

DISCRETENESS AND RATIONALITY OF F-THRESHOLDS

MANUEL BLICKLE, MIRCEA MUSTAȚĂ, AND KAREN E. SMITH

ABSTRACT. The F -thresholds are characteristic p analogues of the jumping coefficients for multiplier ideals in characteristic zero. In this article we give an alternative description of the F -thresholds of an ideal in a regular and F -finite ring R . This enables us to settle two open questions posed in [MTW], namely we show that the F -thresholds are rational and discrete.

1. INTRODUCTION

In recent years multiplier ideals have played an important role in higher dimensional birational geometry. For a given ideal \mathfrak{a} on a smooth variety X , and a real parameter $c > 0$, the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is defined via a log resolution $\pi : X' \rightarrow X$ of the pair (X, \mathfrak{a}) such that $\pi^{-1}(\mathfrak{a})$ defines a simple normal crossing divisor $A = \sum_{i=1}^r a_i E_i$. Then one has

$$(1) \quad \mathcal{J}(\mathfrak{a}^c) := \pi_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor cA \rfloor),$$

and this is an ideal of \mathcal{O}_X that does not depend on the chosen log resolution. A jumping coefficient of \mathfrak{a} is a positive real number c such that $\mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c-\varepsilon})$ for every $\varepsilon > 0$. These invariants have been introduced and studied in [ELSV]. It follows from the formula (1) that if c is a jumping coefficient, then $c \cdot a_i$ is an integer for some i . In particular, every jumping coefficient is a rational number, and the set of jumping coefficients of a given ideal is discrete.

Hara and Yosida introduced in [HY] a positive characteristic analog of multiplier ideals, denoted by $\tau(\mathfrak{a}^c)$. This is a generalized test ideal for a tight closure theory with respect to the pair (X, \mathfrak{a}^c) . One can define similarly jumping coefficients for these test ideals. These invariants were studied under the name of F -thresholds in [MTW], where it was shown that they satisfy many of the formal properties of the jumping coefficients in characteristic zero.

We emphasize that the test ideals are not determined by a log resolution of singularities, even in the cases when such a resolution is known to exist. Instead, the definition uses the Frobenius morphism and requires a priori infinitely many conditions to be checked. This lack of built in finiteness makes the question of rationality and discreteness of the F -thresholds non-trivial, and in fact these properties were left open in [MTW].

2000 *Mathematics Subject Classification*. Primary: 13A35; Secondary: 14B05.

Key words and phrases. F -thresholds, test ideals, jumping coefficients.

The first author was partially supported by the *DFG Schwerpunkt Komplexe Geometrie*. The second author was partially supported by the NSF under grant DMS 0500127.

In the present paper we settle these questions in the case of a ring R that is essentially of finite type over a field: more precisely, we show that for every ideal in such a ring, all F -thresholds are rational and they form a discrete set. Our key result, which implies these two statements, is a finiteness statement for test ideals in a polynomial ring. We show in Proposition 3.2 below that if \mathfrak{a} is an ideal in $k[x_1, \dots, x_n]$ that is generated in degree $\leq d$, then the test ideal $\tau(\mathfrak{a}^e)$ is generated in degree $\leq \lfloor c \cdot d \rfloor$. As far as we know, the analogous statement for multiplier ideals in characteristic zero is not known.

The structure of the paper is as follows. In Section 2 we recall the definition of test ideals and F -thresholds, slightly generalizing the setup in [HY] and [MTW] beyond the case of a local ring. We prefer to work with a different definition of test ideals that is more suitable for our purpose, and show later that in the case of a local ring we recover the definition from [HY]. Due to this alternative definition, and to the slightly different setup, we decided to develop the theory from scratch. We hope that this would be beneficial for the reader, as with the present definition the basic results become particularly transparent. After this initial setup, we prove our main results in Section 3.

2. GENERALIZED TEST IDEALS AND F -THRESHOLDS

In this section we collect some basic results about generalized test ideals as introduced by Hara and Yoshida [HY] (see also [Tak] and [HT]). We restrict ourselves to working over an F -finite regular ring which allows us to use an alternative description of these ideals. We subsequently prove that – at least in the local case – our ideals agree with the ones defined in [HY]. One feature of our construction is that we do not need to work over a local ring and that it is immediately clear (Lemma 2.6) that the test ideals globalize. We also point out that the analogues of the classical theorems about multiplier ideals hold in the characteristic $p > 0$ setting. We include the Restriction Theorem (Remark 2.22), Skoda's Theorem (Proposition 2.20) or the Subadditivity Theorem (Lemma 2.10(4)), all with very elementary proofs.

We also recall the definition of F -thresholds from [MTW], though again, we do not restrict to the case of a *local* ring as in *loc. cit.* In this more general framework we show that the set of F -thresholds coincides with the set of jumping coefficients for the test ideals.

Let us fix the following notation: R denotes a regular F -finite ring of positive characteristic. We stress here that R is not assumed to be local. The regularity of R implies by Kunz [Ku] that the Frobenius morphism $F: R \rightarrow R$ sending $x \in R$ to x^p is flat. This is equivalent to saying that the functor $(F^e)^*$ of extending scalars via F^e is exact. By our assumption that R is F -finite we just mean that F is a finite morphism. Since F is also flat, it follows that R is a finitely generated locally free module over its subring R^p of p^{th} powers.

For an ideal J of R and a positive integer e we put $J^{[p^e]} := (u^{p^e} | u \in J)$ to be the ideal generated by the p^{th} powers of the elements of J . If $J = (u_1, \dots, u_r)$, then $J^{[p^e]} = (u_1^{p^e}, \dots, u_r^{p^e})$. One easily verifies that $(F^e)^*(R/J) \cong R/(J^{[p^e]})$. In particular, since F is (faithfully) flat we see that if u is in R , then $u^p \in J^{[p]}$ if and only if $u \in J$.

Example 2.1. For a field k being F -finite precisely means that $[k : k^p] < \infty$. One easily verifies, by explicitly giving a basis of R over R^p , that for such fields the polynomial ring $k[x_1, \dots, x_n]$ and even the power series ring $k[[x_1, \dots, x_n]]$ in finitely many variables are F -finite.

More generally, if A is an F -finite ring, then every A -algebra that is essentially of finite type over A is again F -finite. To check this observe that if S is a multiplicative system in a ring R , and if a_1, \dots, a_N generate R over R^p , then $\frac{a_1}{1}, \dots, \frac{a_N}{1}$ generate $S^{-1}R$ over $(S^{-1}R)^p$.

Conversely, if k is a field and if R is a k -algebra essentially of finite type over k that is F -finite, then $[k : k^p] < \infty$. Indeed, if \mathfrak{m} is a maximal ideal of R and $K = R/\mathfrak{m}$, then $[K : K^p] < \infty$. Since $[K : k] < \infty$, we deduce that $[k : k^p] < \infty$.

If (R, \mathfrak{m}) is a regular local ring that is F -finite, then its completion \widehat{R} is also F -finite (and regular). In fact, since R is local, R is free over R^p . If a_1, \dots, a_N give a basis of R over R^p , we claim that these elements also give a basis of \widehat{R} over \widehat{R}^p . Indeed, we have canonical isomorphisms

$$F^*(\widehat{R}) = F^*(\operatorname{proj} \lim_{\ell} R/\mathfrak{m}^{\ell}) \simeq \operatorname{proj} \lim_{\ell} F^*(R/\mathfrak{m}^{\ell}) \simeq \operatorname{proj} \lim_{\ell} R/(\mathfrak{m}^{\ell})^{[p]} = \widehat{R}$$

(note that since F is finite, F^* commutes with the above projective limit). Therefore the Frobenius morphism on \widehat{R} is obtained by base extension from the Frobenius morphism on R , which implies our claim.

2.1. The ideals $I_{r,e}(\mathfrak{a})$. Let \mathfrak{a} be an ideal of R . We now introduce certain ideals $I_{r,e}(\mathfrak{a})$ which are crucial for our construction of the generalized test ideals $\tau(\mathfrak{a}^c)$ with exponent $c \in \mathbb{R}_{\geq 0}$. These ideals $I_{r,e}(\mathfrak{a})$ were first introduced in the case of a principal ideal in [AMBL] to study the D -module structure of a localization of R . Also in [AMBL] it was predicted that these ideals might be related to tight closure theory and the present work confirms this prediction. Furthermore, similar ideas also appear in the description of generalized test ideals in [HT, Lemma 2.1].

Definition 2.2. For an ideal \mathfrak{a} of R and integers $r, e \geq 0$ let $I_{r,e}(\mathfrak{a})$ be the unique smallest ideal J of R with respect to inclusion, such that

$$\mathfrak{a}^r \subseteq J^{[p^e]}.$$

By the flatness of the Frobenius morphism, for every family of ideals $\{J_i\}_i$ we have $(\cap_i J_i)^{[p^e]} = \cap_i J_i^{[p^e]}$. Hence $I_{r,e}(\mathfrak{a})$ is well-defined. The following lemma collects some basic properties of these ideals.

Lemma 2.3. *Let \mathfrak{a} and \mathfrak{b} be ideals in R .*

- (1) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $I_{r,e}(\mathfrak{a}) \subseteq I_{r,e}(\mathfrak{b})$.*
- (2) *If $r' \geq r$, then $I_{r',e}(\mathfrak{a}) \subseteq I_{r,e}(\mathfrak{a})$.*
- (3) *$I_{r,e}(\mathfrak{a} \cap \mathfrak{b}) \subseteq I_{r,e}(\mathfrak{a}) \cap I_{r,e}(\mathfrak{b})$ and $I_{r,e}(\mathfrak{a}) + I_{r,e}(\mathfrak{b}) \subseteq I_{r,e}(\mathfrak{a} + \mathfrak{b})$.*
- (4) *$I_{r,e}(\mathfrak{a} \cdot \mathfrak{b}) \subseteq I_{r,e}(\mathfrak{a}) \cdot I_{r,e}(\mathfrak{b})$.*
- (5) *$I_{r,e}(\mathfrak{a}) \subseteq I_{rp,e+1}(\mathfrak{a})$.*

Proof. The assertions in (1) and (2) are straightforward from definition, and (3) follows from (1). For (4), note that since we have $\mathfrak{a}^r \subseteq I_{r,e}(\mathfrak{a})^{[p^e]}$ and $\mathfrak{b}^r \subseteq I_{r,e}(\mathfrak{b})^{[p^e]}$, we deduce

$$(\mathfrak{a} \cdot \mathfrak{b})^r \subseteq I_{r,e}(\mathfrak{a})^{[p^e]} \cdot I_{r,e}(\mathfrak{b})^{[p^e]} = (I_{r,e}(\mathfrak{a}) \cdot I_{r,e}(\mathfrak{b}))^{[p^e]}.$$

The inclusion in (4) then follows from the definition of $I_{r,e}(\mathfrak{a} \cdot \mathfrak{b})$.

For the inclusion in (5), it is enough to show that $\mathfrak{a}^r \subseteq I_{rp,e+1}(\mathfrak{a})^{[p^e]}$, and this is furthermore equivalent with $(\mathfrak{a}^r)^{[p]} \subseteq I_{rp,e+1}(\mathfrak{a})^{[p^{e+1}]}$. This is a consequence of the definition of $I_{rp,e+1}(\mathfrak{a})$ and of the fact that $(\mathfrak{a}^r)^{[p]} \subseteq \mathfrak{a}^{rp}$. \square

When R is free over R^{p^e} one has the following alternative description of these newly defined ideals. Again, in the case of a principal ideal this was already observed in [AMBL].

Proposition 2.4. *Suppose that e_1, \dots, e_N is a basis of R over R^{p^e} and let h_1, \dots, h_s be generators of the r^{th} power \mathfrak{a}^r of an ideal \mathfrak{a} of R . For every $i = 1, \dots, s$ write*

$$(2) \quad h_i = \sum_{j=1}^N a_{i,j}^{p^e} e_j$$

with $a_{i,j} \in R$. Then $I_{r,e}(\mathfrak{a}) = (a_{i,j} \mid i \leq s, j \leq N)$.

Note that the proposition implies in particular that the description therein does not depend on the chosen basis of R over R^{p^e} or on the generators of \mathfrak{a}^r .

Proof of Proposition 2.4. It follows from (2) that $(h_1, \dots, h_s) \subseteq (a_{i,j}^{p^e} \mid i \leq s, j \leq N)$ and therefore $\mathfrak{a}^r \subseteq (a_{i,j} \mid i, j)^{[p^e]}$. Hence the inclusion " \subseteq " in the proposition follows immediately from the definition of $I_{r,e}(\mathfrak{a})$.

For the reverse inclusion, suppose that $\mathfrak{a}^r \subseteq J^{[p^e]}$. If g_1, \dots, g_m generate J , we may write

$$(3) \quad h_i = \sum_{\ell=1}^m b_\ell g_\ell^{p^e}$$

for some $b_\ell \in R$. Consider the dual basis e_1^*, \dots, e_N^* for $\text{Hom}_{R^{p^e}}(R, R^{p^e})$, so $e_i^*(e_j) = \delta_{i,j}$. It follows from (2) that $e_j^*(h_i) = a_{i,j}^{p^e}$. On the other hand, (3) shows that

$$e_j^*(h_i) = \sum_{\ell} g_\ell^{p^e} e_j^*(b_\ell) \in J^{[p^e]}.$$

Therefore $a_{i,j} \in J$ for every i and j , which gives $(a_{i,j} \mid i \leq s, j \leq N) \subseteq I_{r,e}(\mathfrak{a})$. \square

Remark 2.5. In [AMBL] it is shown that the ideals $I_{r,e}(\mathfrak{a})$ have the following D -module theoretic description. If $D_R^{(e)}$ denotes the subring of the ring of differential operators D_R which are linear over R^{p^e} , then $I_{r,e}(\mathfrak{a})^{[p^e]}$ is equal to the $D_R^{(e)}$ -submodule of R generated by \mathfrak{a} . This interesting viewpoint will not be exploited further here, but it might be helpful for generalizing our main results in the next section to the case of a ring that is not essentially of finite type over a field.

The following lemma shows that the formation of the ideals $I_{r,e}(\mathfrak{a})$ commutes with localization and completion. In particular, in order to compute $I_{r,e}(\mathfrak{a})$ we may always localize so that R is free over R^{p^e} and then use Proposition 2.4.

Lemma 2.6. *Let \mathfrak{a} be an ideal in R .*

- (1) *If S is a multiplicative system in R , then $I_{r,e}(S^{-1}\mathfrak{a}) = S^{-1}I_{r,e}(\mathfrak{a})$.*
- (2) *If R is local and \widehat{R} is the completion of R , then $I_{r,e}(\mathfrak{a}\widehat{R}) = I_{r,e}(\mathfrak{a})\widehat{R}$.*

Proof. For (1), note that $\mathfrak{a}^r \subseteq I_{r,e}^{[p^e]}$ implies

$$(S^{-1}\mathfrak{a})^r \subseteq S^{-1}(I_{r,e}(\mathfrak{a})^{[p^e]}) = (S^{-1}I_{r,e}(\mathfrak{a}))^{[p^e]}.$$

Therefore $I_{r,e}(S^{-1}\mathfrak{a}) \subseteq S^{-1}I_{r,e}(\mathfrak{a})$.

For the reverse inclusion, write $I_{r,e}(S^{-1}(\mathfrak{a})) = S^{-1}J$, for some ideal J such that $(J:s) = J$ for every $s \in S$. Using the flatness of F^e we see that $(J^{[p^e]}:s^{p^e}) = J^{[p^e]}$ for every $s \in S$, hence $(J^{[p^e]}:s) = J^{[p^e]}$. Since

$$S^{-1}\mathfrak{a}^r \subseteq (S^{-1}J)^{[p^e]} = S^{-1}(J^{[p^e]}),$$

it follows that $\mathfrak{a}^r \subseteq J^{[p^e]}$. Therefore $I_{r,e}(\mathfrak{a}) \subseteq J$, which gives " \supseteq " in (1).

For (2) we will use Proposition 2.4. Since R is local, R is free over R^{p^e} . Moreover, one shows as in Example 2.1 that if e_1, \dots, e_N give a basis of R over R^{p^e} , then these elements also give a basis of \widehat{R} over $(\widehat{R})^{p^e}$, and the assertion in (2) follows from Proposition 2.4. \square

2.2. Generalized test ideals. We will now use the ideals $I_{r,e}(\mathfrak{a})$ to define the generalized test ideals of Hara and Yoshida [HY]. In Proposition 2.18 it is shown that, at least in the local case, our definition coincides with that in [HY].

Lemma 2.7. *Let \mathfrak{a} be an ideal in R . If r, r', e and e' are such that $\frac{r}{p^e} \geq \frac{r'}{p^{e'}}$ and $e' \geq e$, then*

$$I_{r,e}(\mathfrak{a}) \subseteq I_{r',e'}(\mathfrak{a}).$$

Proof. Note that $r' \leq p^{e'-e}r$. It follows from Lemma 2.3 (2) and (5) that

$$I_{r',e'}(\mathfrak{a}) \supseteq I_{r p^{e'-e}, e'}(\mathfrak{a}) \supseteq I_{r,e}(\mathfrak{a}).$$

\square

Let \mathfrak{a} be an ideal in R and let c be a non-negative real number. If we denote by $[x]$ the smallest integer $\geq x$, then for every e we have $\frac{[cp^e]}{p^e} \geq \frac{[cp^{e+1}]}{p^{e+1}}$. It follows from Lemma 2.7 that

$$I_{[cp^e],e}(\mathfrak{a}) \subseteq I_{[cp^{e+1}],e+1}(\mathfrak{a}).$$

Definition 2.8. In the setup of the preceding paragraph we define the generalized test ideal of \mathfrak{a} with exponent c to be

$$\tau(\mathfrak{a}^c) = \bigcup_{e>0} I_{[cp^e],e}(\mathfrak{a}).$$

Since R is Noetherian, it follows that this union stabilizes after finitely many steps such that for $e \gg 0$ the test ideal $\tau(\mathfrak{a}^c) = I_{\lceil cp^e \rceil, e}(\mathfrak{a})$.

Remark 2.9. We can write $R = R_1 \times \cdots \times R_m$ where all R_i are (F -finite) regular domains. An ideal \mathfrak{a} in R can be written as $\mathfrak{a} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_m$ and it is clear that for every c we have $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}_1^c) \times \cdots \times \tau(\mathfrak{a}_m^c)$. This allows us to assume that R is a domain whenever this is convenient.

Lemma 2.10. *Let \mathfrak{a} and \mathfrak{b} be ideals of R .*

- (1) *If $c_1 < c_2$, then $\tau(\mathfrak{a}^{c_2}) \subseteq \tau(\mathfrak{a}^{c_1})$.*
- (2) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^c) \subseteq \tau(\mathfrak{b}^c)$.*
- (3) *We have $\tau((\mathfrak{a} \cap \mathfrak{b})^c) \subseteq \tau(\mathfrak{a}^c) \cap \tau(\mathfrak{b}^c)$ and $\tau(\mathfrak{a}^c) + \tau(\mathfrak{b}^c) \subseteq \tau((\mathfrak{a} + \mathfrak{b})^c)$.*
- (4) *We have $\tau((\mathfrak{a} \cdot \mathfrak{b})^c) \subseteq \tau(\mathfrak{a}^c) \cdot \tau(\mathfrak{b}^c)$.*

Proof. Lemma 2.3(2) implies that $I_{\lceil c_2 p^e \rceil, e}(\mathfrak{a}) \subseteq I_{\lceil c_1 p^e \rceil, e}(\mathfrak{a})$. By taking $e \gg 0$, we get the assertion in (1). The other assertions follow similarly, by taking the limit in the corresponding assertions from Lemma 2.3. \square

Remark 2.11. The inclusion in (4) above is the analogue of the Subadditivity Theorem for multiplier ideals in characteristic zero (see [Laz], Theorem 9.5.20). See also Theorem 6.10 in [HY] for a different approach.

A direct application of Lemma 2.6 (for $e \gg 0$ and $r = \lceil cp^e \rceil$) shows that the formation of test ideals commutes with localization and completion.

Proposition 2.12. *Let \mathfrak{a} be an ideal in R and c a non-negative real number.*

- (1) *If S is a multiplicative system in R , then $\tau((S^{-1}\mathfrak{a})^c) = S^{-1}\tau(\mathfrak{a}^c)$.*
- (2) *If R is local and \widehat{R} is the completion of R , then $\tau((\mathfrak{a}\widehat{R})^c) = \tau(\mathfrak{a}^c)\widehat{R}$.*

We now proceed to show that the family of test ideals $\tau(\mathfrak{a}^c)$ of a fixed ideal \mathfrak{a} is right continuous in c .

Proposition 2.13. *If \mathfrak{a} is an ideal in R and c is a non-negative real number, then there is $\varepsilon > 0$ such that $\tau(\mathfrak{a}^c) = I_{r, e}(\mathfrak{a})$ whenever $c < \frac{r}{p^e} < c + \varepsilon$.*

Proof. We show first that there is $\varepsilon > 0$ and an ideal I in R such that $I_{r, e}(\mathfrak{a}) = I$ whenever $c < \frac{r}{p^e} < c + \varepsilon$. Indeed, otherwise we can find r_m and e_m for $m \geq 1$ such that $\frac{r_m}{p^{e_m}}$ form a strictly decreasing sequence converging to c and $I_{r_m, e_m}(\mathfrak{a}) \neq I_{r_{m+1}, e_{m+1}}(\mathfrak{a})$ for every m . After replacing this sequence by a subsequence we may assume that $e_m \leq e_{m+1}$ for every m . By Lemma 2.7, we have $I_{r_m, e_m}(\mathfrak{a}) \subseteq I_{r_{m+1}, e_{m+1}}(\mathfrak{a})$ for every m . Since this sequence of ideals does not stabilize, we contradict the fact that R is Noetherian. Therefore we can find an ideal I as required.

We show now that $I = \tau(\mathfrak{a}^c)$. By Remark 2.9 we may clearly assume that R is a domain. If $\mathfrak{a} = (0)$, then $I_{r, e}(\mathfrak{a}) = (0)$ for every r and e , so our assertion is trivial. We assume henceforth that $\mathfrak{a} \neq (0)$. Let e be large enough such that $\tau(\mathfrak{a}^c) = I_{\lceil cp^e \rceil, e}(\mathfrak{a})$ and $\frac{\lceil cp^e \rceil}{p^e} < c + \varepsilon$. If cp^e is not an integer, then $\frac{\lceil cp^e \rceil}{p^e} > c$, and we get $I = \tau(\mathfrak{a}^c)$.

Suppose now that cp^e is an integer. After possibly replacing e by a larger value, we may assume that $c + \frac{1}{p^e} < c + \varepsilon$, so $I = I_{cp^e+1,e} \subseteq I_{cp^e,e} = \tau(\mathfrak{a}^c)$. For the reverse inclusion we need to show that $\mathfrak{a}^{cp^e} \subseteq I^{[p^e]}$. Let $u \in \mathfrak{a}^{cp^e}$. If $e' \geq e$, then $\frac{cp^{e'}+1}{p^{e'}} < c + \varepsilon$, hence $\mathfrak{a}^{cp^{e'}+1} \subseteq I^{[p^{e}]}$. We deduce that if v is a nonzero element in \mathfrak{a} , then for every $e' \gg 0$ we have $vu^{p^{e'}-e} \in (I^{[p^e]})^{[p^{e'}-e]}$. This says that u is in the tight closure of the ideal $I^{[p^e]}$, which is equal to $I^{[p^e]}$ since R is a regular ring (see [HH]). Therefore we conclude that $u \in I^{[p^e]}$. To avoid the explicit appearance of tight closure one could first reduce to the local case and use the freeness of R over $R^{p^{e'}}$ to find, for $e' \gg 0$, a splitting $\varphi : R \rightarrow R$ of the e' th iterate of the Frobenius $F^{e'} : R \rightarrow R$ such that $\varphi(v) = 1$. Applying φ to an equation witnessing the membership $vu^{p^{e'}-e} \in (I^{[p^e]})^{[p^{e'}-e]}$ then immediately shows that $u \in I^{[p^e]}$. \square

Corollary 2.14. *If m is a positive integer and $\mathfrak{b} = \mathfrak{a}^m$, then for every $c \in \mathbb{R}_{\geq 0}$ we have*

$$\tau(\mathfrak{b}^c) = \tau(\mathfrak{a}^{cm}).$$

Proof. It is clear that $I_{r,e}(\mathfrak{b}) = I_{rm,e}(\mathfrak{a})$ for every r and e . Let e be large enough such that

$$\tau(\mathfrak{b}^c) = I_{\lceil cp^e \rceil m, e}(\mathfrak{a}), \text{ and } \tau(\mathfrak{a}^{cm}) = I_{\lceil cmp^e \rceil, e}(\mathfrak{a}).$$

If for some e we have $cp^e \in \mathbb{Z}$, then our assertion is clear. If this is not the case, then for $e \gg 0$ we have $\frac{\lceil cp^e \rceil m}{p^e}$ larger than cm , but close to cm , and we conclude by Proposition 2.13. \square

Corollary 2.15. *For every ideal \mathfrak{a} in R and every non-negative real number c , there is $\varepsilon > 0$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$ for every $c' \in [c, c + \varepsilon)$.*

Proof. It is clear that we may take ε as given by Proposition 2.13. \square

Definition 2.16. A positive real number c is a *jumping exponent* for \mathfrak{a} if $\tau(\mathfrak{a}^c) \neq \tau(\mathfrak{a}^{c-\varepsilon})$ for every positive ε .

Unless explicitly mentioned otherwise, we make the convention that 0 is also a jumping exponent. We will study the basic properties of these numbers in the next section.

Remark 2.17. Suppose that K/k is an extension of perfect fields, and consider the ring extension $R = k[x_1, \dots, x_n] \subseteq S = K[x_1, \dots, x_n]$. If \mathfrak{a} is an ideal in R , then $\tau(\mathfrak{a}^c) \cdot S = \tau((\mathfrak{a} \cdot S)^c)$. Indeed, the monomials of degree at most $p^e - 1$ in each variable give a basis of both R and S over R^{p^e} and S^{p^e} , respectively. It follows from Proposition 2.4 that for every r and e we have $I_{r,e}(\mathfrak{a}) \cdot S = I_{r,e}(\mathfrak{a} \cdot S)$. In particular, we see that \mathfrak{a} and $\mathfrak{a} \cdot S$ have the same jumping exponents.

We show now that at least when R is local, the ideals we have defined coincide with the ideals introduced in [HY]. Suppose that (R, \mathfrak{m}) is local of dimension n and let E denote an injective hull of the residue field of R . We identify E with the top local cohomology module $H_{\mathfrak{m}}^n(R)$. The Frobenius morphism on R induces by functoriality a Frobenius action on $H_{\mathfrak{m}}^n(R)$ and hence also on E . This action, also denoted by F , can be described as follows. If x_1, \dots, x_n form a minimal system of generators of \mathfrak{m} then

$$E \simeq R_{x_1 \cdots x_n} / \sum_{i=1}^n R_{x_1 \cdots \widehat{x}_i \cdots x_n}.$$

Under this isomorphism, the Frobenius morphism on E is induced by the Frobenius morphism on $R_{x_1 \cdots x_n}$.

For every $e \geq 1$ and $r \geq 0$, Hara and Yoshida define

$$Z_{r,e}(\mathfrak{a}) := \{u \in E \mid \mathfrak{a}^r \cdot F^e(u) = 0\}.$$

The generalized test ideal of exponent c , that we temporarily denote by $\tau'(\mathfrak{a}^c)$, is obtained as follows

$$\tau'(\mathfrak{a}^c) := \bigcup_{e \geq 1} \text{Ann}_R Z_{\lceil cp^e \rceil, e}(\mathfrak{a}).$$

In fact, the definition in [HY] is given for rational numbers c , but there is no need for this restriction.

Proposition 2.18. *If \mathfrak{a} is an ideal in a regular, F -finite local ring R , then for every non-negative real number c we have $\tau(\mathfrak{a}^c) = \tau'(\mathfrak{a}^c)$.*

Proof. It is clearly enough to show that for every r and e we have

$$I_{r,e}(\mathfrak{a}) = \text{Ann}_R Z_{r,e}(\mathfrak{a}).$$

We note first that for every η in E , we have $\text{Ann}_R F^e(\eta) = (\text{Ann}_R(\eta))^{[p^e]}$. Indeed, let us consider a regular system of parameters x_1, \dots, x_n in R and write η as the class of $\frac{w}{(x_1 \cdots x_n)^N}$. In this case we have $\text{Ann}_R(\eta) = ((x_1^N, \dots, x_n^N) : w)$. Since the Frobenius morphism on R is flat, forming a colon ideal commutes with Frobenius powers, hence

$$\text{Ann}_R F^e(\eta) = ((x_1^{Np^e}, \dots, x_n^{Np^e}) : w^{p^e}) = (\text{Ann}_R(\eta))^{[p^e]}.$$

We deduce that

$$Z_{r,e}(\mathfrak{a}) = \{\eta \in E \mid \mathfrak{a}^r \subseteq (\text{Ann}_R(\eta))^{[p^e]}\} = \{\eta \in E \mid I_{r,e}(\mathfrak{a}) \subseteq \text{Ann}_R(\eta)\}.$$

Therefore $Z_{r,e}(\mathfrak{a}) = \text{Ann}_E I_{r,e}(\mathfrak{a})$, and we get $I_{r,e}(\mathfrak{a}) = \text{Ann}_R Z_{r,e}(\mathfrak{a})$ by Matlis duality. \square

Remark 2.19. The ideals in [HY] have been defined in a global setting, but the fact that they commute with localization was proved there only under an additional assumption. Whenever this assumption is satisfied, we get $\tau(\mathfrak{a}^c) = \tau'(\mathfrak{a}^c)$.

For future reference, we include the following characteristic p version of Skoda's Theorem, due to Hara and Takagi [HT]. However, since the result in *loc. cit.* is not stated in the generality we will need, we include a proof for the benefit of the reader.

Proposition 2.20. *If \mathfrak{a} is an ideal generated by m elements, then for every $c \geq m$ we have*

$$\tau(\mathfrak{a}^c) = \mathfrak{a} \cdot \tau(\mathfrak{a}^{c-1}).$$

Proof. If e is large enough, then $\tau(\mathfrak{a}^c) = I_{\lceil cp^e \rceil, e}(\mathfrak{a})$ and $\tau(\mathfrak{a}^{c-1}) = I_{\lceil cp^e \rceil - p^e, e}(\mathfrak{a})$. Therefore it is enough to show that for every $r \geq mp^e$ we have

$$I_{r,e}(\mathfrak{a}) = \mathfrak{a} \cdot I_{r-p^e, e}(\mathfrak{a}).$$

The inclusion $\mathfrak{a} \cdot I_{r-p^e, e}(\mathfrak{a}) \subseteq I_{r, e}(\mathfrak{a})$ holds in fact for every $r \geq p^e$. Indeed, this says that $I_{r-p^e, e}(\mathfrak{a}) \subseteq (I_{r, e}(\mathfrak{a}) : \mathfrak{a})$, which is equivalent with

$$\mathfrak{a}^{r-p^e} \subseteq (I_{r, e}(\mathfrak{a}) : \mathfrak{a})^{[p^e]} = (I_{r, e}(\mathfrak{a})^{[p^e]} : \mathfrak{a}^{[p^e]}).$$

This holds since $\mathfrak{a}^{r-p^e} \cdot \mathfrak{a}^{[p^e]} \subseteq \mathfrak{a}^r \subseteq I_{r, e}(\mathfrak{a})^{[p^e]}$.

Suppose now that $\mathfrak{a} = (h_1, \dots, h_m)$. In order to prove the reverse inclusion, note that if $r \geq m(p^e - 1) + 1$, then in the product of r of the h_i , at least one of these appears p^e times. Therefore $\mathfrak{a}^r = \mathfrak{a}^{[p^e]} \cdot \mathfrak{a}^{r-p^e}$. We deduce that

$$\mathfrak{a}^r \subseteq \mathfrak{a}^{[p^e]} \cdot I_{r-p^e, e}^{[p^e]} = (\mathfrak{a} \cdot I_{r-p^e, e}(\mathfrak{a}))^{[p^e]},$$

which implies $I_{r, e}(\mathfrak{a}) \subseteq \mathfrak{a} \cdot I_{r-p^e, e}(\mathfrak{a})$. \square

The following lemma shows that the test ideals of \mathfrak{a} depend only on the integral closure of \mathfrak{a} (see also [HY] and [HT]).

Lemma 2.21. *If $\bar{\mathfrak{a}}$ denotes the integral closure of \mathfrak{a} , then $\tau(\mathfrak{a}^c) = \tau(\bar{\mathfrak{a}}^c)$ for every c .*

Proof. The inclusion $\tau(\mathfrak{a}^c) \subseteq \tau(\bar{\mathfrak{a}}^c)$ is obvious. For the reverse inclusion, note that by usual properties of integral closure, there is m such that $\bar{\mathfrak{a}}^{m+\ell} \subseteq \mathfrak{a}^\ell$ for every ℓ . Corollary 2.15 gives $c' > c$ such that $\tau(\mathfrak{a}^c) = \tau(\mathfrak{a}^{c'})$ and $\tau(\bar{\mathfrak{a}}^c) = \tau(\bar{\mathfrak{a}}^{c'})$.

Using Corollary 2.14, we see that

$$\tau(\bar{\mathfrak{a}}^{c'}) = \tau((\bar{\mathfrak{a}}^{m+\ell})^{\frac{c'}{m+\ell}}) \subseteq \tau((\mathfrak{a}^\ell)^{\frac{c'}{m+\ell}}) = \tau(\mathfrak{a}^{c' - \frac{c'm}{m+\ell}}).$$

If $\ell \gg 0$, then $c < c' - \frac{c'm}{m+\ell} < c'$, hence by our choice of c' we get $\tau(\bar{\mathfrak{a}}^c) \subseteq \tau(\mathfrak{a}^c)$. \square

Remark 2.22. If $\varphi: R \rightarrow S$ is a morphism of regular, F -finite rings of positive characteristic, and if \mathfrak{a} is an ideal in R , then $I_{r, e}(\mathfrak{a} \cdot S) \subseteq I_{r, e}(\mathfrak{a}) \cdot S$ for every r and e . Indeed, since $\mathfrak{a}^r \subseteq I_{r, e}(\mathfrak{a})^{[p^e]}$ we have $(\mathfrak{a} \cdot S)^r \subseteq (I_{r, e}(\mathfrak{a}) \cdot S)^{[p^e]}$.

We deduce that for every non-negative c we have $\tau((\mathfrak{a} \cdot S)^c) \subseteq \tau(\mathfrak{a}^c) \cdot S$. This is an analogue of the Restriction Theorem for multiplier ideals in characteristic zero (see [Laz], Examples 9.5.4 and 9.5.8). For a different argument in a more general (characteristic p) framework, see [HY], Theorems 4.1 and 6.10.

2.3. Jumping coefficients and F -thresholds. In [MTW] jumping exponents of an ideal \mathfrak{a} were described as F -thresholds. Since the statements and the proofs in *loc. cit.* were given in the local case, we review them here for the reader's convenience.

Let \mathfrak{a} be an ideal in R . For a fixed ideal J in R such that $\mathfrak{a} \subseteq \text{rad}(J)$ and for an integer $e > 0$ we define $\nu_{\mathfrak{a}}^J(p^e)$ to be the largest non-negative integer r such that $\mathfrak{a}^r \not\subseteq J^{[p^e]}$ (if there is no such r , then we put $\nu_{\mathfrak{a}}^J(p^e) = 0$). If $\mathfrak{a}^r \not\subseteq J^{[p^e]}$, then $\mathfrak{a}^{pr} \not\subseteq J^{[p^{e+1}]}$. Indeed, otherwise we get $(\mathfrak{a}^r)^{[p]} \subseteq J^{[p^{e+1}]}$, a contradiction. Therefore

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq \frac{\nu_{\mathfrak{a}}^J(p^{e+1})}{p^{e+1}},$$

hence we may define the F -threshold of \mathfrak{a} with respect to J as

$$c^J(\mathfrak{a}) := \lim_{e \rightarrow \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \sup_{e \geq 1} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

Note that if \mathfrak{a} is generated by s elements, then $\mathfrak{a}^{s(p^e-1)+1} \subseteq \mathfrak{a}^{[p^e]}$. If $\mathfrak{a}^\ell \subseteq J$, then $\nu_{\mathfrak{a}}^J(p^e) \leq \ell(s(p^e-1)+1) - 1$ for every e . Therefore $c^J(\mathfrak{a}) \leq s\ell$, in particular $c^J(\mathfrak{a})$ is finite.

The following proposition relates the F -thresholds with the generalized test ideals of \mathfrak{a} .

Proposition 2.23. *Let \mathfrak{a} be an ideal in R .*

(1) *If J is an ideal in R such that $\mathfrak{a} \subseteq \text{rad}(J)$, then*

$$\tau(\mathfrak{a}^{c^J(\mathfrak{a})}) \subseteq J.$$

(2) *If c is a non-negative real number, then $\mathfrak{a} \subseteq \text{rad}(\tau(\mathfrak{a}^c))$ and*

$$c^{\tau(\mathfrak{a}^c)}(\mathfrak{a}) \leq c.$$

Proof. For (1), note that by Corollary 2.15 there is $c' > c^J(\mathfrak{a})$ such that $I := \tau(\mathfrak{a}^{c^J(\mathfrak{a})}) = \tau(\mathfrak{a}^{c'})$. Suppose now that $e \gg 0$, so $\tau(\mathfrak{a}^{c'}) = I_{\lceil c'p^e \rceil, e}(\mathfrak{a})$.

Since $c' > c$ and e is large enough, we have $\lceil c'p^e \rceil \geq \nu_{\mathfrak{a}}^J(p^e) + 1$, hence

$$\mathfrak{a}^{\lceil c'p^e \rceil} \subseteq J^{[p^e]}.$$

This implies that $I \subseteq J$, as required.

For (2), let e be large enough, so $\tau(\mathfrak{a}^c) = I_{\lceil cp^e \rceil, e}(\mathfrak{a})$. By definition, we have $\mathfrak{a}^{\lceil cp^e \rceil} \subseteq \tau(\mathfrak{a}^c)^{[p^e]}$, which implies

$$\nu_{\mathfrak{a}}^{\tau(\mathfrak{a}^c)}(p^e) \leq \lceil cp^e \rceil - 1.$$

Dividing by p^e and letting e go to infinity, we get the required inequality. \square

Corollary 2.24. *For every ideal \mathfrak{a} in R , the set of jumping exponents for \mathfrak{a} is equal to the set of F -thresholds of \mathfrak{a} (obtained for all possible ideals J).*

Proof. We show first that if α is a jumping exponent for \mathfrak{a} , then $\alpha = c^J(\mathfrak{a})$ for $J = \tau(\mathfrak{a}^\alpha)$. Indeed, Proposition 2.23(2) gives $c^J(\mathfrak{a}) \leq \alpha$. Therefore $J = \tau(\mathfrak{a}^\alpha) \subseteq \tau(\mathfrak{a}^{c^J(\mathfrak{a})})$, and the reverse inclusion follows from Proposition 2.23(1). Since α is a jumping exponent, we must have $\alpha = c^J(\mathfrak{a})$.

Suppose now that $\alpha = c^J(\mathfrak{a})$ and we need to show that α is a jumping exponent. If this is not the case, then there is $\alpha' < \alpha$ such that $\tau(\mathfrak{a}^\alpha) = \tau(\mathfrak{a}^{\alpha'})$. Using Proposition 2.23(1) we get $\tau(\mathfrak{a}^{\alpha'}) \subseteq J$. If e is large enough, then $\tau(\mathfrak{a}^{\alpha'}) = I_{\lceil \alpha'p^e \rceil, e}(\mathfrak{a})$. Therefore $\mathfrak{a}^{\lceil \alpha'p^e \rceil} \subseteq J^{[p^e]}$, hence $\nu_{\mathfrak{a}}^J(p^e) \leq \lceil \alpha'p^e \rceil - 1$. Dividing by p^e and letting e go to infinity, we get $c^J(\mathfrak{a}) \leq \alpha'$, a contradiction. This completes the proof of the corollary. \square

3. DISCRETENESS AND RATIONALITY

In this section we prove our main result.

Theorem 3.1. *Let k be a field of characteristic $p > 0$ and let R be a regular, F -finite ring, essentially of finite type over k . Suppose that \mathfrak{a} is an ideal in R .*

- (1) *The set of jumping exponents of \mathfrak{a} is discrete (in every finite interval there are only finitely many such numbers).*
- (2) *Every jumping exponent of \mathfrak{a} is a rational number.*

We will reduce the proof of the theorem to the case $R = k[x_1, \dots, x_n]$. We start with some preliminary results. The first proposition, of independent interest, gives an effective bound for the degrees of the generators of the ideals $\tau(\mathfrak{a}^c)$ in terms of the degrees of the generators of \mathfrak{a} . It is our main ingredient for the proof of the theorem in the polynomial ring case. For a real number t we will denote by $[t]$ the largest integer $\leq t$, and by $\{t\}$ the fractional part $t - [t]$.

Proposition 3.2. *Let \mathfrak{a} be an ideal in the polynomial ring $k[x_1, \dots, x_n]$, where k is a field of characteristic p such that $[k : k^p] < \infty$. If \mathfrak{a} can be generated by polynomials of degree at most d , then for every non-negative real number c , the ideal $\tau(\mathfrak{a}^c)$ can be generated by polynomials of degree at most $[cd]$.*

Proof. Fix first r and e . The ideal \mathfrak{a}^r is generated by polynomials of degree at most rd . Choose such generators h_1, \dots, h_s for \mathfrak{a}^r .

Let b_1, \dots, b_m be a basis of k over k^{p^e} , and consider the basis of $R = k[x_1, \dots, x_n]$ over R^{p^e} given by

$$\{b_i x^u \mid i \leq m, u \in \mathbb{N}^n, 0 \leq u_j \leq p^e - 1\}$$

(if $u = (u_1, \dots, u_n)$, then we put $x^u = x_1^{u_1} \cdots x_n^{u_n}$). If we write

$$h_j = \sum_{i,u} a_{i,u}^{p^e} b_i x^u,$$

with $a_{i,u} \in R$, then for every i and u we have $\deg(a_{i,u}^{p^e}) \leq \deg(h_j) \leq rd$. It follows from Proposition 2.4 that $I_{r,e}(\mathfrak{a})$ can be generated by polynomials of degree at most $\frac{rd}{p^e}$.

Given c , let e be large enough, such that $\tau(\mathfrak{a}^c) = I_{[cp^e],e}(\mathfrak{a})$ and $\frac{d}{p^e} < 1 - \{dc\}$. The above argument shows that $\tau(\mathfrak{a}^c)$ can be generated by polynomials of degree at most

$$\frac{d[cp^e]}{p^e} \leq \frac{d(cp^e + 1)}{p^e} \leq dc + \frac{d}{p^e} < [dc] + 1.$$

Since the degree is an integer, this completes the proof of the proposition. \square

Lemma 3.3. *Let \mathfrak{a} be an ideal in a regular, F -finite ring R .*

- (1) *If α is a jumping exponent for \mathfrak{a} , then $p\alpha$ is a jumping exponent, too.*
- (2) *If \mathfrak{a} can be generated by m elements, and if $\alpha > m$ is a jumping exponent for \mathfrak{a} , then $\alpha - 1$ is a jumping exponent, too.*

Proof. For (1), note that by Corollary 2.24 there is an ideal J containing \mathfrak{a} in its radical such that $\alpha = c^J(\mathfrak{a})$. It is clear that $\nu_{\mathfrak{a}}^{J^{[p]}}(p^e) = \nu_{\mathfrak{a}}^J(p^{e+1})$, so $c^{J^{[p]}}(\mathfrak{a}) = p \cdot c^J(\mathfrak{a})$. Hence $p\alpha$ is a jumping exponent by Corollary 2.24.

For (2), suppose that $\alpha - 1$ is not a jumping exponent. Let $\varepsilon > 0$ be such that $\tau(\mathfrak{a}^{\alpha-1}) = \tau(\mathfrak{a}^{\alpha-1-\varepsilon})$ and $\alpha - \varepsilon > m$. It follows from Proposition 2.20 that $\tau(\mathfrak{a}^{\alpha}) = \tau(\mathfrak{a}^{\alpha-\varepsilon})$, hence α is not a jumping exponent, a contradiction. \square

The following lemma relates the generalized test ideals of two different ideals defining the same scheme. For the characteristic zero analogue in the context of multiplier ideals, see Proposition 2.3 in [Mu].

Lemma 3.4. *Let R be a regular, F -finite ring of positive characteristic, and I an ideal in R of pure codimension r , such that $S = R/I$ is regular. If \mathfrak{a} is an ideal in R containing I , then for every non-negative real number c we have*

$$\tau((\mathfrak{a}/I)^c) = \tau(\mathfrak{a}^{c+r}) \cdot S.$$

Proof. By Proposition 2.12 forming generalized test ideals commutes with localization and completion. Therefore it is enough to prove the lemma when R is local and complete. Since I is generated by part of a regular system of parameters for R , by doing induction on r we see that it is enough to prove the case $r = 1$. Therefore we may assume that $R = k[[x_1, \dots, x_n]]$ and $I = (x_n)$.

We claim that $I_{r+p^e, e}(\mathfrak{a}) \cdot S = I_{r+1, e}(\mathfrak{a}/(x_n))$. This implies the assertion in the lemma. Indeed, if e is large enough, then we have

$$\tau(\mathfrak{a}^{c+1}) \cdot S = I_{\lceil cp^e \rceil + p^e, e}(\mathfrak{a}) \cdot S = I_{\lceil cp^e \rceil + 1, e}(\mathfrak{a}/(x_n)) = \tau((\mathfrak{a}/(x_n))^c).$$

The last equality follows since $\frac{\lceil cp^e \rceil + 1}{p^e}$ is larger than c , but close to c (for e large enough), applying Proposition 2.13.

We prove now the above claim using the description in Proposition 2.4. Let a_1, \dots, a_m be a basis of k over k^{p^e} , and consider the basis of R over R^{p^e} given by

$$\{a_i x^u \mid i \leq m, u = (u_j) \in \mathbb{N}^n, 0 \leq u_j \leq p^e - 1\}.$$

Write $\mathfrak{a} = (x_n) + \mathfrak{b}$, where \mathfrak{b} is generated by power series in $k[[x_1, \dots, x_{n-1}]]$. We have

$$\mathfrak{a}^{r+p^e} = \sum_{i=0}^{r+p^e} x_n^i \mathfrak{b}^{r+p^e-i}.$$

The generators of $\tau(\mathfrak{a}^c)$ that come from writing in the above basis the generators of $x_n^i \mathfrak{b}^{r+p^e-i}$ are divisible by x_n if $i \geq p^e$, hence map to zero in S . On the other hand, the generators coming from $x_n^i \mathfrak{b}^{r+p^e-i}$ for $i \leq p^e - 1$ are the same as the ones obtaining from writing the generators of \mathfrak{b}^{r+p^e-i} in the corresponding basis of $k[[x_1, \dots, x_{n-1}]]$ over $k^{p^e}[[x_1^{p^e}, \dots, x_{n-1}^{p^e}]]$. Moreover, it is clear that it is enough to consider only the largest such ideal, namely \mathfrak{b}^{r+1} . This shows that $I_{r+p^e, e}(\mathfrak{a}) \cdot S = I_{r+1, e}(\mathfrak{a}/(x_n))$, as claimed. \square

Corollary 3.5. *If R , S and \mathfrak{a} are as in the above proposition, and if $c > 0$ is a jumping exponent for $\mathfrak{a} \cdot S$, then $c + r$ is a jumping exponent for \mathfrak{a} .*

We can give now the proof of our main result.

Proof of Theorem 3.1. For (1), suppose that we have a sequence of jumping exponents $\{\alpha_m\}_m$ for \mathfrak{a} having a finite accumulation point α . By Corollary 2.15, we have $\alpha_m < \alpha$ for $m \gg 0$. After replacing this sequence by a subsequence, we may assume that $\alpha_m < \alpha_{m+1}$ for every m .

Let us write $R = R_1 \times \cdots \times R_s$, where all R_i are domains. We have $\mathfrak{a} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_s$, and for every m there is j such that α_m is a jumping exponent for \mathfrak{a}_j . After replacing our sequence by a subsequence, we may replace R by some R_j and therefore assume that R is a domain.

By hypothesis, we can write $R \simeq S^{-1}(k[x_1, \dots, x_n]/I)$ for some ideal I and some multiplicative system S . Note that $[k: k^p] < \infty$ (see Example 2.1). Let us write $\mathfrak{a} = S^{-1}(\mathfrak{b}/I)$. Note that $S^{-1}I$ is a prime ideal, hence it has pure codimension, let us denote it by r , in $S^{-1}k[x_1, \dots, x_n]$. It follows from Corollary 3.5 that $r + \alpha$ is an accumulation point for the jumping exponents of $S^{-1}\mathfrak{b}$. Moreover, Proposition 2.12(1) implies that $r + \alpha$ is an accumulation point for the jumping exponents of \mathfrak{b} . Therefore, in order to achieve a contradiction we may assume that $R = k[x_1, \dots, x_n]$.

Suppose now that \mathfrak{a} is generated by polynomials of degree at most d . In this case Proposition 3.2 implies that every $\tau(\mathfrak{a}^{\alpha_m})$ is generated by polynomials of degree at most $\lfloor \alpha d \rfloor$. Since the ideals $\tau(\mathfrak{a}^{\alpha_m})$ form a strictly decreasing sequence of ideals, we get a strictly decreasing sequences of vector subspaces of the vector space $k[x_1, \dots, x_n]_{\leq \lfloor \alpha d \rfloor}$ of all polynomials of degree at most $\lfloor \alpha d \rfloor$. This gives a contradiction and completes the proof of (1).

Suppose now that $\alpha > 0$ is a jumping exponent for \mathfrak{a} . Lemma 3.3(1) implies that all $p^e \alpha$ are jumping exponents for \mathfrak{a} . Suppose that \mathfrak{a} is generated by m elements and let e_0 be such that $p^{e_0} \alpha > m$. We deduce from Lemma 3.3(2) that $\{p^e \alpha\} + m - 1$ is a jumping exponent for every $e \geq e_0$. Since all these numbers lie in $[m - 1, m)$, it follows from (1) that there are only finitely many such numbers. Therefore we can find $e_1 \neq e_2$ such that $p^{e_1} \alpha - p^{e_2} \alpha$ is an integer. Hence α is a rational number. \square

The ideas in the above proof can be used to explicitly estimate the jumping exponents of an ideal in a polynomial ring in terms of the degrees of its generators. Since we know that all jumping exponents are rational, it is enough to bound their denominators.

Proposition 3.6. *Let $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ be an ideal generated by m polynomials of degree at most d . If e_0 is such that $p^{e_0} > md$ and $N = \binom{md+n}{n}$, then for every jumping exponent α of \mathfrak{a} we have $p^a(p^b - 1)\alpha \in \mathbb{N}$ for some $a \leq e_0 + N$ and $b \leq N$.*

Proof. By Lemma 3.3(2) it is enough to consider the case when $\alpha \leq m$. Since $\tau(\mathfrak{a}^c)$ is generated by polynomials of degree at most md for every $c \leq m$, it follows that we have at most $N = \dim_k k[x_1, \dots, x_n]_{\leq md}$ jumping exponents of \mathfrak{a} in $[0, m]$.

If $\alpha > 0$, then $\alpha \geq 1/d$ (note that $\tau(\mathfrak{a}^c) = R$ for $c < 1/d$ by Proposition 3.2). Since $p^{e_0} \alpha \geq \frac{p^{e_0}}{d} > m$, Lemma 3.3 implies that if $e \geq e_0$, then $\{p^e \alpha\} + m - 1$ is a

jumping exponent of \mathfrak{a} . By letting e vary between e_0 and $e_0 + N$, we see that two of these numbers have to be equal. Therefore there are $e_1 < e_2$ in $\{e_0, \dots, e_0 + N\}$ such that $p^{e_1}(p^{e_2-e_1} - 1)\alpha \in \mathbb{N}$, which completes the proof. \square

Remark 3.7. We can get a bound independent of m in the above proposition by taking $m = n$. Indeed, note first that we may assume that k is infinite: otherwise take an infinite perfect extension K of k , and replace \mathfrak{a} by $\mathfrak{a} \cdot K[x_1, \dots, x_n]$ using Remark 2.17.

If h_1, \dots, h_m are generators of \mathfrak{a} of degree at most d , and if $g_i = \sum_{j=1}^m a_{i,j}h_j$ for $1 \leq i \leq j$, where the $a_{i,j}$ are general elements in k , then the ideal \mathfrak{b} generated by the g_i has the same integral closure as \mathfrak{a} (see, for example [Laz], Example 9.6.19; note that in the proof therein one does not use the assumption that the base field is \mathbb{C}). By Lemma 2.21, the ideals \mathfrak{a} and \mathfrak{b} have the same jumping exponents, so we may apply the argument in Proposition 3.6 to \mathfrak{b} .

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FB MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, STANDORT ESSEN, 45117 ESSEN, GERMANY

E-mail address: `manuel.blickle@uni-essen.de`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, USA

E-mail address: `mmustata@umich.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, USA

E-mail address: `kesmith@umich.edu`