# EXAMPLES OF CALABI-YAU 3-MANIFOLDS WITH COMPLEX MULTIPLICATION

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## Introduction

By string theoretical considerations, one is interested in Calabi-Yau manifolds, since Calabi-Yau 3-manifolds provide conformal field theories (CFT). One is especially interested in Calabi-Yau 3-manifolds with complex multiplication, since such a manifold has many symmetries and mirror pairs of Calabi-Yau 3-manifolds with complex multiplication yield rational conformal field theories (RCFT) (see [5]).

E. Viehweg and K. Zuo [10] have constructed a family of Calabi-Yau 3-manifolds with dense set of complex multiplication fibers. This construction is given by a tower of cyclic coverings. We will use a similar construction to obtain $K3$ surfaces with complex multiplication.

C. Voisin [11] has described a method to obtain Calabi-Yau 3-manifolds by using involutions on $K3$ surfaces. C. Borcea [3] has independently arrived at a more general version of the latter method, which allows to construct Calabi-Yau manifolds in arbitrary dimension. By using this method and $K3$ surfaces with complex multiplication, we will obtain our concrete examples of Calabi-Yau 3-manifolds with complex multiplication.

Our methods are very similar to the methods in [8], which contains different concrete examples of Calabi-Yau 3-manifolds with complex multiplication. The examples of [8] occur as fibers of a family with a dense set of complex multiplication fibers. Here we give some examples, which are not necessarily fibers of a non-trivial family with a dense set of complex multiplication fibers. The first two sections give two different classes of examples by using involutions on $K3$ surfaces.

In the third section we will prove that a $K3$ surface with a degree 3 automorphism has complex multiplication. By using methods, which has been introduced in [8] Section 9.1 and Section 9.2, we will use this automorphism and the Fermat curve of degree 3 for the construction of a Calabi-Yau 3-manifold with complex multiplication.

We use the same methods as in [8] Chapter 10 to determine the Hodge numbers of our examples.

## Contents



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## 1. Construction by degree 2 coverings of a ruled surface

We use similar methods as in [8]. Hence we start by finding curves with complex multiplication. The following theorem yields some examples:

**Theorem 1.1.** *Let $0 < d_1, d < m$, and $\xi_k$ denote a primitive $k$-th. root of unity for all $k \in \mathbb{N}$. Then the curve $C$, which is locally given by*

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

*is covered by the Fermat curve $\mathbb{F}_{(n-2)m}$ locally given by*

$$y^{(n-2)m} + x^{(n-2)m} + 1 = 0$$

*and has complex multiplication.*

*Proof.* (see [8] Theorem 2.4.4) □

**Example 1.2.** By the preceding theorem, the curves locally given by

$$y^4 = x_1^8 + x_0^8, \quad y^4 = x_1(x_1^7 + x_0^7), \quad y^4 = x_1(x_1^6 + x_0^6)x_0$$

have complex multiplication. These curves are degree 4 covers of the projective line and have the genus 9 as one can easily calculate by the Hurwitz formula.

The curves of the preceding example have a natural interpretation as cyclic covers of $\mathbb{P}^1$ of degree 4. One can identify these covers with the set of their 8 branch points in $\mathbb{P}^1$. Thus let $\mathcal{M}_8$ denote the configuration space of 8 different points in $\mathbb{P}^1$.[1] We use a modified version of the construction in [10], Section 5 to construct $K3$ surfaces with complex multiplication by Example 1.2 in a first step. This method is nearly the same method as in [8] Section 8.2.

For our application, it is sufficient to work with $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ resp., with rational ruled surfaces. Let $\pi_n : \mathbb{P}_n \to \mathbb{P}^1$ denote the rational ruled surface given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ and $\sigma$ denote a non-trivial global section of $\mathcal{O}_{\mathbb{P}^1}(8)$, which has the 8 different zero points represented by a point $q \in \mathcal{M}_8$. The sections $E_\sigma$, $E_0$ and $E_\infty$ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(8))$ are induced by

$$\mathrm{id} \oplus \sigma : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(8), \quad \mathrm{id} \oplus 0 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(8)$$

$$\text{and} \quad 0 \oplus \mathrm{id} : \mathcal{O}(8) \to \mathcal{O} \oplus \mathcal{O}(8)$$

resp., by the corresponding surjections onto the cokernels of these embeddings as described in [6], **II**. Proposition 7.12.

**Remark 1.3.** The divisors $E_\sigma$ and $E_0$ intersect each other transversally over the 8 zero points of $\sigma$. Recall that $\mathrm{Pic}(\mathbb{P}_8)$ has a basis given by a fiber and an arbitrary section. Hence by the fact that $E_\sigma$ and $E_0$ do not intersect $E_\infty$, one concludes that they are linearly equivalent with self-intersection number 8. Since $E_\infty$ is a section, it intersects each fiber transversally. Thus one has that $E_\infty \sim E_0 - (E_0.E_0)F$, where $F$ denotes a fiber. Hencefore one concludes

$$E_\infty.E_\infty = E_\infty.(E_0 - (E_0.E_0)F) = -(E_0.E_0) = -8.$$

[1] Note that this is not the same notation as in [8].

Next we establish a morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_8$ over $\mathbb{P}^1$. By [6], **II**. Proposition 7.12., this is the same as to give a surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(8)) \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $\mathbb{P}_2$. By the composition

$$\pi_2^*(\mathcal{O} \oplus \mathcal{O}(8)) = \pi_2^*(\mathcal{O}) \oplus \pi_2^*\mathcal{O}(8) \hookrightarrow \bigoplus_{i=0}^{4} \pi_2^*\mathcal{O}(2i) = Sym^4(\pi_2^*(\mathcal{O} \oplus \mathcal{O}(8))) \to \mathcal{O}_{\mathbb{P}_2}(4),$$

where the last morphism is induced by the natural surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathcal{O}_{\mathbb{P}_2}(1)$ (see [6], **II**. Proposition 7.11), we obtain a morphism $\mu^*$ of sheaves. This morphism $\mu^*$ is not a surjection onto $\mathcal{O}_{\mathbb{P}_2}(4)$, but onto its image $\mathcal{L} \subset \mathcal{O}_{\mathbb{P}_2}(4)$. Over $\mathbb{A}^1 \subset \mathbb{P}^1$ all rational ruled surfaces are locally given by $\mathrm{Proj}(\mathbb{C}[x])[y_1, y_2]$, where $x$ has the weight 0. Hence we have locally that $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(8)) = \mathcal{O}e_1 \oplus \mathcal{O}e_2$. Over $\mathbb{A}^1$ the morphism $\mu^*$ is given by

$$e_1 \to y_1^4, e_2 \to y_2^4$$

such that the sheaf $\mathcal{L} = im(\mu^*) \subset \mathcal{O}_{\mathbb{P}_2}(4)$ is invertible. Thus the morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_8$ corresponding to $\mu^*$ is locally given by the ring homomorphism

$$(\mathbb{C}[x])[y_1, y_2] \to (\mathbb{C}[x])[y_1, y_2] \text{ via } y_1 \to y_1^8 \text{ and } y_2 \to y_2^8.$$

**Construction 1.4.** One has a commutative diagram

$$\begin{array}{ccccc}
\mathcal{Y}' & \xrightarrow{\tau'} & \mathbb{P}_2' & \xrightarrow{\mu'} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta_2} & & \uparrow{\scriptstyle \delta_8} \\
\hat{\mathcal{Y}} & \xrightarrow{\hat{\tau}} & \hat{\mathbb{P}}_2 & \xrightarrow{\hat{\mu}} & \hat{\mathbb{P}}_8 \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_8} \\
\mathcal{Y} & \xrightarrow[\sqrt[2]{\frac{\mu^* E_\sigma}{3\cdot(\mu^* E_0)_{red}}}]{\tau} & \mathbb{P}_2 & \xrightarrow[\sqrt[4]{\frac{E_\infty + 8\cdot F}{E_0}}]{\mu} & \mathbb{P}_8 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_8} \\
\mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1
\end{array}$$

of morphisms between normal varieties with:

(a) $\delta$, $\delta_2$, $\delta_8$, $\rho$, $\rho_2$ and $\rho_8$ are birational.
(b) $\pi$ is a family of curves, $\pi_2$ and $\pi_8$ are $\mathbb{P}^1$-bundles.

*Proof.* One must only explain $\delta_8$ and $\rho_8$. Recall that $E_\sigma$ is a section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(8))$, which intersects $E_0$ transversally in exactly 8 points. The morphism $\rho_8$ is the blowing up of the 8 intersection points of $E_0 \cap E_\sigma$. The preimage of the 8 points given by $q \in \mathcal{M}_8$ with respect to $\pi_8 \circ \rho_8$ consists of the exceptional divisor $\hat{D}_1$ and the proper transform $\tilde{D}_2$ of the preimage of these 8 points with respect to $\rho_8$ given by 8 rational curves with self-intersection number $-1$. The morphism $\delta_8$ is obtained by blowing down $\tilde{D}_2$. □

**Remark 1.5.** The section $\sigma$ has the zero divisor given by some $q \in \mathcal{M}_8$. Hence one obtains $\mu^*(E_\sigma) \cong C$, where $C \to \mathbb{P}^1$ is the cyclic cover of degree 4 as in Example 1.2 ramified over the 8 points given by $\sigma$. The surface $\mathcal{Y}$ is a cyclic degree 2 cover of $\mathbb{P}_2$ ramified over $C$. Thus it has an involution. It is given by the invertible sheaf

$$\mathcal{L} = \omega_{\mathbb{P}_2}^{-1}$$

and the divisor

$$B = \mu^*(E_\sigma), \quad \text{where} \quad \mathcal{O}(B) \cong \mathcal{L}^2,$$

with the notation of [1] **I**. 17. Thus [1] **I**. Lemma 17.1 implies that $\mathcal{Y}$ is a $K3$ surface.

By [8] Lemma 10.4.1, there is only one elliptic curve with a cyclic degree 4 cover onto $\mathbb{P}^1$. Let $\mathbb{E}$ denote this curve, which is locally given by

$$y^4 = x(x-1)^2.$$

One can easily see that $\mathbb{E}$ has the $j$ invariant 1728. Thus $\mathbb{E}$ has complex multiplication.

We introduce a new notation. Let $n \in \mathbb{N}$, let $\xi$ be a fixed primitive $n$-th. root of unity and let $C_1$ and $C_2$ be curves locally given by

$$y^n = f_1(x) \quad \text{and} \quad y^n = f_2(x),$$

where $f_1, f_2 \in \mathbb{C}[x]$. By $(x, y) \to (x, \xi y)$, one can define an automorphism $\gamma_i$ on $C_i$ for $i = 1, 2$. The surface $C_1 \times C_2/\langle(1, 1)\rangle$ is the quotient of $C_1 \times C_2$ by $\langle(\gamma_1, \gamma_2)\rangle$.

**Proposition 1.6.** *The surface $\mathcal{Y}$ is birationally equivalent to $C \times \mathbb{E}/\langle(1, 1)\rangle$.*[2]

*Proof.* Let $\tilde{E}_\bullet$ denote the proper transform of the section $E_\bullet$ with respect to $\rho_8$. Then $\hat{\mu}$ is the Kummer covering given by

$$\sqrt[4]{\frac{\tilde{E}_\infty + 8 \cdot F}{\tilde{E}_0 + \hat{D}_1}},$$

where $\hat{D}_1$ denotes the exceptional divisor of $\rho_8$. Thus the morphism $\mu'$ is the Kummer covering

$$\sqrt[4]{\frac{(\delta_8)_*\tilde{E}_\infty + 8 \cdot (\delta_8)_*F}{(\delta_8)_*\tilde{E}_0 + (\delta_8)_*\hat{D}_1}} = \sqrt[4]{\frac{\mathbb{P}^1 \times \{\infty\} + 8 \cdot (P \times \mathbb{P}^1)}{\mathbb{P}^1 \times \{0\} + \Delta \times \mathbb{P}^1}},$$

where $\Delta$ is the divisor of the 8 different points in $\mathbb{P}^1$ given by $q \in \mathcal{M}_8$ and $P \in \mathbb{P}^1$ is the point with the fiber $F$. Since $E_0 + E_\sigma$ is a normal crossing divisor, $\tilde{E}_\sigma$ neither meets $\tilde{E}_0$ nor $\tilde{D}_2$, where $\tilde{D}_2$ is the proper transform of $\pi_8^*(\Delta)$. Therefore $(\delta_8)_*\tilde{E}_\sigma$ neither meets

$$(\delta_8)_*\tilde{E}_0 = \mathbb{P}^1 \times \{0\} \quad \text{nor} \quad (\delta_8)_*\tilde{E}_\infty = \mathbb{P}^1 \times \{\infty\}.$$

Hence one can choose coordinates in $\mathbb{P}^1$ such that $(\delta_8)_*\tilde{E}_\sigma = \mathbb{P}^1 \times \{1\}$.

By the definition of $\tau$, we obtain that $\hat{\tau}$ is given by

$$\sqrt[2]{\frac{\rho_2^*\mu^*(E_\sigma)}{\rho_2^*\mu^*(E_0)}} = \sqrt[2]{\frac{\hat{\mu}^*(\tilde{E}_\sigma)}{\hat{\mu}^*(\tilde{E}_0)}},$$

and $\tau'$ is given by

$$\sqrt[2]{\frac{\mu'^*(\mathbb{P}^1 \times \{1\})}{\mu'^*(\mathbb{P}^1 \times \{0\})}}.$$

By the fact that the last function is the root of the pullback of a function on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to $\mu'$, it is possible to reverse the order of the field extensions corresponding to $\tau'$ and $\mu'$ such that the resulting varieties obtained by Kummer coverings are birationally equivalent. Hence we have the composition of $\beta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sqrt[2]{\frac{\mathbb{P}^1 \times \{1\}}{\mathbb{P}^1 \times \{0\}}}$$

with

$$\sqrt[4]{\frac{\beta^*(\mathbb{P}^1 \times \{\infty\}) + 8 \cdot (P \times \mathbb{P}^1)}{\beta^*(\mathbb{P}^1 \times \{0\}) + (\Delta \times \mathbb{P}^1)}},$$

which yields the covering variety isomorphic to $\mathbb{E} \times C/\langle(1, 1)\rangle$. □

[2]Similarly to [10], Construction 5.2, we show that $\mathcal{Y}'$ is birationally equivalent to $C \times \mathbb{E}/\langle(1, 1)\rangle$.

As in [8] Section 8.2 we conclude:

**Corollary 1.7.** *If the curve $C$ has complex multiplication, the $K3$-surface $\mathcal{Y}$ has only commutative Hodge groups.*

By the the preceding corollary, our Example 1.2 yields 3 $K3$ surfaces with complex multiplication locally given by

$$y_2^2 + y_1^4 + x_1^8 + x_0^8, \quad y_2^2 + y_1^4 + x_1(x_1^7 + x_0^7), \quad y_2^2 + y_1^4 + x_1(x_1^6 + x_0^6)x_0.$$

**Proposition 1.8.** *For $i = 1, 2$ assume that $C_i$ is a Calabi-Yau $i$-manifold with complex multiplication endowed with the involution $\iota_i$ such that $\iota_i$ acts by $-1$ on $\Gamma(\omega_{C_i})$. By blowing up the singular locus of $C_1 \times C_2/\langle(\iota_1, \iota_2)\rangle$, one obtains a Calabi-Yau 3-manifold with complex multiplication.*

*Proof.* It is well-known that an involution on a Calabi-Yau 2-manifold resp., a $K3$ surface, which acts by $-1$ on $\Gamma(\omega)$, has a smooth divisor of fixed points or it has not any fixed point. Thus the proof follows from [8] Section 7.2. □

Now we need some elliptic curves with complex multiplication:

**Example 1.9.** Elliptic curves with $CM$ has been well studied by number theorists. Some examples of elliptic curves with complex multiplication are given by the following list:

| equation | $j$ invariant |
|---|---|
| $y_1^2x_0 = x_1^3 - x_0^3$ | 0 |
| $y_1^2x_0 = x_1(x_1 - x_0)(x_1 - 2x_0)$ | 1728 |
| $y^2x_0 = x_1(x_1 - x_0)(x_1 - (1 + \sqrt{2})^2x_0)$ | 8000 |
| $y^2x_0 = x_1(x_1 - x_0)(x_1 - \frac{1}{4}(3 + i\sqrt{7})^2x_0)$ | $-3375$ |
| $y^2x_0 = x_1^3 - 15x_1x_0^2 + 22x_0^3$ | 54000 |
| $y^2x_0 = x^3 - 595x_1x_0^2 + 5586x_0^3$ | 16581375 |

Note that the equations allow an explicit definition of an involution on these elliptic curves. (see [8] Section 7.4)

**1.10.** By combining our 3 examples of $K3$ surfaces and the 6 elliptic curves and using Propostion 1.8, we have 18 examples of Calabi-Yau 3-manifolds with complex multiplication. It seems to be quite easy to describe these examples by explicit equations. By [11], one has equations to determine the Hodge numbers of these examples. Let $C_2$ be a $K3$ surface satisfying the assumptions of Proposition 1.8, let $N$ be the number of curves in the ramification locus of the quotient map $C_2 \to C_2/\iota_2$ and let $N'$ be given by

$$N' = g_1 + \ldots + g_N,$$

where $g_i$ denotes the genus of the $i$-th. curve in the ramification locus. Then one has for the Calabi-Yau 3-manifold, which results by Proposition 1.8:

$$h^{1,1} = 11 + 5N - N'$$

$$h^{2,1} = 11 + 5N' - N$$

Thus in our case the Hodge numbers are given by

$$h^{1,1} = 7 \text{ and } h^{2,1} = 55.$$

**Example 2.1.** By Theorem 1.1, the projective curves given by

$$y^6 = x_1^6 + x_0^6, \quad y^6 = x_1(x_1^5 + x_0^5), \quad y^6 = x_1(x_1^4 + x_0^4)x_0$$

have complex multiplication. These curves have the genus 10 as one can easily calculate by the Hurwitz formula.

Let $\mathcal{M}_6$ denote the configuration space of 6 different points in $\mathbb{P}^1$. Again we use a modified version of the construction in [10], Section 5.

Here the sections $E_\sigma$, $E_0$ and $E_\infty$ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(6))$ are induced by

$$\mathrm{id} \oplus \sigma : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6), \quad \mathrm{id} \oplus 0 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6)$$
$$\text{and} \quad 0 \oplus \mathrm{id} : \mathcal{O}(6) \to \mathcal{O} \oplus \mathcal{O}(6)$$

resp., by the corresponding surjections onto the cokernels of these embeddings as described in [6], **II**. Proposition 7.12.

One concludes similarly to the preceding section that

$$E_\infty.E_\infty = E_\infty.(E_0 - (E_0.E_0)F) = -(E_0.E_0) = -6.$$

By the composition

$$\pi_1^*(\mathcal{O} \oplus \mathcal{O}(6)) = \pi_1^*(\mathcal{O}) \oplus \pi_1^*\mathcal{O}(6) \hookrightarrow \bigoplus_{i=0}^{6} \pi_1^*\mathcal{O}(i) = Sym^6(\pi_1^*(\mathcal{O} \oplus \mathcal{O}(6))) \to \mathcal{O}_{\mathbb{P}_1}(6),$$

where the last morphism is induced by the natural surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathcal{O}_{\mathbb{P}_1}(1)$ (see [6], **II**. Proposition 7.11), we obtain a morphism $\mu^*$ of sheaves as in the preceding section. The morphism $\mu : \mathbb{P}_1 \to \mathbb{P}_1$ corresponding to $\mu^*$ is locally given by the ring homomorphism

$$(\mathbb{C}[x])[y_1, y_2] \to (\mathbb{C}[x])[y_1, y_2] \quad \text{via} \quad y_1 \to y_1^6 \quad \text{and} \quad y_2 \to y_2^6.$$

**Construction 2.2.** One has a commutative diagram

$$\begin{array}{ccccc}
\mathcal{Y}' & \xrightarrow{\tau'} & \mathbb{P}_1' & \xrightarrow{\mu'} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta_1} & & \uparrow{\scriptstyle \delta_6} \\
\hat{\mathcal{Y}} & \xrightarrow{\hat{\tau}} & \hat{\mathbb{P}}_1 & \xrightarrow{\hat{\mu}} & \hat{\mathbb{P}}_6 \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_6} \\
\mathcal{Y} & \underset{\sqrt[2]{\frac{\mu^* E_\sigma}{3\cdot(\mu^* E_0)_{red}}}}{\xrightarrow{\tau}} & \mathbb{P}_1 & \underset{\sqrt[6]{\frac{E_\infty + 6\cdot F}{E_0}}}{\xrightarrow{\mu}} & \mathbb{P}_6 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_6} \\
\mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1
\end{array}$$

of morphisms between normal varieties with:

(a) $\delta$, $\delta_1$, $\delta_1$, $\rho$, $\rho_1$ and $\rho_6$ are birational.
(b) $\pi$ is a family of curves, $\pi_1$ and $\pi_6$ are $\mathbb{P}^1$-bundles.

*Proof.* One must only explain $\delta_6$ and $\rho_6$. These morphisms are given by blowing up morphisms similar to the preceding section. ☐

**Remark 2.3.** The section $\sigma$ has the zero divisor given by some $q \in \mathcal{M}_6$. Hence one obtains $\mu^*(E_\sigma) \cong C$, where $C \to \mathbb{P}^1$ is a cyclic cover of degree 6 as in Example 2.1 ramified over the 6 points given by $\sigma$. The surface $\mathcal{Y}$ is a cyclic degree 2 cover of $\mathbb{P}_1$

ramified over $C$. Thus it is birationally equivalent to the $K3$ surface given the degree 2 cover of $\mathbb{P}^2$ ramified over $C$.

Let $C'$ denote the projective smooth curve locally given by

$$y^6 = x(x-1).$$

By Theorem 1.1, it has complex multiplication.

**Proposition 2.4.** *The surface $\mathcal{Y}$ is birationally equivalent to $C \times C'/\langle(1,1)\rangle$.*

*Proof.* Let $\tilde{E}_\bullet$ denote the proper transform of the section $E_\bullet$ with respect to $\rho_6$. Then $\hat{\mu}$ is the Kummer covering given by

$$\sqrt[6]{\frac{\tilde{E}_\infty + 6 \cdot F}{\tilde{E}_0 + \hat{D}_1}},$$

where $\hat{D}_1$ denotes the exceptional divisor of $\rho_6$. Thus the morphism $\mu'$ is the Kummer covering

$$\sqrt[6]{\frac{(\delta_6)_*\tilde{E}_\infty + 6 \cdot (\delta_6)_* F}{(\delta_6)_*\tilde{E}_0 + (\delta_6)_*\hat{D}_1}} = \sqrt[6]{\frac{\mathbb{P}^1 \times \{\infty\} + 6 \cdot (P \times \mathbb{P}^1)}{\mathbb{P}^1 \times \{0\} + \Delta \times \mathbb{P}^1}},$$

where $\Delta$ is the divisor of the 6 different points in $\mathbb{P}^1$ given by $q \in \mathcal{M}_6$ and $P \in \mathbb{P}^1$ is the point with the fiber $F$. Since $E_0 + E_\sigma$ is a normal crossing divisor, $\tilde{E}_\sigma$ neither meets $\tilde{E}_0$ nor $\tilde{D}_2$, where $\tilde{D}_2$ is the proper transform of $\pi_6^*(\Delta)$. Therefore $(\delta_6)_*\tilde{E}_\sigma$ neither meets

$$(\delta_6)_*\tilde{E}_0 = \mathbb{P}^1 \times \{0\} \quad \text{nor} \quad (\delta_6)_*\tilde{E}_\infty = \mathbb{P}^1 \times \{\infty\}.$$

Hence one can choose coordinates in $\mathbb{P}^1$ such that $(\delta_6)_*\tilde{E}_\sigma = \mathbb{P}^1 \times \{1\}$.

By the definition of $\tau$, we obtain that $\hat{\tau}$ is given by

$$\sqrt[2]{\frac{\rho_1^*\mu^*(E_\sigma)}{\rho_1^*\mu^*(E_0)}} = \sqrt[2]{\frac{\hat{\mu}^*(\tilde{E}_\sigma)}{\hat{\mu}^*(\tilde{E}_0)}},$$

and $\tau'$ is given by

$$\sqrt[2]{\frac{\mu'^*(\mathbb{P}^1 \times \{1\})}{\mu'^*(\mathbb{P}^1 \times \{0\})}}.$$

By the fact that the last function is the root of the pullback of a function on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to $\mu'$, it is possible to reverse the order of the field extensions corresponding to $\tau'$ and $\mu'$ such that the resulting varieties obtained by Kummer coverings are birationally equivalent. Hence we have the composition of $\beta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sqrt[2]{\frac{\mathbb{P}^1 \times \{1\}}{\mathbb{P}^1 \times \{0\}}}$$

with

$$\sqrt[6]{\frac{\beta^*(\mathbb{P}^1 \times \{\infty\}) + 6 \cdot (P \times \mathbb{P}^1)}{\beta^*(\mathbb{P}^1 \times \{0\}) + (\Delta \times \mathbb{P}^1)}},$$

which yields the covering variety isomorphic to $C' \times C/\langle(1,1)\rangle$. □

Hence $\mathcal{Y}$ is birationally equivalent to $C' \times C/\langle(1,1)\rangle$. As in [8] Section 8.2 we conclude:

**Corollary 2.5.** *If the curve $C$ has complex multiplication, the $K3$-surface $\mathcal{Y}$ has only commutative Hodge groups.*

**2.6.** By the preceding corollary, our Example 2.1 yields 3 different $K3$ surfaces with complex multiplication as degree 2 covers of $\mathbb{P}^2$, which are locally given by

$$y_2^2 + y_1^6 = x_1^6 + x_0^6, \quad y_2^2 + y_1^6 = x_1(x_1^5 + x_0^5), \quad y_2^2 + y_1^6 = x_1(x_1^4 + x_0^4)x_0.$$

By an elliptic curve with complex multiplication, these $K3$ surfaces yield Calabi-Yau 3-manifolds with complex multiplication. We obtain 18 Calabi-Yau 3-manifolds with complex multiplication by using Example 1.9. By the same methods as in 1.10, one calculates easily that the resulting Calabi-Yau 3-manifolds have the Hodge numbers

$$h^{1,1} = 6 \text{ and } h^{2,1} = 60$$

## 3. Construction by a degree 3 quotient

Consider the $K3$ surface

$$S = V((y_2^3 - y_1^3)y_1 + (x_1^3 - x_0^3)x_0) \subset \mathbb{P}^3.$$

By using the partial derivatives of the defining equation, one can easily verify that $S$ is smooth. First we will prove that this surface has complex multiplication. In a second step we consider an automorphism of degree 3 on this surface, which allows the construction of a Calabi-Yau 3-manifold with complex multiplication.

**Proposition 3.1.** *The $K3$ surface $S$ has complex multiplication.*

*Proof.* Consider the isomorphic curves

$$C_1 = V(z_1^4 - (y_2^3 - y_1^3)y_1) \subset \mathbb{P}^2,$$

$$C_2 = V(z_2^4 - (x_1^3 - x_0^3)x_0) \subset \mathbb{P}^2.$$

Since the elliptic curve with $j$ invariant 0 given by

$$V(y^2x_0 + x_1^3 + x_0^3) \subset \mathbb{P}^2$$

has complex multiplication, one concludes as in [8] Remark 7.4.2 that $C_1$ and $C_2$ have complex multiplication, too. The $K3$ surface $S$ is birationally equivalent to $C_1 \times C_2/\langle(1,1)\rangle$. This follows from the rational map $C_1 \times C_2 \to S$ given by

$$((z_1 : y_2 : y_1), (z_2 : x_1 : x_0)) \to (\frac{z_2}{z_1}y_2 : \frac{z_2}{z_1}y_1 : x_1 : x_0).$$

Thus $S$ has complex multiplication.[3] □

**3.2.** Let $\xi$ denote $e^{\frac{2\pi i}{3}}$. The $K3$ surface $S$ has an automorphism $\gamma$ of degree 3 given by

$$(y_2 : y_1 : x_1 : x_0) \to (\xi y_2 : y_1 : \xi x_1 : x_0).$$

On $\{x_0 = 1\}$ we have the 4 fixed points given by

$$(0 : \sqrt[4]{-1} : 0 : 1).$$

Now assume $x_0 = 0$. This yields

$$(y_2^3 - y_1^3)y_1 = 0.$$

Thus in addition the line given by $y_1 = x_0 = 0$ is fixed.

**Proposition 3.3.** *The automorphism $\gamma$ acts via pullback by $\xi^2$ on $\Gamma(\omega_S)$.*

[3] In [3] Section 5 one finds a similar rational map

*Proof.* The $-1$ eigenspace in $\Gamma(\omega_{C_1})$ and $\Gamma(\omega_{C_2})$ comes from the cohomology of the elliptic curve $E_0$ given by

$$y^2x_0 = x_1^3 - x_0^3$$

(see [8] Section 4.2). By the rational map in the proof of Proposition 3.1, one concludes that $\Gamma(\omega_S)$ is given by tensor product of the $-1$ eigenspace in $\Gamma(\omega_{C_1})$ and $\Gamma(\omega_{C_2})$.

The automorphism $\gamma_{\mathbb{F}_3} : E_0 \to E_0$ given by $x_1 \to \xi x_1$ is the generator of the Galois group of the degree 3 cover, which allows an identification of $E_0$ with the Fermat curve $\mathbb{F}_3$ of degree 3. It acts via pullback by $\xi$ on $\Gamma(\omega_{\mathbb{F}_3})$. Thus the corresponding automorphisms $\varphi_{C_1}$ and $\varphi_{C_2}$ act by $\xi$ on the $-1$ eigenspace with respect to $C_1$ and $C_2$. Note that $(\varphi_{C_1}, \varphi_{C_2})$ yields an automorphism of $(C_1, C_2)/\langle(1,1)\rangle$. By the birational map to $S$, this automorphism corresponds to $\gamma$ and one verifies easily that $\gamma$ acts via pullback by $\xi^2$ on $\Gamma(\omega_S)$. □

**3.4.** Consider the blowing up $\tilde{\mathbb{P}}^3$ of $\mathbb{P}^3$ with respect to $\{y_2 = x_1 = 0\}$. Let $\tilde{S}$ denote the proper transform of blowing up of $S$ with respect to the latter blowing up, which has the exceptional divisor $E$ consisting of four $-1$ curves over the 4 points given by $(0 : \sqrt[4]{-1} : 0 : 1)$. Consider the projection

$$p : S \setminus \{y_2 = x_1 = 0\} \hookrightarrow \mathbb{P}^3 \setminus \{y_2 = x_1 = 0\} \to \mathbb{P}^1 \text{ given by } (y_2 : y_1 : x_1 : x_0) \to (y_2 : x_1).$$

Over $\{x_0 = 1\}$ one has an embedding of an open subset of $\tilde{\mathbb{P}}^3$ into $\mathbb{P}^1 \times \mathbb{A}^3$, which yields an open embedding $e$ of an open subset $U$ of $\tilde{S}$ into $\mathbb{P}^1 \times \mathbb{A}^3$. Note that $\mathbb{P}^1 \times \mathbb{A}^3$ is endowed with a natural projection $pr_1 : \mathbb{P}^1 \times \mathbb{A}^3 \to \mathbb{P}^1$. Over $U \setminus \{y_2 = x_1 = 0\}$ one has

$$p = pr_1 \circ e.$$

Thus by glueing, $p$ extends to a morphism $\tilde{S} \to \mathbb{P}^1$, which is a family of projective curves of degree 4. This family has a section $D = \{y_1 = x_0 = 0\}$. One checks easily the singular loci of the fibers do not meet $D$. By $\tilde{S} \times \mathbb{F}_3 \to \mathbb{P}^1$, we have a family of surfaces. Let $\gamma_{\mathbb{F}_3}$ denote the generator of the Galois group of $\mathbb{F}_3 \to \mathbb{P}^1$, which acts via pullback by $\xi$ on $\omega_{\mathbb{F}_3}$. The quotient map onto $\tilde{S} \times \mathbb{F}_3/(\gamma, \gamma_{\mathbb{F}_3})$ yields a quotient singularity of type $A_{3,2}$ (with the notation of [1]). As in [8] Section 9.2 described one must blow up $D$ and in a second step one blows up the fixed locus of the exceptional divisor over $D$. By blowing down the image of the proper transform of the exceptional divisor over $D$, one obtains a Calabi-Yau 3-manifold, which has obviously complex multiplication.

**3.5.** The automorphism $\gamma$ acts on $\tilde{S}$, too. The quotient map $\varphi$ onto $M = \tilde{S}/\gamma$ is ramified over $E$ and $D = \{y_1 = x_0 = 0\}$. Since $D$ is a rational curve on a $K3$ surface, the adjunction formula implies that $D.D = -2$. By the Hurwitz formula, one has

$$\varphi^* K_M \sim -2D - E.$$

Since

$$3 \cdot K_M^2 = (\varphi^* K_M)^2,$$

one concludes that

$$c_1(M)^2 = K_M^2 = -4.$$

Thus the Noether formula tells us that $b_2(M) = 14$. Since we have blown up 4 points, one concludes that $h_0^{1,1}(S) = 10$. Thus

$$h_1^{1,1}(S) = h_2^{1,1}(S) = 5.$$

By the fact that one has an exceptional divisor consisting of 12 copies of $\mathbb{P}^2$ and 6 rational ruled surfaces, one obtains similarly to [8] Section 10.3:

$$h^{1,1}(X) = h_0^{1,1}(S) + h_0^{1,1}(E) + 18 = 29$$

$$h^{2,1}(X) = h_1^{1,0}(E) \cdot h_2^{1,1}(S) = 5$$

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