**Lemma 5.7.** *Let  $X$  be a reduced projective variety of pure dimension  $n$  over an algebraically closed field,  $A$  a very ample divisor on  $X$ . Then for any  $m > n$  and any general  $(E_1, \dots, E_m)$  in  $|A|$ , we have that*

- (1) *in the local ring of any point of  $X$ , the local equations defining any cardinality  $n$  subset of  $E_1, \dots, E_m$  form a regular sequence*
- (2) *the intersection of any  $n + 1$  of the  $E_i$ 's is empty.*

*In a similar vein, for every  $1 \leq m \leq n$  and general ordered  $m$ -tuple  $(E_1, \dots, E_m)$  in  $|A|$ , one has that in the local ring of any point of  $X$ , the local equations defining  $E_1, \dots, E_m$  form a regular sequence. Moreover, in each case, we have that the scheme-theoretic intersections of the  $E_j$ 's are geometrically reduced and equidimensional.*

*Proof.* The same principles of constructibility, fibration and flatness that were used in the previous lemma imply that in order to prove that the corresponding subset of the parameter space of ordered  $m$ -tuples of divisors is open and dense, it is enough to see that it is nonempty. We will proceed by induction. In the case  $m = 1$  we will pick  $E_1 \in |A|$  which is geometrically reduced and equidimensional (we can actually pick  $E_1$  to be irreducible).

Assume we have  $E_1, \dots, E_m \in |A|$  as required. Take any reduced equidimensional Cartier divisor  $E_{m+1}$  that does not contain any of the associated primes of the reduced equidimensional  $(n - k)$ -dimensional scheme-theoretic intersections

$$E_{i_1} \cap \dots \cap E_{i_k}$$

where  $1 \leq i_1 < \dots < i_k \leq m$  and  $1 \leq k \leq n$ . Then on one hand,  $E_{m+1}$  will avoid all of the  $n$ -fold intersections. On the other hand,  $E_{m+1}$  is not a zero divisor in any of intersections  $E_{i_1} \cap \dots \cap E_{i_{n-1}}$  (where  $i_1, \dots, i_{n-1} \leq m$ ) hence locally forms a regular sequence with them. This proves both statements.  $\square$

**Remark 5.8.** Although we imposed the hypothesis of equidimensionality on  $X$  and only obtained an equidimensional condition in the conclusion, when  $X$  is irreducible and we consider generic intersections

$$E_{i_1} \cap \dots \cap E_{i_k}$$

with  $k < n$  in Lemma 5.7 (for  $m > n$  or  $m \leq n$ ), then these intersections are even irreducible by [14], Corollary 6.7.

**Lemma 5.9.** *Let  $X$  be an irreducible projective variety of dimension  $n$ ,  $A$  a very ample,  $D$  an arbitrary divisor on  $X$ . Then for every  $r \geq n$ , there exist divisors  $E_1, \dots, E_r \in |A|$ , such that for every  $1 \leq k \leq n$  and every choice  $1 \leq j_1 < \dots < j_k \leq r$ , the intersection*

$$E_{j_1} \cap \dots \cap E_{j_k}$$

is reduced, irreducible if  $k < n$  (zero dimensional, if  $k = n$ ). Furthermore, for all  $0 \leq i \leq n$  and  $m \geq 1$ , the values

$$h^i(E_{j_1} \cap \dots \cap E_{j_k}, mD)$$

are minimal in the family parametrizing  $k$ -fold intersections of divisors.

*Proof.* Pick the divisors  $E_1, \dots, E_n \in |A|$  in such a way that for any  $1 \leq k \leq n$  and any  $1 \leq j_1 < \dots < j_k \leq r$ , the intersection

$$E_{j_1} \cap \dots \cap E_{j_k}$$

is reduced, irreducible if  $k < n$  (0-dimensional if  $k = n$ ), and for all  $m \geq 1$ , the values of

$$h^i(E_{j_1} \cap \dots \cap E_{j_k}, mpD|_{E_{j_1} \cap \dots \cap E_{j_k}})$$

are all minimal, ie. they are equal to the value which is taken up over a nonempty open set of the variety parametrizing  $k$ -fold intersections in Proposition 5.5. Note that Proposition 5.5 is applied separately for each  $m \geq 1$ . Naturally, as  $m$  grows, the open locus may shrink. However, the Baire category theorem implies that there is a nonempty intersection of all the open loci.  $\square$

**Proposition 5.10** (Rational continuity). *Let  $X$  be an irreducible projective variety of dimension  $n$  over  $\mathbb{C}$ ,  $D$  an arbitrary divisor,  $A$  an ample divisor,  $i \geq 0$ . Then*

$$\frac{1}{p^n} \left| \widehat{h}^i(X, pD - A) - \widehat{h}^i(X, pD) \right| \rightarrow 0$$

as  $p \rightarrow \infty$ .

*Proof.* We first reduce to the case when  $A$  is very ample. Let  $m_0A$  be a fixed large integral multiple of  $A$ , which is very ample. By the homogeneity of the asymptotic cohomological functions we see that

$$\frac{1}{p^n} \left| \widehat{h}^i(X, pm_0D - m_0A) - \widehat{h}^i(X, pm_0D) \right| = \frac{m_0^n}{p^n} \left| \widehat{h}^i(X, pD - A) - \widehat{h}^i(X, pD) \right|.$$

The proposition for  $A$  very ample then implies that the right-hand side converges to 0 as  $p \rightarrow \infty$ . Hence we can assume from now on that  $A$  is very ample.

Fix a positive integer  $p$ , and divisors  $E_1, \dots, E_n \in |A|$  as in Lemma 5.9, such that for all  $m \geq 1$ , the values of

$$h^i(E_{j_1} \cap \dots \cap E_{j_k}, mpD|_{E_{j_1} \cap \dots \cap E_{j_k}})$$

are all minimal. Next, fix a natural number  $m \geq n$ . Then by Lemma 5.7 and Proposition 5.5 we can find general divisors  $F_1, \dots, F_m \in |A|$  which satisfy the conditions of Corollary

4.4. Furthermore, by Lemma 5.9, we can assume that for all  $1 \leq k \leq n$  and all  $1 \leq j_1 < \dots < j_k \leq m$

$$h^i(F_{j_1} \cap \dots \cap F_{j_k}, mpD) = h^i(E_1 \cap \dots \cap E_k, mpD) .$$

By putting  $mpD$  in place of  $D$  and  $F_1, \dots, F_m$  in place of  $A_1, \dots, A_m$  in Corollary 4.4, we obtain that

$$\begin{aligned} |h^i(X, mpD - mA) - h^i(X, mpD)| &= \left| h^i\left(X, mpD - \sum_{j=1}^m F_j\right) - h^i(X, mpD) \right| \\ &\leq 2 \cdot \sum_{k=1}^{i+1} \sum_{1 \leq l_1 < \dots < l_k \leq m} h^{i+1-k}(F_{l_1} \cap \dots \cap F_{l_k}, mpD) \\ &= 2 \cdot \sum_{k=1}^{i+1} \binom{m}{k} h^{i+1-k}(E_1 \cap \dots \cap E_k, mpD) . \end{aligned}$$

Let us divide both sides by  $\frac{m^n}{n!}$  and take upper limits. Note that the choice of the  $F$ 's depends on  $m$ , therefore it is crucial that they got replaced by the  $E$ 's.

Using Corollary 6.2, Lemma 6.1, and the homogeneity of asymptotic cohomological functions, we arrive at

$$\begin{aligned} \left| \widehat{h}^i(X, pD - A) - h^i(X, pD) \right| &\leq C_n \cdot \sum_{k=1}^{i+1} \limsup_m \left( \frac{h^{i+1-k}(E_1 \cap \dots \cap E_k, mpD)}{m^{n-k}/(n-k)!} \right) \\ &= C_n \sum_{k=1}^{i+1} \widehat{h}^{i+1-k}(E_1 \cap \dots \cap E_k, pD) \\ &= C_n \sum_{k=1}^{i+1} p^{n-k} \widehat{h}^{i+1-k}(E_1 \cap \dots \cap E_k, D) , \end{aligned}$$

where  $C_n$  is a positive constant only depending on  $n$ . Upon dividing by  $p^n$  we arrive at the statement of the proposition.  $\square$

**5.2. Numerical invariance of asymptotic cohomological functions.** The crucial ingredient in the proof of the numerical invariance of asymptotic cohomological functions is the fact that numerically trivial divisors form a bounded family. The specific version of this theme which we employ is formulated in Proposition 5.12.

As a first step, we show that one can give uniform estimates on the cohomology of multiples of divisors in families. Note that this statement does not follow from the semicontinuity theorem in [12], Section III.12.

**Proposition 5.11** (Cohomology estimate in families). *Let  $f : \mathcal{X} \rightarrow T$  be a projective map of noetherian schemes,  $\mathcal{F}$  a coherent sheaf on  $\mathcal{X}$ ,  $\mathcal{L}$  an invertible sheaf on  $\mathcal{X}$ .*

*Then there exists a positive constant  $C$  depending only on  $f, T, \mathcal{L}, \mathcal{F}$  for which*

$$h^i(\mathcal{X}_t, \mathcal{F}_t \otimes \mathcal{L}_t^{\otimes m}) \leq C \cdot m^{\dim \mathcal{X}_t}$$

*for all  $m \geq 1, i \geq 0$  and all  $t \in T$ .*

*Proof.* We will proceed by induction on the maximal fibre dimension of  $f$  and noetherian induction on  $T$ . The case  $\dim \mathcal{X} = 0$  is straightforward to check.

For the inductive step, we can assume that the base is reduced and irreducible, as the fibres are unaffected by nilpotents in the base, and we can deal with the irreducible components one at a time.

Our strategy is to find a non-empty open subset  $U_1 \subseteq T$  over which the proposition holds with a certain constant  $C_1$ . Starting from there, we can (by noetherian induction) construct a stratification of  $T$  into finitely many irreducible subschemes  $U_q \subseteq T, 1 \leq q \leq r$ , such that the proposition holds over  $U_q$  with a constant  $C_q$ . Then we reach the desired conclusion by setting

$$C \stackrel{\text{def}}{=} \max \{ C_1, \dots, C_r \} .$$

Let  $\eta$  denote the generic point of  $T$  and consider  $\mathcal{X}_\eta$ , the fibre over the generic point. Since  $\mathcal{X}_\eta$  is projective, we are able to find very ample Cartier divisors  $\mathcal{A}_\eta, \mathcal{B}_\eta$  on  $\mathcal{X}_\eta$  such that

$$\mathcal{L}_\eta = \mathcal{O}_{\mathcal{X}_\eta}(\mathcal{A}_\eta) \otimes \mathcal{O}_{\mathcal{X}_\eta}(-\mathcal{B}_\eta) ,$$

and  $\mathcal{A}_\eta, \mathcal{B}_\eta$  have the properties that

- (1) none of  $\mathcal{A}_\eta, \mathcal{B}_\eta$  contain any of the associated subvarieties of  $\mathcal{F}_\eta$  on  $\mathcal{X}_\eta$ ,
- (2) the local equation of  $\mathcal{D}_\eta$  in  $\mathcal{X}_\eta$  is not a zero-divisor in  $\mathcal{A}_\eta, \mathcal{B}_\eta \subseteq \mathcal{X}_\eta$ .

This way, we obtain the short exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}_\eta}(-\mathcal{B}_\eta) \rightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^{\otimes(m+1)} \rightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^{\otimes(m+1)}|_{\mathcal{A}_\eta} \rightarrow 0 \\ 0 \rightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}_\eta}(-\mathcal{B}_\eta) \rightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^{\otimes m} \rightarrow \mathcal{F}_\eta \otimes \mathcal{L}_\eta^{\otimes m}|_{\mathcal{B}_\eta} \rightarrow 0 . \end{aligned}$$

By generic flatness and denominator chasing, it is possible to extend  $\mathcal{A}_\eta$  and  $\mathcal{B}_\eta$  to  $U$ -ample  $U$ -flat divisors  $\mathcal{A}, \mathcal{B}$  over a non-empty open neighbourhood  $U \subseteq T$  of  $\eta$  in such a way that  $\mathcal{F}|_U$  is  $U$ -flat, the divisors  $\mathcal{A}_u, \mathcal{B}_u$  are very ample for every  $u \in U$ ,

$$\mathcal{L}_u = \mathcal{O}_{\mathcal{X}_u}(\mathcal{A}_u) \otimes \mathcal{O}_{\mathcal{X}_u}(-\mathcal{B}_u)$$

and  $\mathcal{A}_u, \mathcal{B}_u$  do not contain the associated primes of  $\mathcal{F}_u$ . Here  $U$ -flatness ensures that formation of the ideals of  $\mathcal{A}, \mathcal{B}$  respects base change on  $U$ . Moreover, it follows from Proposition 9.4.2. in [11] that (by possibly shrinking the open subset  $U \subseteq T$ ) the following sequences are exact for all  $u \in U$ :

$$\begin{aligned} 0 \rightarrow \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}_u}(-\mathcal{B}_u) \rightarrow \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)} \rightarrow \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{A}_u} \rightarrow 0 \\ 0 \rightarrow \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}_u}(-\mathcal{B}_u) \rightarrow \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes m} \rightarrow \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes m}|_{\mathcal{B}_u} \rightarrow 0 . \end{aligned}$$

From the corresponding long exact sequences we obtain that

$$\begin{aligned} \left| h^i \left( \mathcal{X}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)} \right) - h^i \left( \mathcal{X}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes m} \right) \right| \leq \\ h^{i-1} \left( \mathcal{A}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{A}_u} \right) + h^{i-1} \left( \mathcal{B}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{B}_u} \right) + \\ h^i \left( \mathcal{A}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{A}_u} \right) + h^i \left( \mathcal{B}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{B}_u} \right) . \end{aligned}$$

The schemes  $\mathcal{A}, \mathcal{B}$  are over  $U$ , whose fibre dimension is strictly less than the maximal fibre dimension of  $\mathcal{X}_U \rightarrow U$ ,  $\mathcal{F}|_{\mathcal{A}}, \mathcal{F}|_{\mathcal{B}}$  are coherent sheaves on the respective schemes, and  $\mathcal{L}|_{\mathcal{A}}, \mathcal{L}|_{\mathcal{B}}$  are both relative ( $U$ -flat) Cartier divisors on  $\mathcal{A}, \mathcal{B}$ , respectively. This latter fact follows from the  $U$ -flatness of  $\mathcal{A}, \mathcal{B}$ . Hence we can apply the induction hypothesis to the projective families  $\mathcal{A} \rightarrow U, \mathcal{B} \rightarrow U$ , and obtain that

$$h^j \left( \mathcal{A}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{A}_u} \right) \leq C' m^{\dim \mathcal{X}_u - 1}$$

for all  $j \geq 0$ , where the constant  $C'$  is independent of  $u$  (it only depends on  $\mathcal{A}_U \rightarrow U, \mathcal{F}|_{\mathcal{A}_U}$  and  $\mathcal{L}|_{\mathcal{A}_U}$ ), and similarly

$$h^j \left( \mathcal{B}_u, \mathcal{F}_u \otimes \mathcal{L}_u^{\otimes(m+1)}|_{\mathcal{B}_u} \right) \leq C'' m^{\dim \mathcal{X}_u - 1}$$

Consequently,

$$h^i(\mathcal{X}_u, \mathcal{F}_u \otimes \mathcal{L}_{\mathcal{X}_u}^{\otimes m}) \leq C \cdot m^{\dim \mathcal{X}_u}$$

for all  $i \geq 0$  and all  $u \in U$ , where the positive constant  $C$  is again independent of  $u$ .  $\square$

The uniform behaviour of numerically trivial divisors enters the picture in the form of a vanishing theorem of Fujita (see the reference in the proof).

**Proposition 5.12.** *Let  $X$  be an irreducible projective variety of dimension  $n$ . Then there exists a family*

$$\begin{array}{ccc} V & \subseteq & X \times T \\ \downarrow \phi & & \\ T & & \end{array}$$

with  $T$  a quasi-projective variety (not necessarily irreducible),  $V \subseteq X \times T$  a closed subscheme and  $\phi$  flat, together with a very ample divisor  $A$  on  $X$ , such that

- (1)  $A + N$  is very ample for every numerically trivial divisor  $N$  on  $X$ ,
- (2) if  $D \in |A + N|$  for some  $N \sim_{num} 0$ , then  $D = V_t$  for some  $t \in T$ .

*Proof.* We start by showing that there exists a very ample line bundle  $A$  on  $X$  such that  $A + N$  is very ample for every numerically trivial divisor  $N$ .

According to Fujita's vanishing theorem ([19] Section 1.4.D.), for any fixed ample divisor  $B$  there exists  $m_0 > 0$  such that

$$H^i(X, \mathcal{O}_X(mB + E)) = 0$$

for all  $m \geq m_0$ ,  $i \geq 1$  and all nef divisors  $E$ . In particular, vanishing holds for every numerically trivial divisor. Take any very ample divisor  $B$ , let  $m_0$  be as in Fujita's vanishing theorem. Consider  $A' = (m_0 + n)B + N$  where  $N$  is an arbitrary numerically trivial divisor. Then by Fujita's theorem,

$$H^i(X, \mathcal{O}_X(A' - iB)) = H^i(X, \mathcal{O}_X(N + (m_0 + n - i)B)) = 0$$

for all  $i \geq 0$  hence  $A'$  is 0-regular with respect to  $B$ . By Theorem 1.8.3 in [19],  $A'$  is globally generated. But then  $A' + B = (m_0 + n + 1)B + N$  is very ample. Observe that the coefficient of  $B$  is independent of  $N$  hence the choice

$$A \stackrel{\text{def}}{=} (m_0 + n + 1)B$$

will satisfy the requirements for  $A$ .

To prove the Proposition, pick  $A$  as above to begin with. According to 1.4.36 in [19], Section 1.4.D, there exists a scheme  $Q$  of finite type over  $\mathbb{C}$ , and a line bundle  $\mathcal{L}$  on  $X \times Q$ , with the property that for every line bundle  $\mathcal{O}_X(A + N)$  with  $N$  numerically trivial, there exists  $q \in Q$  for which

$$\mathcal{O}_{X \times q}(A + N) = \mathcal{L}|_{X \times \{q\}}.$$

Let  $p, \pi$  denote the projection maps from  $Q \times X$  to  $Q$  and  $X$ , respectively. We can arrange by possibly twisting  $\mathcal{L}$  further by  $\pi^* \mathcal{O}_X(mA)$ , that  $R^j p_* \mathcal{L} = 0$  for  $j > 0$ , and that the natural map  $\rho : p^* p_* \mathcal{L} \rightarrow \mathcal{L}$  is surjective.

By the theorem on cohomology and base change ([12], Section III.12)

$$\mathcal{E} \stackrel{\text{def}}{=} p_* \mathcal{L}$$

is a vector bundle on  $Q$  whose formation commutes with base change over  $Q$ . As  $\mathcal{E}(q) = H^0(X, \mathcal{L}|_{X \times q})$ , its projectivization  $\mathbb{P}(\mathcal{E})$  will parametrize the divisors  $D \in |N + A|$ , with  $N$  numerically trivial.

The universal divisor over  $\mathbb{P}(\mathcal{E})$  is constructed as follows. The kernel  $\mathcal{M}$  of the natural map  $\rho$  (which is surjective in our case) is a vector bundle whose formation respects base change on  $Q$ . By restricting to the fibre of  $p$  over  $q$ , one has an exact sequence

$$0 \rightarrow (M_X)_q \rightarrow H^0(\{q\} \times X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L \rightarrow 0$$

where  $L = \mathcal{L}|_{\{q\} \times X}$  and  $(M_X)_q = H^0(X, \mathcal{M}|_{\{q\} \times X})$ . Hence we see that

$$\mathbb{P}_{\text{sub}}(M_X) = \{ (s, x) \mid x \in X, s \in H^0(X, L), s(x) = 0 \} ,$$

ie.  $V = \mathbb{P}_{\text{sub}}(\mathcal{M})$  and  $T = \mathbb{P}(\mathcal{E})$  will satisfy the requirements of the proposition.  $\square$

**Corollary 5.13.** *With notation as above, fix a positive integer  $1 \leq k \leq n$ , and a very ample divisor  $A$  as in Proposition 5.12. Then there exists a projective family*

$$f : \mathcal{X} \rightarrow T'$$

which parametrizes (possibly with repetitions)  $k$ -fold intersections  $E_1 \cap \dots \cap E_k$  of all divisors such that  $\mathcal{O}_X(E_j - A)$  is numerically trivial for  $1 \leq j \leq k$ .

*Proof.* Consider the family  $V \rightarrow T$  constructed in the Proposition, and proceed as in the proof of Proposition 5.5 using the same principles of flatness, fibration and constructibility.  $\square$

**Lemma 5.14.** *With notation as above, let  $P, A, D$  be divisors on  $X$ , with  $P$  numerically trivial,  $A$  a very ample divisor for which  $A + P'$  is also very ample for any choice of a numerically trivial divisor  $P'$ ,  $D$  arbitrary. Fix  $1 \leq k \leq n$ , and take  $E_1, \dots, E_k \in |A + P|$  such that  $E_1 \cap \dots \cap E_k$  is geometrically reduced and of pure dimension  $n - k$ . Then there exists a positive constant  $C$  only depending on  $D$  and  $A$ , such that*

$$h^i(E_1 \cap \dots \cap E_k, mD) \leq C \cdot m^{n-k}$$

for all  $m \geq 1$ .

*Proof.* As we saw in Corollary 5.13, there exists a projective family

$$f : \mathcal{X} \rightarrow T$$

which parametrizes  $k$ -fold intersections of divisors  $E_1 \cap \dots \cap E_k$ , with  $\mathcal{O}_X(E_j - A)$  numerically trivial for  $1 \leq j \leq k$ . In particular, all the intersections which occur in the lemma are fibres of this family. By restricting  $f$  to the inverse image of those points in the base whose fibres have pure dimension  $n - k$ , we obtain an equidimensional family. Note that by [11] 9.9.3. applied to [11] 9.9.2 (1),(3) and (4), the set of points in the base whose fibres are of pure dimension  $n - k$  is constructible, hence can be stratified by locally closed subsets.

The claim of the lemma then follows from Proposition 5.11 applied to the line bundle

$$\mathcal{L} \stackrel{\text{def}}{=} i^* pr_X^* \mathcal{O}_X(D) ,$$

where  $pr_X$  is the projection map  $\mathcal{X} \rightarrow X$ , and  $i$  is the inclusion of  $\mathcal{X}$  into  $T \times X$ .  $\square$

**Proposition 5.15** (Invariance with respect to numerical equivalence). *Let  $X$  be an  $n$ -dimensional irreducible projective variety,  $D$  an arbitrary,  $P$  a numerically trivial divisor on  $X$ . Then*

$$\widehat{h}^i(X, D + P) = \widehat{h}^i(X, D) .$$

*In particular, the functions  $\widehat{h}^i$  are well-defined on both  $N^1(X)$ , and  $N^1(X)_{\mathbb{Q}}$ .*

*Proof.* Fix a very ample divisor  $A$  as in Lemma 5.12, and fix a positive integer  $p$ . In particular, for such  $A$ , the divisor  $A + N$  is very ample for any numerically trivial divisor  $N$ . We want to estimate the difference

$$|h^i(X, m(p(D + P) - A)) - h^i(X, mpD)|$$

in order to prove that

$$(2) \quad \frac{1}{p^n} \widehat{h}^i(X, p(D + P) - A) \rightarrow \widehat{h}^i(X, D)$$

as  $p \rightarrow \infty$ . We know from Proposition 5.10 that

$$\frac{1}{p^n} \cdot \widehat{h}^i(X, p(D + P) - A) \rightarrow \widehat{h}^i(X, D + P)$$

as  $p \rightarrow \infty$ . By the uniqueness of limits, (2) then implies  $\widehat{h}^i(X, D + P) = \widehat{h}^i(X, D)$ .

Observe that

$$|h^i(X, m(p(D + P) - A)) - h^i(X, mpD)| = \left| h^i\left(X, mpD - \sum_{j=1}^m E_j^{(p)}\right) - h^i(X, mpD) \right|$$

where  $E_j^{(p)} \in |A - pP|$  for every  $1 \leq j \leq m$  are general divisors. From Corollary 4.4 we obtain

$$\begin{aligned} & \left| h^i\left(X, mpD - \sum_{j=1}^m E_j^{(p)}\right) - h^i(X, mpD) \right| \\ & \leq 2 \cdot \sum_{k=1}^{i+1} \sum_{1 \leq j_1 < \dots < j_k \leq m} h^{i+1-k}(E_{j_1}^{(p)} \cap \dots \cap E_{j_k}^{(p)}, mpD). \end{aligned}$$

Lemma 5.14 ensures the existence of a positive constant  $C_{A,D}$ , for which

$$h^s(E_{j_1}^{(p)} \cap \dots \cap E_{j_k}^{(p)}, mpD) \leq C_{A,D} (mp)^{n-k},$$

and which might depend on  $D$  and  $A$ , but is independent of  $pP$ , and the particular choices of the divisors  $E_{j_i}^{(p)}$ . Therefore,

$$\begin{aligned} \left| h^i\left(X, mpD - \sum_{j=1}^m E_j^{(p)}\right) - h^i(X, mpD) \right| & \leq C_{A,D} \sum_{k=1}^{i+1} \sum_{1 \leq j_1 < \dots < j_k \leq m} (mp)^{n-k} \\ & = C_{A,D} \sum_{k=1}^{i+1} \binom{m}{k} \cdot (mp)^{n-k} \end{aligned}$$

hence

$$|h^i(X, m(p(D + P) - A)) - h^i(X, mpD)| \leq C' m^n p^{n-1}.$$

After dividing by  $\frac{m^n}{n!}$  and taking upper limits we arrive at

$$\left| \widehat{h}^i(X, p(D + P) - A) - \widehat{h}^i(X, pD) \right| \leq C'' p^{n-1}$$

which, upon dividing by  $p^n$  gives the desired conclusion.  $\square$

**5.3. Local uniform continuity in ample directions.** The subsection serves to prove the estimate of Theorem 5.1 in ample directions. This forms the basis for the proof of the general case.

**Proposition 5.16.** *Let  $X$  be an irreducible projective variety of dimension  $n$ ,  $D$  an arbitrary,  $A$  an ample  $\mathbb{Q}$ -divisor. Assume that the continuity of asymptotic cohomological functions holds for varieties of dimension less than  $n$ . Fix an arbitrary norm on  $N^1(X)_{\mathbb{Q}}$ . Then there exists a constant  $C_A$  independent of  $D$  and such that*

$$\left| \widehat{h}^i(X, D - bA) - \widehat{h}^i(X, D) \right| \leq C_A \cdot \sum_{k=1}^n \|D\|^{n-k} \cdot b^k \cdot \|A\|^k .$$

for all integers  $b \geq 1$ .

*Proof.* Both sides of the inequality are homogeneous of degree  $n$ , therefore we can assume that both  $D$  and  $A$  are integral and  $A$  is very ample. Also, assume without loss of generality that  $\|A\| = 1$ .

Fix divisors  $E_1, \dots, E_n$ , as in Lemma 5.9, and choose norms on  $N^1(Y)_{\mathbb{Q}}$  where  $Y$  runs through all the possible intersections of the  $E_j$ 's,  $1 \leq j \leq n$ , such that for any choice of  $Y$  and any divisor  $E$  on  $X$

$$\|E|_Y\| \leq \|E\| .$$

As restriction of divisors induces a map of vector spaces  $N^1(X)_{\mathbb{Q}} \rightarrow N^1(Y)_{\mathbb{Q}}$ , it is possible to choose norms as required.

Next, fix a natural number  $m \geq n$ , and pick divisors  $F_1, \dots, F_m$  as in Lemmas 5.7 and 5.9, such that for all choices  $1 \leq k \leq n$  and  $1 \leq j_1 < \dots < j_k \leq m$

$$h^i(F_{j_1} \cap \dots \cap F_{j_k}, mD) = h^i(E_1 \cap \dots \cap E_k, mD) .$$

Then by Corollary 4.4, one has

$$\begin{aligned} \left| h^i(X, mD - mA) - h^i(X, mD) \right| &= \left| h^i \left( X, mD - \sum_{j=1}^{mb} F_j \right) - h^i(X, mD) \right| \\ &\leq 2 \cdot \sum_{k=1}^{i+1} \sum_{1 \leq l_1 < \dots < l_k \leq mb} h^{i-k+1}(F_{l_1} \cap \dots \cap F_{l_k}, mD) \\ &= 2 \cdot \sum_{k=1}^{i+1} \binom{mb}{k} h^{i-k+1}(E_1 \cap \dots \cap E_k, mD) . \end{aligned}$$

Let us divide both sides by  $\frac{m^n}{n!}$  and take upper limits. Then by Corollary 6.2 and Lemma 6.1, one has

$$\begin{aligned} \left| \widehat{h}^i(X, D - A) - \widehat{h}^i(X, D) \right| &\leq 2 \cdot \sum_{k=1}^{i+1} b^k \cdot \limsup \frac{h^{i-k+1}(E_1 \cap \dots \cap E_k, mD)}{m^{n-k}/(n-k)!} \\ &= C_n \sum_{k=1}^{i+1} b^k \widehat{h}^{i-k+1}(E_1 \cap \dots \cap E_k, D) . \end{aligned}$$

The hypothesis of the Proposition — in the form of Corollary 5.4 — implies that

$$\widehat{h}^{i-k+1}(E_1 \cap \dots \cap E_k, mD) \leq C_k \cdot \|D|_{E_1 \cap \dots \cap E_k}\|^{n-k} .$$

We remark that the  $k$ -fold intersections of the divisors  $E_j$  are reduced of the expected dimension, and either irreducible or zero-dimensional, hence the induction hypothesis indeed applies to them. Hence

$$\left| \widehat{h}^i(X, D - bA) - \widehat{h}^i(X, D) \right| \leq C \cdot \sum_{k=1}^{i+1} b^k \cdot C_k \cdot \|D|_{E_1 \cap \dots \cap E_k}\|^{n-k} = C_A \sum_{k=1}^n b^k \cdot \|D\|^{n-k}$$

as required.  $\square$

## 6. AUXILIARY RESULTS

First we collect some simple properties of upper limits that we need.

**Lemma 6.1** (Properties of  $\limsup$ ). *Let  $a_n, b_n, c_n$  be sequences of nonnegative real numbers. Then*

- (1)  $\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$ .
- (2)  $\limsup (a_n b_n) \leq (\limsup a_n)(\limsup b_n)$ .
- (3) *If  $|a_n - b_n| \leq c_n$  then*

$$|\limsup a_n - \limsup b_n| \leq \limsup c_n .$$

*Proof.* The first two statements are well-known. As for the third one, we observe that  $|a_n - b_n| \leq c_n$  is equivalent to  $a_n - b_n \leq c_n$  and  $b_n - a_n \leq c_n$ , that is,  $a_n \leq b_n + c_n$  and  $b_n \leq a_n + c_n$ . But (1) implies that

$$\limsup a_n \leq \limsup b_n + \limsup c_n \text{ and } \limsup b_n \leq \limsup a_n + \limsup c_n$$

hence

$$|\limsup a_n - \limsup b_n| \leq \limsup c_n .$$

$\square$

**Corollary 6.2.** *Let  $X$  be an irreducible projective variety on dimension  $n$ ,  $D, D'$  arbitrary divisors on  $X$ , such that*

$$|h^i(X, \mathcal{O}_X(mD)) - h^i(X, \mathcal{O}_X(mD'))| \leq a_m$$

for a sequence  $a_m$  of nonnegative real numbers. Then

$$\left| \widehat{h}^i(X, \mathcal{O}_X(D)) - \widehat{h}^i(X, \mathcal{O}_X(D')) \right| \leq \limsup_m \frac{a_m}{m^n/n!} .$$

Although a short exact sequence of sheaves does not give rise to anything similar to a long exact sequence for the asymptotic cohomological functions, one typical usage of the cohomology long exact sequence can to some extent be simulated. Namely, there is a substitute for the principle that the vanishings of certain cohomology groups give extra relations between others.

**Lemma 6.3.** *Let  $X$  be an irreducible variety,*

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$$

be a short exact sequence of coherent sheaves on  $X$ ,  $L$  a line bundle on  $X$ . Then if

$$\dim \operatorname{supp} \mathcal{A} \leq n - 1 ,$$

then

$$\limsup_m \frac{h^i(X, \mathcal{B} \otimes L^{\otimes m})}{m^n/n!} = \limsup_m \frac{h^i(X, \mathcal{C} \otimes L^{\otimes m})}{m^n/n!} ,$$

while if

$$\dim \operatorname{supp} \mathcal{C} \leq n - 1 ,$$

then

$$\limsup_m \frac{h^i(X, \mathcal{A} \otimes L^{\otimes m})}{m^n/n!} = \limsup_m \frac{h^i(X, \mathcal{B} \otimes L^{\otimes m})}{m^n/n!} .$$

*Proof.* We will treat the case when  $\dim \operatorname{supp} \mathcal{A} \leq n - 1$ , the other case can be dealt with the exact same way. Consider the short exact sequence

$$0 \rightarrow \mathcal{A} \otimes L^{\otimes m} \rightarrow \mathcal{B} \otimes L^{\otimes m} \rightarrow \mathcal{C} \otimes L^{\otimes m} \rightarrow 0 .$$

From the corresponding long exact sequence, we obtain

$$|h^i(X, \mathcal{B} \otimes L^{\otimes m}) - h^i(X, \mathcal{C} \otimes L^{\otimes m})| \leq h^{i-1}(X, \mathcal{A} \otimes L^{\otimes m}) + h^i(X, \mathcal{A} \otimes L^{\otimes m}) .$$

As  $\dim \operatorname{supp} \mathcal{A} \leq n - 1$ , we have that

$$h^{i-1}(X, \mathcal{A} \otimes L^{\otimes m}) \leq C \cdot m^{n-1} \text{ and } h^i(X, \mathcal{A} \otimes L^{\otimes m}) \leq C \cdot m^{n-1}$$

for some positive constant  $C$ . Then Corollary 6.2 implies

$$\left| \limsup_m \frac{h^i(X, \mathcal{B} \otimes L^{\otimes m})}{m^n/n!} - \limsup_m \frac{h^i(X, \mathcal{C} \otimes L^{\otimes m})}{m^n/n!} \right| \leq \limsup_m \frac{2 \cdot C \cdot m^{n-1}}{m^n/n!} = 0 .$$

□

The following result forms a part of the proof of the main theorem of this paper.

**Proposition 6.4.** *Let  $V$  be an  $r$ -dimensional normed rational vector space,  $A_1, \dots, A_r$  a basis for  $V$ ,  $f : V \rightarrow \mathbb{R}^{\geq 0}$  a homogeneous function on  $V$ . Assume furthermore, that for every  $1 \leq i \leq r$  there exists a constant  $C_i$  such that for all  $D \in V$ , and all natural numbers  $b \geq 1$ ,*

$$|f(D - bA_i) - f(D)| \leq C_i \cdot \sum_{k=1}^n \|D\|^{n-k} \cdot b^k .$$

*Then there exists a constant  $C > 0$ , such that for every  $D, D' \in V$  one has*

$$|f(D) - f(D')| \leq C \cdot \sum_{k=1}^n (\max\{\|D\|, \|D'\|\})^{n-k} \cdot \|D - D'\|^k .$$

*Proof.* Assume that the given norm is the maximum norm with respect to the basis  $A_1, \dots, A_r$  of  $V$ . Let

$$D - D' = \sum_{j=1}^r b_j A_j ,$$

where by the homogeneity of both sides of the inequality in the proposition, we can assume that for all  $1 \leq j \leq r$ , the coordinates  $b_j$  are integers (not necessarily nonnegative).

We will show that

$$|f(D') - f(D)| \leq C \cdot \sum_{k=1}^n \|D'\|^{n-k} \cdot \|D - D'\|^k$$

with some constant  $C$  independent of  $D$  and  $D'$ , from which the proposition follows by  $\|D'\| \leq \max\{\|D\|, \|D'\|\}$ . Write

$$f\left(D - \sum_{i=1}^r b_i A_i\right) - f(D) = \sum_{l=1}^r \left( f\left(D - \sum_{j=1}^l b_j A_j\right) - f\left(D - \sum_{j=1}^{l-1} b_j A_j\right) \right) ,$$

ie. as a telescoping sum of terms, where each summand is a difference of two terms by a nonnegative multiple of a basis vector  $A_j$ . By the triangle inequality and the assumption of the lemma,

$$\begin{aligned} & \left| f \left( D - \sum_{j=1}^r b_j A_j \right) - f(D) \right| \leq \sum_{l=1}^r \left| f \left( \left( D - \sum_{j=1}^{l-1} b_j A_j \right) - b_l A_l \right) - f \left( D - \sum_{j=1}^{l-1} b_j A_j \right) \right| \\ & \leq \sum_{l=1}^r C \cdot \sum_{k=1}^n \max \left\{ \left\| D - \sum_{j=1}^{l-1} b_j A_j \right\|^{n-k}, \left\| D - \sum_{j=1}^l b_j A_j \right\|^{n-k} \right\} \cdot \|b_l A_l\|^k. \end{aligned}$$

Applying the triangle inequality again, we obtain that

$$\begin{aligned} \left\| D - \sum_{j=1}^{l-1} b_j A_j \right\| & \leq \left\| D - \sum_{j=1}^r b_j A_j \right\| + \left\| \sum_{j=l}^r b_j A_j \right\|, \\ \left\| D - \sum_{j=1}^l b_j A_j \right\| & \leq \left\| D - \sum_{j=1}^r b_j A_j \right\| + \left\| \sum_{j=l+1}^r b_j A_j \right\|, \end{aligned}$$

therefore, by Newton's binomial theorem and collecting terms, one has

$$|f(D - A) - f(D)| \leq C \cdot \sum_{l=1}^r \sum_{k=1}^n \sum_{s=0}^{n-k} \binom{n-k}{s} \left\| D - \sum_{j=1}^r b_j A_j \right\|^{n-k-s} \left\| \sum_{j=l}^r b_j A_j \right\|^s \cdot \|b_l A_l\|^k.$$

Observe, that as we chose the maximum norm relative to the basis  $A_1, \dots, A_r$  on  $N^1(X)_{\mathbb{Q}}$ , one has

$$\left\| \sum_{j=l}^r b_j A_j \right\|^s = \max \{b_l^s, \dots, b_r^s\} \quad \text{and} \quad b_l^k \cdot \max_{l \leq j \leq r} \{b_j^s\} \leq \max_{l \leq j \leq r} \{b_j^{k+s}\} \leq \left\| \sum_{j=1}^r b_j A_j \right\|^{k+s}$$

for every  $1 \leq l \leq r$  and  $1 \leq k \leq n$  and  $1 \leq s \leq n - k$ . This implies

$$\begin{aligned} |f(D - A) - f(D)| & \leq C \cdot \sum_{l=1}^r \sum_{k=1}^n \sum_{s=0}^{n-k} \binom{n-k}{s} \left\| D - \sum_{j=1}^r b_j A_j \right\|^{n-k-s} \cdot \left\| \sum_{j=1}^r b_j A_j \right\|^{k+s} \\ & \leq C r n(n!) \sum_{p=1}^n \left\| D - \sum_{j=1}^r b_j A_j \right\|^{n-p} \cdot \left\| \sum_{j=1}^r b_j A_j \right\|^p \\ & = C r n(n!) \sum_{k=1}^n \|D'\|^{n-k} \cdot \|D - D'\|^k. \end{aligned}$$

□

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