A BRIANÇON–SKODA TYPE THEOREM FOR GRADED SYSTEMS OF IDEALS

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1. Introduction

A celebrated result of Briançon and Skoda [1] states that for every coherent sheaf of ideals $\mathfrak{a} \subseteq \mathcal{O}_X$ on a smooth irreducible complex variety, a certain fixed power of its integral closure is actually contained in $\mathfrak{a}$. We establish an analogue of this statement for stable graded systems of ideals.

The Briançon–Skoda theorem, originally proved via L^2-methods in complex analysis, has sparkled a great deal of interest among algebraists. Algebraic proofs were given by Lipman and Teissier for ideals in a regular local ring [6], Hochster and Huneke [4] using tight closure theory (one of the first major victories of this method), and Ein–Lazarsfeld [2, 5], via multiplier ideals. Here we will follow this latter path, and prove our result by means of multiplier ideal theory. For a thorough treatment of this topic, the reader is invited to consult [5, Chapters 9–11].

Let X be a smooth irreducible complex variety. A graded system of ideals on X is a sequence ideal sheaves $\mathfrak{a}_m \subseteq \mathcal{O}_X$, for which $\mathbf{a}_m\mathbf{a}_n \subseteq \mathbf{a}_{m+n}$ for all natural numbers $m, n \geq 1$. Experience shows (see [9]) that unlike the powers of a fixed ideal, graded systems of ideals in general behave very pathologically. In an attempt to remedy this situation, we introduce the concept of stability, and show that stable graded systems of ideals act in a more controlled manner.

We define a stable graded system of ideals to be one, where all elements differ from $\mathcal{O}_X$, and for every irreducible subvariety Z of X, which occurs in the support of $\mathbf{a}_m$ for infinitely many m, the asymptotic order of vanishing of $\mathbf{a}_\bullet$ along Z is positive. This notion is in a way an algebraic counterpart of the notion of a stable divisor (for a precise definition of this notion the reader is invited to consult [3]); the graded system of ideals formed by the base ideal sheaves of the multiples of a stable Cartier divisor is certainly stable.

A distinguishing property of stable graded systems of ideals is that their cosupports — and the cosupports of the corresponding sequence of asymptotic multiplier ideals — stabilize to an algebraic set.

Our main result is the following.

Theorem. *Let $\mathbf{a}_\bullet$ be a stable graded system of ideals. Then there exists an integer C so that for all $n \gg 0$,*

$$\mathcal{J}(C \cdot \mathbf{a}_\bullet) \subseteq \mathbf{a}_n.$$

As asymptotic multiplier ideals are integrally closed, the fact that $\mathbf{a}_m \subseteq \mathcal{J}(m \cdot \mathbf{a}.)$ then implies the following generalization of the aforementioned result of Briançon and Skoda.

Corollary. *Let $\mathbf{a}_\bullet$ be a stable graded system of ideals. Then there exists a positive integer C, such that for all $n \gg 0$*

$$\overline{\mathbf{a}}_{Cn} \subseteq \mathbf{a}_n \ .$$

Note that as a by-product, our results give a relatively elementary treatment (modulo standard properties of asymptotic multiplier ideals) of a weak version of the original Briançon–Skoda theorem with the exception of finitely many initial terms.

The organization of the paper goes as follows. The definition and basic properties of stable graded systems of ideals are found in Section 2, while Section 3 hosts the generalized Briançon–Skoda theorem along with its proof and some illustrating examples.

The authors would like to thank Manuel Blickle, Lawrence Ein, and Robert Lazarsfeld for helpful discussions.

2. Stable Graded Systems of Ideals

Throughout this paper we will work on a smooth irreducible complex variety X of dimension d.

Definition 2.1. *Let $\mathbf{a}_\bullet = \{\mathbf{a}_n\}_{n\in\mathbf{N}}$ be a sequence of ideal sheaves on X. We say that $\mathbf{a}_\bullet$ is a* graded system of ideals *if for all $m, n \in \mathbf{N}$,*

$$\mathbf{a}_m\mathbf{a}_n \subseteq \mathbf{a}_{m+n}.$$

Setting $\mathbf{a}_0 = \mathcal{O}_X$, we note that the definition is equivalent to asking that $\oplus_{n\geq 0}\mathbf{a}_n$ have a natural structure of a graded algebra of sheaves on X. This is called the Rees algebra associated to the graded system $\mathbf{a}_\bullet$. Notable examples of graded systems of ideals are the system of symbolic powers of a prime ideal, or the system formed by the base ideals of the tensor powers of a line bundle on a projective variety.

If $\mathbf{a}_\bullet$ is a graded system of ideals, the set

$$\{n \in \mathbf{N} : \mathbf{a}_n \neq 0\}$$

is a subsemigroup $S(\mathbf{a}_\bullet)$ of $\mathbf{N}$. It is well-known that this forces all sufficiently large elements of $S(\mathbf{a}_\bullet)$ to be multiples of $e(\mathbf{a}_\bullet) \stackrel{\text{def}}{=} \gcd S(\mathbf{a}_\bullet)$. We call $S(\mathbf{a}_\bullet)$ the semigroup of $\mathbf{a}_\bullet$, and $e(\mathbf{a}_\bullet)$ the exponent of $\mathbf{a}_\bullet$. In the following, we shall deal exclusively with graded systems of ideals with exponent $e(\mathbf{a}_\bullet) = 1$, so that $\mathbf{a}_n \neq 0$ for all $n \gg 0$.

Let $Z \subset X$ be a (reduced and irreducible) subvariety, and $\mathbf{a}$ an ideal sheaf on X. Then the order of vanishing of the ideal $\mathbf{a}$ along the subvariety Z is

$$\operatorname{ord}_Z(\mathbf{a}) \stackrel{\text{def}}{=} \max\{n \in \mathbf{N} : \mathbf{a} \subseteq I_Z^{\langle n\rangle}\} \ ,$$

where $I_Z^{\langle n\rangle}$ denotes the nth symbolic power of the ideal I_Z.

If Z is a point $p \in X$, then $\operatorname{ord}_p(\mathbf{a}) = n$ simply says that all partial derivatives of order at most $n-1$ of functions in $\mathbf{a}$ vanish at p. In fact, the case $Z = p$ is already in some sense the general case, since

$$\operatorname{ord}_Z(\mathbf{a}) = \min\{\operatorname{ord}_p(\mathbf{a}) : p \in Z\} \ ,$$

and this minimum is attained at general points $p \in Z$. Along similar lines, one can see that it suffices to test order of vanishing generically: $\operatorname{ord}_Z(\mathbf{a}) = n$ if and only if $\operatorname{ord}_p(\mathbf{a}) = n$ for general $p \in Z$, $\operatorname{ord}_p(\mathbf{a}) = n$.

Recall that [3] shows that one may attach an asymptotic order of vanishing to a graded system of ideals $\mathbf{a}_\bullet$ via the following formula:

Definition 2.2. *Let $\mathbf{a}_\bullet$ be a graded system of ideals. Then*

$$\operatorname{ord}_Z(\mathbf{a}_\bullet) := \lim_{n\to\infty} \frac{\operatorname{ord}_Z(\mathbf{a}_n)}{n}.$$

More precisely, it is established in [3] that the limit in question always exists.

We now arrive at the key technical definition.

Definition 2.3. *We say the graded system $\mathbf{a}_\bullet$ is* stable *if*

(1) $e(\mathbf{a}_\bullet) = 1$
(2) *for all subvarieties Z that occur as components of $V(\mathbf{a}_n)$ for infinitely many $n \in \mathbf{N}$, $\operatorname{ord}_Z(\mathbf{a}_\bullet) > 0$.*

Stable graded systems of ideals are ubiquitious. Ordinary or symbolic powers of a prime ideal certainly form stable graded systems, but as we will see, nontrivial examples of them occur naturally in various contexts.

Remark 2.4. Let X be a smooth projective variety, D a stable Cartier divisor (in the sense of [3]) on X. Then the graded system of ideals $\mathfrak{b}_m(D) = \mathfrak{b}(mD)$ is stable (see [3, Proposition 3.8.]).

Example 2.5. The following is a classical example of a non-stable graded system of ideals which goes back to Zariski. In [10] he shows the existence of a a smooth projective surface X, an irreducible curve $C \subseteq X$, and a divisor L on X, such that for every $m \geq 1$, C is in the base locus of mL, but $mL - C$ is base point free. In this case, the base ideal $\mathfrak{b}(mL)$ associated to mL vanishes to order exactly one along C, hence the graded system consisting of the base ideals $\mathfrak{b}(mL)$ is not stable.

A key fact about stable graded systems is that their cosupports stabilize to an algebraic set, the asymptotic cosupport of $\mathbf{a}_\bullet$, which we will denote $\mathbf{B}(\mathbf{a}_\bullet)$. Later we show that $\mathbf{B}(\mathbf{a}_\bullet)$ also equals the asymptotic cosupport of the corresponding sequence of asymptotic multiplier ideals.

Lemma 2.6. *Let $\mathbf{a}_\bullet$ be a stable graded system of ideals. Then there exists an algebraic set $\mathbf{B}(\mathbf{a}_\bullet) \subset X$ so that for all $n \gg 0$, $V(\mathbf{a}_n) = \mathbf{B}(\mathbf{a}_\bullet)$.*

Proof. In fact, we will show that

$$\mathbf{B}(\mathbf{a}_\bullet) = \bigcap_{n\in\mathbf{N}} V(\mathbf{a}_n).$$

Since X is a noetherian topological space, there is some $N \in \mathbf{N}$ so that $\mathbf{B}(\mathbf{a}_\bullet) = \cap_{i=1}^N V(\mathbf{a}_i)$. In particular, $\mathbf{B}(\mathbf{a}_\bullet) \subseteq V(\mathbf{a}_n)$ for all $n \geq N$. Stability will guarantee the reverse inclusion.

We first show that the subsequence $n_k = k \cdot N!$ satisfies $V(\mathbf{a}_{n_k}) = \mathbf{B}(\mathbf{a}_\bullet)$ for all k. By definition of graded systems, for all i, $1 \leq i \leq N$, one has $\mathbf{a}_i^{N!/i} \subseteq \mathbf{a}_{N!}$, so $V(\mathbf{a}_{N!}) \subseteq V(\mathbf{a}_i)$. Thus

$$V(\mathbf{a}_{N!}) \subseteq \bigcap_{i=1}^N V(\mathbf{a}_i) = \mathbf{B}(\mathbf{a}_\bullet).$$

So for all $k \in \mathbf{N}$, one has

$$V(\mathbf{a}_{k\cdot N!}) \subseteq V(\mathbf{a}_{N!}^k) = \mathbf{B}(\mathbf{a}_\bullet)\ ,$$

as desired.

Now, let $p \in \mathbb{N}$ be a prime greater than N. Then p and $N!$ generate a subsemigroup of $S(\mathbf{a}_\bullet)$ under addition which includes all $n \gg 0$. Hence one has $n = n_1 p + n_2 N!$ for all $n \gg 0$, and

$$V(\mathbf{a}_n) \subseteq V(\mathbf{a}_p^{n_1}) \cup V(\mathbf{a}_{N!}^{n_2}) = V(\mathbf{a}_p) \cup V(\mathbf{a}_{N!}) \ .$$

We now claim that there exists $n_0 \in \mathbf{N}$ depending on $\mathbf{a}_\bullet$ and p such that if $n \geq n_0$ then in fact

$$V(\mathbf{a}_n) \subset V \cup V(\mathbf{a}_{N!}) \ ,$$

where $V \subset V(\mathbf{a}_p)$ and $\dim(V) < \dim(V(\mathbf{a}_p))$.

Granting the claim, we may then repeat the same argument with $V(\mathbf{a}_p)$ replaced by V; since $V(\mathbf{a}_p)$ is itself a Noetherian topological space, after finitely many iterations of this process we obtain that $V(\mathbf{a}_n) = \mathbf{B}(\mathbf{a}_\bullet)$ for $n \gg 0$.

To prove the claim, let Z be a top-dimensional component of $V(\mathbf{a}_p)$ which does not occur in $V(\mathbf{a}_{N!})$. Then Z does not appear in $V(\mathbf{a}_{k \cdot N!})$ for all k, so calculating along that subsequence we see that $\mathrm{ord}_Z(\mathbf{a}_\bullet) = 0$. By definition of stability, we see that Z can occur as a component of $V(\mathbf{a_n})$ for at most finitely many n. So after finitely many steps $V(\mathbf{a_n})$ is supported along some proper algebraic subset of Z. Repeating this argument for each top-dimensional component of $V(\mathbf{a}_p)$, the claim follows. □

Recall that the level-n asymptotic multiplier ideal $\mathcal{J}(n\mathbf{a}_\bullet)$ is defined to be the maximal element of the set of multiplier ideals

$$\left\{ \mathcal{J}(\frac{n}{p} \cdot \mathbf{a}_p) \right\} \ .$$

For a proof of the fact that this maximum exists and other properties of asymptotic multiplier ideals, the reader is referred to [5, Chapter 11].

One thus obtain a sequence of ideals

$$\mathbf{b}_n := \mathcal{J}(n\mathbf{a}_\bullet),$$

indexed by the natural numbers. It is not graded; however, the subadditivity theorem for asymptotic multiplier ideals shows that the ideals $\mathbf{b}_n$ form a reverse graded system (in the sense of [7]):

$$\mathbf{b}_{n+m} \subseteq \mathbf{b}_n \cdot \mathbf{b}_m \ ,$$

which we will denote by $\mathbf{b}_\bullet$. In [7] it is shown that the following limit defining $\mathrm{ord}_Z(\mathbf{b}_\bullet)$ exists:

$$\mathrm{ord}_Z(\mathbf{b}_\bullet) := \lim_{n \to \infty} \frac{\mathrm{ord}_Z(\mathbf{b}_n)}{n} \ .$$

Analogously to the case of stable graded systems of ideals, we now verify that the cosupports of an asymptotic multiplier ideal sequence corresponding to a stable graded system also stabilize.

Lemma 2.7. *Let $\mathbf{a}_\bullet$ be a stable graded system of ideals with asymptotic cosupport $\mathbf{B}(\mathbf{a}_\bullet)$. Then for all $n \gg 0$, $V(\mathcal{J}(n\mathbf{a}_\bullet)) = \mathbf{B}(\mathbf{a}_\bullet)$.*

Proof. By the definition of multiplier ideals one has that for all $n \in \mathbf{N}$, $\mathbf{a}_n \subset \mathcal{J}(n\mathbf{a}_\bullet)$. Therefore $V(\mathcal{J}(n\mathbf{a}_\bullet)) \subseteq \mathbf{B}(\mathbf{a}_\bullet)$ for $n \gg 0$, and

$$\mathrm{ord}_Z(\mathbf{b}_\bullet) \leq \mathrm{ord}_Z(\mathbf{a}_\bullet)$$

for all subvarieties Z of X. We claim that in fact

$$\mathrm{ord}_Z(\mathbf{b}_\bullet) = \mathrm{ord}_Z(\mathbf{a}_\bullet)$$

for every subvariety Z of X. This will show in particular that for $n \gg 0$, $\mathcal{J}(n\mathbf{a}_\bullet)$ is co-supported along every irreducible component of $\mathbf{B}(\mathbf{a}_\bullet)$ and hence $V(\mathcal{J}(n\mathbf{a}_\bullet)) = \mathbf{B}(\mathbf{a}_\bullet)$. To establish the claim, the key will be to compare orders of vanishing on X and on suitable log resolutions of the ideals in the graded system.

Fix a subvariety $Z \subseteq X$, and let $X_0 = X$ and for all $i \geq 1$, pick

$$\mu_i : X_i \to X$$

to be log resolutions of $\mathbf{a}_i$ such that

(1) μ_i is a log resolution of Z,
(2) for all $j < i$, μ_i factors through μ_j.

As the μ_i's are log resolutions of Z for all i, $\mu_i^{-1}(Z)$ is a SNC divisor on X_i for all i, and there is a unique prime divisor $E_i \subseteq X_i$ dominating the smooth locus of Z.Write b_i for the coefficient of E_i in the relative canonical divisor $K_{X_i/X}$, and a_i for the coefficient of E_i in $F_i = \mu_i^*\mathbf{a}_i$.

Observe firstly that $b_i = b_1$ for all $i \geq 0$, as $\mu_1^{-1}(Z)$ is already a SNC divisor on X_1. Then for some $p = p(n)$ which we may take to be arbitrarily large,

$$\begin{aligned}\mathrm{ord}_Z(\mathcal{J}(n\mathbf{a}_\bullet)) &\geq \mathrm{ord}_{E_p}(K_{X_p/X} - [\frac{n}{p}F_i]) \\ &= -b_1 + [n\frac{a_p}{p}] \\ &= -b_1 + [n\frac{\mathrm{ord}_Z(\mathbf{a}_p)}{p}].\end{aligned}$$

Fix a sequence $p(n)$ as above, with the additional property that $p(n) \to \infty$ as $n \to \infty$. Estimating the asymptotic order of $\mathbf{b}_\bullet$ gives

$$\begin{aligned}\mathrm{ord}_Z(\mathbf{b}_\bullet) &\geq \lim_{n\to\infty} \frac{-b_1 + [n\frac{\mathrm{ord}_Z(\mathbf{a}_{p(n)})}{p(n)}]}{n} \\ &= \lim_{n\to\infty} \frac{n \cdot \frac{\mathrm{ord}_Z(\mathbf{a}_{p(n)})}{p(n)} - \left\{n\frac{\mathrm{ord}_Z(\mathbf{a}_{p(n)})}{p(n)}\right\}}{n} \\ &= \lim_{n\to\infty} \frac{n \cdot \frac{\mathrm{ord}_Z(\mathbf{a}_{p(n)})}{p(n)}}{n} \\ &= \mathrm{ord}_Z(\mathbf{a}_\bullet)\ ,\end{aligned}$$

as we wanted. □

3. A generalized statement of Briancon-Skoda type

In analogy with the aformentioned result of Briancon and Skoda, we are interested in finding functions $f : \mathbf{N} \to \mathbf{N}$ so that for all $n \gg 0$,

$$\overline{\mathbf{a}}_{f(n)} \subseteq \mathbf{a}_n\ .$$

Our plan is to establish the corresponding statement, ie.

$$\mathcal{J}(\mathbf{a}_{f(n)}) \subseteq \mathbf{a}_n$$

if $n \gg 0$, for asymptotic multiplier ideals, and appeal to their integral closure.

We note first of all that such functions may not exist if the graded system $\mathbf{a}_\bullet$ is not stable.

Remark 3.1. Take $X = \mathbf{C}^2$, $\ell \subset X$ a line and $p \in \ell$ a point. Let $\mathbf{a}_\bullet$ be a graded system generated by $\mathbf{a}_1$ and $\mathbf{a}_2$, where $\mathbf{a}_1 = I_\ell$ and $\mathbf{a}_2 = I_p$. Then

$$\mathbf{a}_n = \begin{cases} I_p^{n/2} & \text{if } n \text{ is even} \\ I_p^{(n-1)/2} I_\ell & \text{if } n \text{ is odd.} \end{cases}$$

For any k, we may always calculate the level-k asymptotic multiplier ideal at an ideal $\mathbf{a}_{p(k)}$ where $p(k)$ is even, which shows that $\mathcal{J}(k\mathbf{a}_\bullet)$ is cosupported at p. Therefore, if n is any odd integer, for all $k \in \mathbf{N}$, $\mathcal{J}(k\mathbf{a}_\bullet) \not\subseteq \mathbf{a}_n$.

We remark that, as we have seen, stability of $\mathbf{a}_\bullet$ rules out the troublesome appearances of I_ℓ to bounded order in $\mathbf{a}_n$ for infinitely many n. □

On the other hand, in the case of stable graded systems of ideals, we may find functions f which have linear growth and satisfy the desired containment.

Remark 3.2. Consider first the case of finitely generated graded systems, ie. graded systems of ideals for which the associated Rees algebra $\bigoplus_{m \geq 0} \mathbf{a}_m$ is finitely generated. Then by Proposition 2.4.27. in [5], there exists an integer $N > 0$ such that for all $m \geq 1$ one has

$$\mathbf{a}_N^m = \mathbf{a}_{Nm} \ .$$

If $\mathbf{a}_\bullet$ is stable then

$$\mathcal{J}((Nm + D) \cdot \mathbf{a}_\bullet) \subseteq \mathbf{a}_m$$

for a suitable positive constant D. Note that finitely generated graded systems are not necessarily stable.

Theorem 3.3. *Let $\mathbf{a}_\bullet$ be a stable graded system of ideals. Then there exist positive constants C and D so that for all $n \gg 0$,*

$$\mathcal{J}((\lceil Cn + D \rceil)\mathbf{a}_\bullet) \subseteq \mathbf{a}_n.$$

In particular,

$$\mathcal{J}(C'n \cdot \mathbf{a}_\bullet) \subseteq \mathbf{a}_n$$

for a suitable positive integer C'.

Proof. Fix two relatively prime integers $n_1 < n_2$, so large that $V(\mathbf{a}_i) = V(\mathcal{J}(n_1\mathbf{a}_\bullet)) = \mathbf{B}(\mathbf{a}_\bullet)$ for $i = 1, 2$. The semigroup generated by the n_i includes all $n \gg 0$. By Hilbert's Nullstellensatz, there exist integers ν_i so that for $i = 1, 2$

$$I_{\mathbf{B}(\mathbf{a}_\bullet)}^{\nu_i} \subseteq \mathbf{a}_{n_i}.$$

We are looking for a function $f : \mathbb{N} \to \mathbb{N}$ such that f satisfies $\mathcal{J}(f(n)\mathbf{a}_\bullet) \subset \mathbf{a}_n$ for all n in the semigroup generated by the n_i. Observe that

$$\mathcal{J}(f(n)\mathbf{a}_\bullet) \subseteq \mathcal{J}(n_1\mathbf{a}_\bullet)^{[f(n)/n_1]} \subseteq I_{\mathbf{B}(\mathbf{a}_\bullet)}^{[f(n)/n_1]}.$$

Write $n = m_1 n_1 + m_2 n_2$. Then if $[f(n)/n_1] \geq m_1\nu_1 + m_2\nu_2$,

$$\mathcal{J}(f(n)\mathbf{a}_\bullet) \subseteq I_{\mathbf{B}(\mathbf{a}_\bullet)}^{m_1\nu_1 + m_2\nu_2} \subseteq \mathbf{a}_1^{m_1}\mathbf{a}_2^{m_2} \subseteq \mathbf{a}_n.$$

Since in any case the $m_i \leq \frac{n}{n_i}$, one checks that $f(n) = \lceil (\nu_1 + \frac{\nu_2 n_1}{n_2})n \rceil + n_1$ satisfies the containment. □

Corollary 3.4. *Let $\mathbf{a}_\bullet$ be a stable graded system of ideals. Then there exists a positive integer C, such that for all $n \gg 0$,*

$$\overline{\mathbf{a}}_{Cn} \subseteq \mathbf{a}_n \ .$$

.

It is natural to ask what numbers C and D may occur in Theorem 3.3. In many situations of geometric interest, for instance, $C = 1$ will do. However, simple examples show that this is not necessarily the case in general.

Example 3.5. Let $X = \mathrm{Spec}(\mathbf{C}[x])$ and $I = (x)$. One checks that $\mathbf{a}_n = I^{n+\lceil\sqrt{n}\rceil}$ is a graded system of ideals. However, $\mathcal{J}((n+D)\mathbf{a}_\bullet) = I^{n+D} \not\subseteq \mathbf{a}_n$ for all $n \gg 0$.

Example 3.6. Here we show that in certain cases the set of possible candidates for the constant C is bounded away from 1. Let $\mathfrak{m} = (x, y) \subseteq \mathbb{C}^2$ and let $\epsilon > 0$ be arbitrary. Set

$$\mathbf{a}_n = \mathfrak{m}^{(\epsilon n)}(x^n, y) \ .$$

Since $y \in (\mathbf{a}_n : \mathfrak{m}^{\epsilon n})$ for all n, the multiplier ideals of the graded system $\mathbf{a}_\bullet$ and $\mathfrak{m}^{\epsilon n}$ are the same. So if $\mathcal{J}((Cn+D)\mathbf{a}_\bullet) \subseteq \mathbf{a}_n$, one also has

$$\mathfrak{m}^{\epsilon(Cn+D)} \subseteq \mathbf{a}_n \ .$$

However, compare the powers of x that occur in both sides of the containment directly above. On the right-hand side, $x^{n+\epsilon n}$ is the lowest power. So $\epsilon(Cn+D) \geq n + \epsilon n$, and taking a limit as $n \to \infty$, one has $C > 1/\epsilon$.

Finally, we explore what happens for sums and products of graded systems of ideals.

Proposition 3.7. *Let $\mathbf{a}_\bullet$, $\mathbf{b}_\bullet$ be two stable graded systems of ideals on X, C_A, C_B two positive integers which satisfy the conclusion of the theorem. If $\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet$ and $\mathbf{a}_\bullet + \mathbf{b}_\bullet$ denote the product and sum of the two graded systems (in the sense of* [5, 2.4.24.]*), then for large enough n, one has*

$$\mathcal{J}((C_A + C_B)n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)) \subseteq (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)_n,$$

and

$$\mathcal{J}((C_A + C_B)n \cdot (\mathbf{a}_\bullet + \mathbf{b}_\bullet)) \subseteq (\mathbf{a}_\bullet + \mathbf{b}_\bullet)_n \ .$$

Proof. Let us first consider the product $(\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)_n = \mathbf{a}_n\mathbf{b}_n$ of two graded systems. Pick $p > 0$ large enough so that

$$\mathcal{J}(\frac{C_A n}{p} \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)_p) \ , \ \mathcal{J}(\frac{C_B n}{p} \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)_p) \ , \ \mathcal{J}(\frac{C_A n}{p} \cdot \mathbf{a}_p) \ , \ \mathcal{J}(\frac{C_B n}{p} \cdot \mathbf{b}_p)$$

compute the respective asymptotic multiplier ideals

$$\mathcal{J}(C_A n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)) \ , \ \mathcal{J}(C_B n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)) \ , \ \mathcal{J}(C_A n \cdot \mathbf{a}_\bullet) \ , \ \mathcal{J}(C_B n \cdot \mathbf{b}_\bullet) \ .$$

Then

$$\begin{aligned} \mathcal{J}(C_A n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)) &\subseteq \mathcal{J}(\frac{C_A n}{p} \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)_p) = \mathcal{J}(\frac{C_A n}{p} \cdot \mathbf{a}_p\mathbf{b}_p) \subseteq \mathcal{J}(\frac{C_A n}{p} \cdot \mathbf{a}_p) \\ &= \mathcal{J}(C_A n \cdot \mathbf{a}_\bullet) \subseteq \mathbf{a}_n \end{aligned}$$

by the choice of C_A. By the subadditivity theorem for asymptotic multiplier ideals this implies

$$\mathcal{J}((C_A+C_B)n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)) \subseteq \mathcal{J}(C_A n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet))\mathcal{J}(C_B n \cdot (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)) \subseteq \mathbf{a}_n\mathbf{b}_n = (\mathbf{a}_\bullet \cdot \mathbf{b}_\bullet)_n$$

as required.

To verify the statement about the sum of the two graded systems, we employ Mustaţă's summation theorem ([8] or [5, 11.2.7]), according to which

$$\mathcal{J}((C_A+C_B)n\cdot(\mathbf{a}_\bullet+\mathbf{b}_\bullet)) \subseteq \sum_{\lambda+\mu=(C_A+C_B)n} J(\lambda\cdot\mathbf{a}_\bullet)\mathcal{J}(\mu\cdot\mathbf{b}_\bullet)\ ,$$

where λ and μ are positive rational numbers. Observe that

$$\lambda+\mu=(C_A+C_B)n$$

implies that either $\lambda\geq C_A n$ or $\mu\geq C_B n$. In the first case, we obtain by the choice of C_A that

$$J(\lambda\cdot\mathbf{a}_\bullet)\mathcal{J}(\mu\cdot\mathbf{b}_\bullet)\subseteq J(\lambda\cdot\mathbf{a}_\bullet)\subseteq J(C_A n\cdot\mathbf{a}_\bullet)\subseteq\mathbf{a}_n\ ,$$

in the second case we analogously conclude that $J(\lambda\cdot\mathbf{a}_\bullet)\mathcal{J}(\mu\cdot\mathbf{b}_\bullet)\subseteq\mathbf{b}_n$. Therefore,

$$\mathcal{J}((C_A+C_B)n\cdot(\mathbf{a}_\bullet+\mathbf{b}_\bullet))\subseteq\mathbf{a}_n+\mathbf{b}_n\subseteq(\mathbf{a}_\bullet+\mathbf{b}_\bullet)_n\ .$$

□

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