Logarithmic Surfaces and Hyperbolicity

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1 Introduction and results

We first introduce some notations (see section 2 for the precise definitions). Let $\bar{X}$ be a projective manifold and $D \subset \bar{X}$ a normal crossing divisor. We call the pair $(\bar{X}, D)$ a log manifold and denote $X = \bar{X} \setminus D$. It is called a log surface if $\dim \bar{X} = 2$. Let $T^*\bar{X}$ be its cotangent bundle and $\bar{T}^*X$ its log cotangent bundle. We denote by $q_{\bar{X}} = \dim H^0(\bar{X}, T^*\bar{X})$ its irregularity and $\bar{q}_X = \dim H^0(\bar{X}, \bar{T}^*X)$ its log irregularity. We denote its log canonical bundle by $\bar{K}X = \bigwedge^{\dim \bar{X}} \bar{T}^*X = K_{\bar{X}}(D)$ and its log Kodaira dimension by $\bar{\kappa}_X = \kappa(\bar{X}, \bar{K}X)$, the L-dimension of $\bar{K}X$. We call $(\bar{X}, D)$ to be of log general type if $\bar{\kappa}_X = \dim \bar{X}$. Finally let $\alpha_X : X \to \mathcal{A}_X$ be the quasi-Albanese map. It is a holomorphic map which extends to a rational map $\bar{\alpha}_X : \bar{X} - \to \bar{\mathcal{A}}_X$ (Iitaka '76 [14]).

It is known that for any log manifold such that $\bar{q}_X > \dim \bar{X}$, any entire holomorphic curve $f : \mathbb{C} \to X$ is algebraically degenerate (this means contained in a proper algebraic subvariety of $\bar{X}$). More generally, by results of Noguchi '81 [20] and Noguchi-Winkelmann '02 [22] one has (defining the log structure and $\bar{q}_X$ on a Kähler manifold in a similar way as above):

Theorem 1 (Noguchi, Noguchi-Winkelmann) *Let $\bar{X}$ be a compact Kähler manifold and D be a hypersurface in $\bar{X}$. If $\bar{q}_X > \dim \bar{X}$, then any entire holomorphic curve $f : \mathbb{C} \to X$ is analytically degenerate (this means contained in a proper analytic subset of $\bar{X}$).*

In this paper we are interested in the case of log manifolds $(\bar{X}, D)$ with $\bar{q}_X = \dim \bar{X}$. We restrict our attention to Brody curves, this means entire

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curves $f: \mathbb{C} \to X$ where its derivative f' is bounded in $\bar{X}$, for which we resolve the problem completely in dimension 2:

Theorem 2 *Let $(\bar{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$ and with log Kodaira dimension $\bar{\kappa}_X \geq 1$. Then every Brody curve $f: \mathbb{C} \to X$ is algebraically degenerate.*

The proof will be given in section 3.

We would like to remark that, since Theorem 12 can be proved for Kähler surfaces, since semi-Albanese and log Bloch theorem exist also in the Kähler case by Noguchi-Winkelmann '02 [22] and since Kähler surfaces with algebraic dimension two are projective ([1]), Theorem 2 follows more generally for $\bar{X}$ a Kähler surface.

Corollary 3 *Let $(\bar{X}, D)$ a log surface with log irregularity $\bar{q}_X = 2$ and log Kodaira dimension $\bar{\kappa}_X \geq 1$. If X does not contain non-hyperbolic algebraic curves and if D is hyperbolically stratified (this means that every irreducible component minus all the others is a hyperbolic curve), then X is complete hyperbolic and hyperbolically imbedded in $\bar{X}$.*

Proof This follows from Theorem 2 and from the main result of Green '77 [12]. □

Another consequence of Theorem 2 is the "best possible" result for algebraic degeneracy in the three component case of complements of plane curves (see Dethloff-Schumacher-Wong '95 [9] and [10], Bertheloot-Duval '01 [2]):

Corollary 4 *Let $D \subset \mathbb{P}^2$ be a curve of degree at least four consisting of three components. Then every Brody curve $f: \mathbb{C} \to \mathbb{P}^2 \setminus D$ is algebraically degenerate.*

Proof $\bar{K}(\mathbb{P}^2 \setminus D) = \mathcal{O}((\deg D) - 3)$ is very ample and, hence, big. Let $D_i = \{f_i = 0\}$, where f_i is a homogeneous equation for D_i, for for $i = 1, 2, 3$. Then

$$\omega_j = d \log \left(\frac{f_j^{\deg f_3}}{f_3^{\deg f_j}} \right) \ , \ j = 1, 2,$$

are linearly independent log forms, and it is easy to see that there are no others. Hence, $\bar{q}_{(\mathbb{P}^2 \setminus D)} = 2$. □

Remark 5 *a) In view of hyperbolicity questions, the restriction to Brody curves is not essential, see Corollary 3.*

b) The conditions that the log surface has log Kodaira dimension $\bar{\kappa}_X \geq 1$ in Theorem 2 or that $\deg D \geq 4$ in Corollary 4 are essential, as the following counterexample show.

Proposition 6 *Let E be an elliptic curve, $\bar{X} = E \times E$ respectively $\bar{X} = E \times \mathbb{P}_1$ respectively $\bar{X} = (\mathbb{P}_1)^2$ or $\mathbb{P}_2$ and $X = \bar{X} \setminus D = E \times E$ respectively $X = E \times \mathbb{C}^*$ respectively $X = (\mathbb{C}^*)^2$. Then $\bar{q}_X = 2$ and $q_{\bar{X}} = 2$ respectively $q_{\bar{X}} = 1$ respectively $q_{\bar{X}} = 0$, but $\bar{K}X = 0$ is trivial and, hence, these log surfaces do not satisfy $\bar{\kappa}_X \geq 1$. The curve $f : \mathbb{C} \to X \subset \bar{X}$ obtained by the curve*

$$\hat{f} : \mathbb{C} \to \mathbb{C}^2; t \to (t, at + b) \text{ with } a \in \mathbb{R} \setminus \mathbb{Q},\ b \in \mathbb{C} \tag{1}$$

on the universal cover $\mathbb{C}^2$ of X is Brody with respect to any metric on $\bar{X}$, but not algebraically degenerate.

For a proof, if necessary, see the appendix.

Remark 7 *When $\bar{X}$ is the projective plane, the problem even in the cases of log irregularity zero (for example Demailly-ElGoul '00 [7], ElGoul '03 [11]) or one (Rousseau '03 [24]) are quite well understood for the general boundary.*

Remark 8 *It is easily obtained from Hodge theory due to Deligne '71 [6] (see, for example, Catanese '84 [5]) that we have, for $(\bar{X}, D)$ a log surface:*

$$\operatorname{rank}_{\mathbb{Z}} \mathrm{NS}(\bar{X}) \geq \operatorname{rank}_{\mathbb{Z}} \{c_1(D_i)\}_{i=1}^{k} = k - \bar{q}_X + q_{\bar{X}} \tag{2}$$

where $D_1, ..., D_k$ are the irreducible components of D and $NS(\bar{X})$ denotes the Neron-Severi group of $\bar{X}$. This may be seen from the exact sequence (6) *and can also be deduced from the proof of Theorem 1.2(i) of Noguchi-Winkelmann '02 [22], p.605. But there does not seem to be an easy way to profit from this, unless one assumes some bound on the Neron-Severi group of $\bar{X}$.*

Connected with this, we would also like to mention again the work of Noguchi-Winkelmann '02 [22] which deals with this question in all dimensions and even with Kähler manifolds, especially with log tori or with log manifolds having small Neron-Severi groups, under the additional condition that all irreducible components D_i of D are ample. But as can be seen from the equation (2) above, their results never concern the case of log surfaces with log irregularity $\bar{q}_X \leq 2$.

We now give an indication of our method of proof. By the result of Kawamata '81 [15] and Kawamata-Viehweg [16] on finite coverings of semi-abelian varieties, we may assume X is of log general type. In the case $q_{\bar{X}} = 2$, the compactified quasi-Albanese map $\bar{\alpha}_X$ is a morphism and so this theorem is a direct consequence of theorems of McQuillan '98 and El Goul '03 ([17], [18], [11]) concerning foliations. If $q_{\bar{X}} < 2$, $\bar{\alpha}_X$ can have points of indeterminacy so that Brody curves are not preserved by $\bar{\alpha}_X$ in general. But from value distribution theory, the order of growth of a holomorphic curve is preserved under rational maps and Brody curves are of order at most two. Using this, the key analysis in this paper consists of a detailed study of the geometry of the quasi-Albanese map (in particular at its points of indeterminacy) with respect to f to reduce the order of $\bar{\alpha}_X \circ f$ to one or less. Then $\bar{\alpha}_X \circ f$ is either constant or a leaf of a linear foliation on $\mathcal{A}_X$ and our theorem follows again by the theorems of McQuillan and El Goul. We do this componentwise where in the case $q_{\bar{X}} = 1$, we use the fact from Noguchi-Winkelmann-Yamanoi '02 [23] that one can choose a metric on $\bar{\mathcal{A}}_X$ which lifts to the product metric on the universal cover $\mathbb{C} \times \mathbb{P}_1$ of $\bar{\mathcal{A}}_X$. In the case $q_{\bar{X}} = 0$ we take rational monomials of the components of $\bar{\alpha}_X$ motivated by arranging residues in a way that allows us to control the points of indeterminacy of the resulting map with respect to f. This linear algebra of residues and the corresponding rational maps seem to be a new tool in understanding the rational endomorphisms of projective space.

We finally remark that we can give an elementary proof of the result of McQuillan and El Goul in the case of linear foliations on $\mathcal{A}_X$ by using techniques similar to those given in Bertheloot-Duval '01 [2].

2 Some basics

2.1 Log manifolds and residues of log 1-forms

Let $\bar{X}$ be a complex manifold with a normal crossing divisor D. This means that around any point x of $\bar{X}$, there exist local coordinates $z_1, ..., z_n$ centered at x such that D is defined by $z_1 z_2 ... z_l = 0$ in some neighborhood of x and for some $0 \leq l \leq n$. The pair $(\bar{X}, D)$ will be called a *log-manifold*. Let $X = \bar{X} \setminus D$.

Following Iitaka '82 [13], we define the logarithmic cotangent sheaf

$$\bar{T}^* X = \bar{\Omega} X = \Omega(\bar{X}, \log D)$$

as the locally free subsheaf of the sheaf of meromorphic 1-forms on $\bar{X}$, whose restriction to X is $TX = \Omega X$ (where we identify vector bundles and their sheaves of sections) and whose localization at $x \in \bar{X}$ is of the form

$$\bar{\Omega} X_x = \sum_{i=1}^{l} \mathcal{O}_{\bar{X},x} \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} \mathcal{O}_{\bar{X},x} dz_j, \tag{3}$$

where the local coordinates $z_1, ..., z_n$ around x are chosen as before. Its dual, the logarithmic tangent sheaf $\bar{T}^* X = T(\bar{X}, -\log D)$, is a locally free subsheaf of the holomorphic tangent bundle $T\bar{X}$ over $\bar{X}$. Its restriction to X is identical to TX, and its localization at $x \in \bar{X}$ is of the form

$$\bar{T} X_x = \sum_{i=1}^{l} \mathcal{O}_{\bar{X},x} z_i \frac{\partial}{\partial z_i} + \sum_{j=l+1}^{n} \mathcal{O}_{\bar{X},x} \frac{\partial}{\partial z_j}. \tag{4}$$

Let $x \in \bar{X}$ be a point and $z_1, ..., z_n$ be local coordinates centered at x such that D is defined by $z_1 z_2 ... z_l = 0$ in some neighborhood of x and for some $l \leq n$. Let ω be a log 1-form defined around x, so that by (3) we have

$$\omega_x = \sum_{i=1}^{l} (h_i)_x \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} (h_j)_x dz_j \tag{5}$$

Then we call, for $i = 1, ..., l$, the complex number $(h_i)_x(x) \in \mathbb{C}$ the *residue* of ω in x on the local irreducible branch of D given by $z_i = 0$. Since we do not assume simple normal crossing, we may have several such local irreducible components for D at x even if D is irreducible. But if $\bar{X}$ is compact, it is easy to see that for any (global) irreducible component D_j of D, the residue is constant on D_j (meaning it is the same for all points $x \in D_j$). In fact, whether $\bar{X}$ is compact or not, we have the exact sequence of sheaves:

$$0 \to \Omega_{\bar{X}} \to \Omega_{\bar{X}}(\log D) \overset{\text{Res}}{\to} \mathcal{O}(\hat{D}) \to 0 \, , \tag{6}$$

where $\hat{D}$ is the normalization of D.

We will use the following result of McQuillan '98 [17] and Brunella '99 [3] extended to the log context by El Goul '03 [11, Theorem 2.4.2]:

Theorem 9 *Let $(\bar{X}, D)$ be a log surface of log general type. Let $f : \mathbb{C} \to X$ be an entire curve. Suppose that there exists a foliation $\mathcal{F}$ on $\bar{X}$ such that f is contained in a leaf of $\mathcal{F}$. Then f is algebraically degenerate in $\bar{X}$.*

2.2 Quasi-Albanese maps

We first recall the definition and some basic facts on semi-abelian varieties (see Iitaka '76 [14]).

A quasi-projective variety G is called a *semi-abelian variety* if it is a complex commutative Lie group which admits an exact sequence of groups

$$0 \to (\mathbb{C}^*)^l \to G \xrightarrow{\pi} A \to 0 \, , \tag{7}$$

where A is an abelian variety of dimension m.

From the standard compactification $(\mathbb{C}^*)^l \subset (\mathbb{P}_1)^l$, which is equivariant with respect to the $(\mathbb{C}^*)^l$ action, we obtain a completion $\bar{G}$ of G as the $(\mathbb{P}_1)^l$ fiber bundle associated to the $(\mathbb{C}^*)^l$ principal bundle $G \to A$. This is a smooth compactification of G with a simple normal crossing boundary divisor S. The projection map

$$\bar{\pi} : \bar{G} \to A$$

has the structure of a $(\mathbb{P}_1)^l$-bundle.

We denote the natural action of G on $\bar{G}$ on the right as addition. It follows that the exponential map from the Lie algebra $\mathbb{C}^n$ to G is a group homomorphism and, hence, it is also the universal covering map of $G = \mathbb{C}^n/\Lambda$, where $\Lambda = \pi_1(G)$ is a discrete subgroup of $\mathbb{C}^n$ and $n = m + l$.

Following Iitaka '76 [14], we have the following explicit trivialization of the log tangent and cotangent bundles of $\bar{G}$: Let $z_1, ..., z_n$ be the standard coordinates of $\mathbb{C}^n$. Since $dz_1, ..., dz_n$ are invariant under the group action of translation on $\mathbb{C}^n$, they descend to forms on G. There they extend to logarithmic forms on $\bar{G}$ along S, which are elements of $H^0(\bar{G}, \bar{\Omega} G)$. These logarithmic 1-forms are everywhere linearly independent on $\bar{G}$. Thus, they globally trivialize the vector bundle $\bar{\Omega} G$. Finally, we note that these log 1-forms are invariant under the group action of G on $\bar{G}$, and, hence, the associated trivialization of $\bar{\Omega} G$ over $\bar{G}$ is also invariant.

Let now $(\bar{X}, D)$ be again a log surface and $\alpha_{\bar{X}} : \bar{X} \to \mathcal{A}_{\bar{X}}$ the Albanese map of $\bar{X}$ (it can be constant if $q_{\bar{X}} = 0$). Taking into account also the log 1-forms, Iitaka '76 [14] introduced the quasi-Albanese map $\alpha_X : X \to \mathcal{A}_X$, which is a holomorphic map to the semi-abelian variety $\mathcal{A}_X$, which comes equipped with the exact sequence

$$0 \to (\mathbb{C}^*)^l \to \mathcal{A}_X \xrightarrow{\pi} \mathcal{A}_{\bar{X}} \to 0 \tag{8}$$

and we have the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\alpha_{\bar{X}}"'] & \mathcal{A}_X \arrow[d, "\pi"] \\ & \mathcal{A}_{\bar{X}} \end{tikzcd} \tag{9}$$

He also proved that α_X extends to a rational map $\bar{\alpha}_X : \bar{X} - \to \bar{\mathcal{A}}_X$, and the diagram (9) extends to

$$\begin{tikzcd} \bar{X} \arrow[r, dashed, "\bar{\alpha}_X"] \arrow[dr, "\alpha_{\bar{X}}"'] & \bar{\mathcal{A}}_X \arrow[d, "\bar{\pi}"] \\ & \mathcal{A}_{\bar{X}} \end{tikzcd} \tag{10}$$

In general the $(\mathbb{P}_1)^l$-bundle $\bar{\mathcal{A}}_X \xrightarrow{\bar{\pi}} \mathcal{A}_{\bar{X}}$ is not trivial. But Noguchi-Winkelmann-Yamanoi '02 [23] observed that the transition functions of the $(\mathbb{P}_1)^l$-bundle $\bar{\pi} : \bar{\mathcal{A}}_X \to \mathcal{A}_{\bar{X}}$ (as the structure group $(\mathbb{C}^*)^l$ can always be reduced to the subgroup defined by $|z_i| = 1,\ i = 1, .., l$) can be chosen to be isometries with respect to the product Fubini-Study metric on $(\mathbb{P}^1)^l$.

Proposition 10 *There exists a metric h on $\bar{\mathcal{A}}_X$ so that the universal cover map*

$$\Big(\mathbb{C}^m \times (\mathbb{P}_1)^l, \text{eucl.} \times \text{product FS}\Big) \to (\bar{\mathcal{A}}_X, h)$$

is a local isometry.

Although we will not use it, it is of interest to recall the following result of Bertheloot-Duval '01 [2]:

Theorem 11 *A Brody curve $f : \mathbb{C} \to (\mathbb{C}^*)^t \subset \mathbb{P}_t$ is linear with respect to the standard coordinates coming from the universal cover $\mathbb{C}^t$ of $(\mathbb{C}^*)^t$.*

We also recall the following result of Kawamata '81 [15], Theorem 27:

Theorem 12 *Let X be a normal algebraic variety, $\mathcal{A}$ a semi-abelian variety and let $f : X \to \mathcal{A}$ be a finite morphism. Then $\bar{\kappa}_X \geq 0$ and there exist a semi-abelian subvariety $\mathcal{B} \subset \mathcal{A}$, etale covers $\tilde{X}$ and $\tilde{\mathcal{B}}$ of X and $\mathcal{B}$, respectively, and a normal algebraic variety $\tilde{Y}$ such that*

(1) $\tilde{Y}$ is finite over $\mathcal{A}/\mathcal{B}$.

(2) $\tilde{X}$ is a fiber bundle over $\tilde{Y}$ with fiber $\tilde{\mathcal{B}}$ and translations by $\tilde{\mathcal{B}}$ as structure group.

(3) $\bar{\kappa}_{\tilde{Y}} = \dim \tilde{Y} = \bar{\kappa}_X$.

For the rest of this subsection, we restrict to the case $\dim \bar{X} = 2$. Let $\omega \in H^0(\bar{X}, \bar{T}^* X)$ be a log 1-form with residues $a_j \in \mathbb{Z}$ along the irreducible components D_j of D, $j = 1, ..., k$. We define a holomorphic function

$$\Phi : X \to \mathbb{C}^*; \ \Phi(x) = \exp\left(\int_{x_0}^{x} \omega\right), \tag{11}$$

where x_0 is a fixed point in X. We claim that this function extends to a rational function $\bar{\Phi} : \bar{X} - \to \mathbb{P}_1$: Let $P \in D_1$ (respectively $P \in D_1 \cap D_2$) and let z_1, z_2 be local coordinates around P such that $D_1 = \{z_1 = 0\}$ (respectively $D_1 = \{z_1 = 0\}$ and $D_2 = \{z_2 = 0\}$). Then it follows from equation (11) that there exists a holomorphic function
$h : U(P) \to \mathbb{C}^*$ on a neighborhood $U(P)$ of P such that

$$\bar{\Phi}(z_1, z_2) = z_1^{a_1} h(z_1, z_2) \quad \Big(\text{respectively } \bar{\Phi}(z_1, z_2) = z_1^{a_1} z_2^{a_2} h(z_1, z_2)\Big). \tag{12}$$

In more detail, if $D_1 = \{z_1 = 0\}$ around P, then we have

$$\omega = h_1(z)\frac{dz_1}{z_1} + h_2(z)dz_2 = a_1 \frac{dz_1}{z_1} + \frac{h_1(z) - a_1}{z_1} dz_1 + h_2(z) dz_2 \,.$$

By the first Riemann extension theorem, the function $\frac{h_1(z)-a_1}{z_1}$ extends to a holomorphic function, so we have $\omega = a_1\frac{dz_1}{z_1} + \omega_h$, where ω_h is a holomorphic form. Now by (11) we get

$$\Phi(z) = \exp\left(\int_{x_0}^{z} \omega\right) = \exp\left(\int_{x_0}^{z} a_1\frac{dz_1}{z_1} + \omega_h\right) = \exp\left(a_1(\log z_1 + 2\pi i \mathbb{Z})\right)\cdot\exp(h_1(z))$$

$$= \left(\exp\left(\log z_1 + 2\pi i\mathbb{Z}\right)\right)^{a_1}\cdot\exp\left(h_1(z)\right) = z_1^{a_1}h(z)$$

The other equality follows in the same way.

From this we get by the second Riemann extension theorem, GAGA and by the local description of points of indeterminacy the following.

Proposition 13 *The holomorphic map $\Phi : X \to \mathbb{C}^*$ extends to a rational function $\bar{\Phi} : \bar{X} \to \mathbb{P}_1$. It is a morphism outside the points of intersection of the irreducible components of D. More precisely, a point of $D_{j_1} \cap D_{j_2} \subset \bar{X}$ is a point of indeterminacy of $\bar{\Phi}$ iff $a_{j_1}\cdot a_{j_2} < 0$. In particular points of indeterminacy never occur at self-intersection points of a component.*

Consider the case $\bar{q}_X = 2$, $q_{\bar{X}} = 0$. We now describe the components of the map $\alpha_X = ((\alpha_X)_1, (\alpha_X)_2) : X \to (\mathbb{C}^*)^2$ in more detail. The following basic facts follow from the exact sequence (6) and a duality argument in Hodge theory, and they can be found for example in Noguchi-Winkelmann '02 [22]:

Proposition 14 *We can choose the basis $\omega_1, \omega_2 \in H^0(\bar{X}, \bar{T}^*X)$ such that the residue $a_{ij} \in \mathbb{Z}$ for all $i = 1, 2$ and $j = 1, ..., k$, where a_{ij} is the residue of ω_i along the irreducible component D_j of D. The matrix of residues so obtained has rank 2:*

$$\begin{array}{ccccc} & D_1 & D_2 & ... & D_k \\ \omega_1 & a_{11} & a_{12} & ... & a_{1k} \\ \omega_2 & a_{21} & a_{22} & ... & a_{2k} \end{array} \qquad (13)$$

and we have, as in (11),

$$(\alpha_X)_i(x) = \exp\left(\int_{x_0}^{x} \omega_i\right), \qquad (14)$$

where x_0 is a fixed point in X.

2.3 Brody curves, maps of order 2 and limit sets of entire curves

Let $(\bar{X}, D)$ be a log manifold and $f : \mathbb{C} \to X$ be an entire curve. We recall that f is a Brody curve if the derivative of f with respect to some (and so any) hermitian metric on $\bar{X}$ is bounded.

Following Noguchi-Ochiai '90 [21], we have the characteristic function $T_f(r,\omega) = \int_1^r \frac{dt}{t}\int_{|z|<t} f^*\omega$ of a holomorphic map $f : \mathbb{C} \to Y$ with respect to a real continuous $(1,1)$-form ω on a Kähler manifold Y. If Y is compact and ω_H denotes the $(1,1)$-form associated to a hermitian metric H on Y, then it is easy to see ([21], (5.2.19)) that $\rho_f := \overline{\lim}_{r\to\infty}\frac{\log T_f(r,\omega_H)}{\log r}$ is independant of the hermitian metric H. The map f is said to be of *order at most 2* if $\rho_f \le 2$. Since the derivative of a Brody curve in the projective variety $\bar{X}$ is bounded with respect to hermitian metrics on $\bar{X}$, Brody curves are easily seen to be of order at most 2. By the classical Weierstrass theorem we get also that a curve $f : \mathbb{C} \to \mathbb{P}_N$ of order at most 2 which omits the coordinate hyperplanes can be written as $(1 : \exp(P_1(z)) : ... : \exp(P_n(z)))$, with polynomials P_i in the variable $z \in \mathbb{C}$ of degree at most 2.

We now prove that the property of having order at most 2 is preserved under rational maps (see also [9] for a similar result).

Lemma 15 *Let $f : \mathbb{C} \to \mathbb{P}_N$ be a curve of order at most 2 and*

$$R : \mathbb{P}_N - \to \mathbb{P}_M$$

be a rational map (not necessarily dominant) such that $f(\mathbb{C})$ is not contained in the set of indeterminacy of R. Then the curve $R \circ f : \mathbb{C} \to \mathbb{P}_M$ is of order at most 2.

Proof Let $f = (f_0 : ... : f_N)$ be a reduced representation and

$$R = (R_0 : ... : R_M)$$

be a (not necessarily reduced) representation by polynomials $R_0, ..., R_M$ of degree p. Then

$$(f_0^p : ... : f_N^p : R_0 \circ f : ... : R_M \circ f)$$

is a reduced representation of a curve $F : \mathbb{C} \to \mathbb{P}_{N+M+1}$, and without loss of generality $R_0 \circ f \not\equiv 0$. We have by [21], p.183

$$T_F(r,\omega_{\text{FS}}) = \int_0^{2\pi} \frac{1}{2}\log\left(|f_0^p|^2 + ... + |f_N^p|^2 + |R_0 \circ f|^2 + ... + |R_M \circ f|^2\right) d\theta + O(1) \le$$

$$\int_0^{2\pi} \frac{1}{2}\log\left(|C|(|f_0^p|^2 + ... + |f_N^p|^2)\right) d\theta + O(1) = \int_0^{2\pi} \frac{p}{2}\log\left(|f_0|^2 + ... + |f_N|^2\right) d\theta + O(1)$$

$$= p \cdot T_f(r,\omega_{\text{FS}}) + O(1)$$

By Noguchi-Ochiai '90 [21] (5.2.29) and (5.2.30) we get

$$T_{R\circ f}(r,\omega_{\text{FS}}) \le \sum_{j=1}^M T_{\frac{R_j\circ f}{R_0\circ f}}(r,\omega_{\text{FS}}) \le M\cdot T_F(r,\omega_{\text{FS}}) + O(1) \le M\cdot p\cdot T_f(r,\omega_{\text{FS}}) + O(1)$$

From this the lemma follows. □

Finally, we need the definition and some observations on the limit set of an entire curve $f : \mathbb{C} \to \bar{X}$ as given by Nishino-Suzuki '80 [19]. For $r > 0$, put $\Delta_r^c = \{z \in \mathbb{C} : |z| > r\}$. Let $\overline{f(\Delta_r^c)} \subset \bar{X}$ be the closure (with respect to the usual topology) of $f(\Delta_r^c)$ in $\bar{X}$, and $f(\infty) := \bigcap_{r>0} \overline{f(\Delta_r^c)}$. We remark that $f(\infty)$ is exactly the set of all points $p \in \bar{X}$ such that there exists a sequence $(z_v)_{v \in \mathbb{N}}$ with $|z_v| \to \infty$ and $f(z_v) \to p$.

Proposition 16 *Let $f : \mathbb{C} \to X$ be as above for a log surface $(\bar{X}, D)$.*

a) f extends to a holomorphic map $\hat{f} : \mathbb{P}_1 \to \bar{X}$ iff $f(\infty)$ consists of exactly one point.

b) If $f(\infty) \subset C$ is contained in an algebraic curve $C \subset \bar{X}$ and does not consist of one point, only, then $f(\infty)$ is equal to the union of some of the irreducible components of C.

3 Proof of Theorem 2

3.1 Reduction to the case log general type and linear degeneracy

In this subsection we first reduce the proof of Theorem 2 to the case of log general type and dominant quasi-Albanese map. Then we reduce it further to the existence of a dominant morphism $\Psi : X \to \mathcal{A}_X$ such that the composite map $\Psi \circ f : \mathbb{C} \to \mathcal{A}_X$ is linearly degenerate with respect to the universal cover $\mathbb{C}^2 \to \mathcal{A}_X$ (Proposition 17 below).

We first consider the case where the image $\bar{Y} := \bar{\alpha}_X(\bar{X}) \subset \bar{\mathcal{A}}_X$ of $\bar{X}$ under the compactified quasi-Albanese map $\bar{\alpha}_X : \bar{X} \to \bar{\mathcal{A}}_X$ is a proper algebraic subvariety. Since $\bar{q}_X = 2$, so in particular X admits nontrivial log 1-forms and, hence, $\bar{Y}$ cannot degenerate to a point, we have $\dim \bar{Y} = 1$. By the log Bloch's theorem due to Noguchi '81 [20], we get that the Zariski closure $Z = \overline{(\alpha_X \circ f)(\mathbb{C})}^{\mathrm{Zar}} \subset \mathcal{A}_X$ of the image of $\mathbb{C}$ under the map $\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X$ is a translate of an algebraic subgroup of $\mathcal{A}_X$. But by the universal property of the quasi-Albanese map (see Iitaka '76 [14]), $Y = \bar{Y} \cap \mathcal{A}_X$ cannot be such a translate. Hence, $Z \subset Y$ is a proper algebraic subvariety of the curve Y. So the map $\bar{\alpha}_X \circ f$ is constant, and, hence, f is algebraically degenerate. $\Box$

We next consider the case where the quasi-Albanese map $\alpha_X : X \to \mathcal{A}_X$ is a dominant morphism and the log Kodaira dimension $\bar{\kappa}_X = 1$. For this part of the proof we adopt the notations of Theorem 12, meaning we allow arbitrary boundary divisors. This is not a problem since we do not need the condition that $f : \mathbb{C} \to X$ is Brody for this part of the proof and since we can always lift an entire curve f through a birational map which is biholomorphic on X and the property of f to be algebraically degenerate is of course invariant under such a birational map. By the same token, we may assume that the morphism $\alpha_X : X \to \mathcal{A}_X$ extends to a morphism $\bar{\alpha}_X : \bar{X} \to \bar{\mathcal{A}}_X$ (here we use a desingularization of the rational map $\bar{\alpha}_X$ which again is biholomorphic on X). Let $\bar{X} \xrightarrow{\bar{\beta}} \bar{Y} \xrightarrow{\bar{\gamma}} \bar{\mathcal{A}}_X$ be a Stein factorization of the morphism $\bar{\alpha}_X$. Then $\bar{Y}$ is a normal variety which compactifies $Y = \bar{\beta}(X)$, and f is algebraically degenerate (in X) if and only if the map $\beta \circ f$ is algebraically degenerate (in Y). Now $\gamma : Y \to \mathcal{A}_X$ is a finite morphism, so we can apply Theorem 12 to get that there exists a finite etale cover $\tilde{Y} \to Y$ and a normal algebraic variety $\tilde{Z}$ such that $\tilde{\pi} : \tilde{Y} \to \tilde{Z}$ is a fiber bundle and such that

$$\dim \tilde{Z} = \bar{\kappa}_{\tilde{Z}} = \bar{\kappa}_{\tilde{Y}} = \bar{\kappa}_X = 1.$$

Hence $\tilde{Z}$ is a log curve of log general type, and so hyperbolic. Thus any entire curve in $\tilde{Z}$ is constant and therefore so is the curve $\tilde{\pi} \circ \widetilde{(\beta \circ f)} : \mathbb{C} \to \tilde{Z}$, where $\widetilde{(\beta \circ f)}$ denotes the lift of $\beta \circ f$ through the etale cover $\tilde{Y} \to Y$. But this implies that $\widetilde{(\beta \circ f)}$ and, hence, $\beta \circ f$ is algebraically degenerate. □

So we may assume from now on that $(\bar{X}, D)$ is a log surface of log general type and that the quasi-Albanese map $\alpha_X : X \to \mathcal{A}_X$ is dominant. The rest of section 3 will deal with the proof of this case (which is the difficult one).

Proposition 17 *Let $(\bar{X}, D)$ a log surface of log general type. Let $\Psi : X \to \mathcal{A}_X$ be a dominant morphism which extends to a rational map $\bar{\Psi} : \bar{X} - \to \bar{\mathcal{A}}_X$. Let $f : \mathbb{C} \to X$ be an entire curve. Assume that the map $\Psi \circ f : \mathbb{C} \to \mathcal{A}_X$ is linearly degenerate with respect to the universal cover $\mathbb{C}^2 \to \mathcal{A}_X$. Then f is algebraically degenerate.*

Proof The map $\Psi \circ f : \mathbb{C} \to \mathcal{A}_X$ is linearly degenerate with respect to the universal cover $\mathbb{C}^2 \to \mathcal{A}_X$. Hence, $\Psi \circ f$ is a leaf of the linear foliation given by a 1-form with constant coefficients on $\mathbb{C}^2$, descending to a nowhere vanishing log 1-form ω on $\bar{\mathcal{A}}_X$ (see subsection 2.2). It corresponds to a meromorphic nowhere vanishing 1-form on $\bar{\mathcal{A}}_X$ without poles on $\mathcal{A}_X$. Now $\bar{\Psi} : \bar{X} - \to \bar{\mathcal{A}}_X$ is a dominant rational map, and so the meromorphic form ω pulls back via $\bar{\Psi}^*$ to a nonvanishing meromorphic 1-form $\bar{\Psi}^* \omega$ which is holomorphic over X. By construction $f : \mathbb{C} \to X$ lifts to a leaf of the foliation $\mathcal{F}$ on $\bar{X}$ given by the meromorphic 1-form $\bar{\Psi}^* \omega$ (see Brunella '00 [4]). By Theorem 9 it follows that f is algebraically degenerate, which finishes the proof of Proposition 17. □

3.2 The case $q_{\bar{X}} = 2$

We finish the proof of Theorem 2 in this case by using Proposition 17, where $\Psi = \bar{\Psi} = \alpha_{\bar{X}}$ is the Albanese map.

The euclidean metric of the universal cover $\mathbb{C}^2$ of the Albanese torus $\mathcal{A}_{\bar{X}}$ descends to a metric h on it. We may choose a hermitean metric g on $\bar{X}$ such that $\alpha_{\bar{X}}^* h \leq g$ and we have

$$\mathbb{C} \xrightarrow{f} (\bar{X}, g) \xrightarrow{\alpha_{\bar{X}}} (\mathcal{A}_{\bar{X}}, h) \leftarrow (\mathbb{C}^2, \text{eucl.})$$

Now since f is a Brody curve, we have $g(f') \leq C$. By composing with the Albanese map, we therefore get

$$h\left((\alpha_{\bar{X}} \circ f)'\right) \leq C.$$

After lifting to $\mathbb{C}^2$, the components of $(\alpha_{\bar{X}} \circ f)'$ are bounded holomorphic functions. Hence, by Liouville's theorem, they are constant. So the map $\alpha_{\bar{X}} \circ f : \mathbb{C} \to \mathcal{A}_X$ is linearly degenerate with respect to the universal cover $\mathbb{C}^2 \to \mathcal{A}_X$, and we conclude the proof with Proposition 17. $\square$

3.3 The case $q_{\bar{X}} = 1$

We finish the proof of Theorem 2 in this case by using Proposition 17, where $\bar{\Psi} = \bar{\alpha}_X$ is the quasi-Albanese map.

We have the following diagram (see (5)):

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] & \bar{X} \arrow[r, dashed, "\bar{\alpha}_X"] \arrow[dr, "\alpha_{\bar{X}}"'] & \bar{\mathcal{A}}_X \arrow[d] \\
 & & \mathcal{A}_{\bar{X}}
\end{tikzcd}$$

As in the case $q_{\bar{X}} = 2$, we get that $\alpha_{\bar{X}} \circ f$ is linear with respect to the coordinates from the universal covering $\mathbb{C} \to \mathcal{A}_{\bar{X}}$.

Let $I \subset D \subset \bar{X}$ be the (finite) set of points of indeterminacy of $\bar{\alpha}_X$, $U \subset \mathcal{A}_{\bar{X}}$ a small neighborhood of $\alpha_{\bar{X}}(I)$, $V = \alpha_{\bar{X}}^{-1}(U)$ and $W = f^{-1}(V)$.

Since $\bar{\alpha}_X$ is a morphism on the compact set $\bar{X} \setminus V$, $(\bar{\alpha}_X \circ f)'$ is bounded on $\mathbb{C} \setminus W$ with respect to any hermitian metric h on $\bar{\mathcal{A}}_X$.

Brody curves are of order ≤ 2. Hence, by Lemma 15,

$$\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X \subset \bar{\mathcal{A}}_X$$

is of order ≤ 2.

Let $\mathbb{C} \times \mathbb{P}_1 \to \bar{\mathcal{A}}_X$ be the universal cover, and let

$$(\widetilde{\alpha_X \circ f}) : \mathbb{C} \to \mathbb{C} \times \mathbb{C}^* \subset \mathbb{C} \times \mathbb{P}_1$$

be the lift of the map $\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X \subset \bar{\mathcal{A}}_X$ to $\mathbb{C} \times \mathbb{P}_1$. If $\mathrm{pr}_1 : \mathbb{C} \times \mathbb{P}_1 \to \mathbb{C}$ respectively $\mathrm{pr}_2 : \mathbb{C} \times \mathbb{P}_1 \to \mathbb{P}_1$ denote the projections to the first respectively the second factor, we get that $\mathrm{pr}_1 \circ \widetilde{(\alpha_X \circ f)}$ is the lift of $\alpha_{\bar{X}} \circ f$ through the universal cover $\mathbb{C} \to \mathcal{A}_{\bar{X}}$, which we know already to be a linear function. Define $\Phi := \mathrm{pr}_2 \circ \widetilde{(\alpha_X \circ f)} : \mathbb{C} \to \mathbb{C}^* \subset \mathbb{P}_1$. If we prove that Φ is of the form $\Phi(z) = \exp(P(z))$ with a linear polynomial $P(z)$, then the proof is finished by Proposition 17.

By Proposition 10, there exists a metric h on $\bar{\mathcal{A}}_X$ such that the universal covering map $(\mathbb{C} \times \mathbb{P}_1, \text{eucl.} \times \text{FS}) \to (\bar{\mathcal{A}}_X, h)$ is a local isometry. By this, we get the existence of of a constant C such that

$$||\Phi'|| \leq C \tag{15}$$

Furthermore, we get the following estimate for the characteristic function.

$$\begin{aligned} T_\Phi(r, \omega_{\mathrm{FS}}) &= \int_1^r \frac{dt}{t} \int_{|z|<t} \Phi^* \omega_{\mathrm{FS}} \leq \int_1^r \frac{dt}{t} \int_{|z|<t} \widetilde{(\alpha_X \circ f)}^* \omega_{\mathrm{eucl.} \times \mathrm{FS}} \\ &= \int_1^r \frac{dt}{t} \int_{|z|<t} (\alpha_X \circ f)^* \omega_h = T_{(\alpha_X \circ f)}(r, \omega_h) \end{aligned}$$

Hence, from $\rho_{(\alpha_X \circ f)} \leq 2$, it follows $\rho_\Phi \leq 2$. So $\Phi \circ f(z) = \exp(P(z))$ with $\deg P \leq 2$. If $\deg P \leq 1$, the proof is finished. So we may assume $\deg P = 2$. We get, by a linear coordinate change in $\mathbb{C}$ and a multiplicative transformation in $\mathbb{C}^* \subset \mathbb{P}_1$, that

$$P(z) = z^2 \, .$$

Since $U \subset \mathcal{A}_{\bar{X}}$ is a small neighborhood of the finite number of points $\alpha_{\bar{X}}(I)$ and the map $\alpha_{\bar{X}} \circ f$ is **linear** with respect to the coordinates from the universal cover $\mathbb{C} \to \mathcal{A}_{\bar{X}}$, we get that $W = (\alpha_{\bar{X}} \circ f)^{-1}(U)$ is the union of all translations by the lattice of a finite number of small open sets.

Hence, there exists a sequence on the diagonal

$$(z_v = x_v + i x_v)_{v \to \infty} \subset \mathbb{C} \setminus W \text{ with } x_v \to \infty$$

We have

$$|(\Phi \circ f)'|_{FS}(z) = \frac{|2z| \exp(x^2 - y^2)}{1 + \exp\left(2(x^2 - y^2)\right)}$$

and, hence, since $(\Phi \circ f)'$ is bounded on $\mathbb{C} \setminus W$ by (15):

$$C \geq |(\Phi \circ f)'|_{FS}(z_v) = \frac{|2z_v| \exp(0)}{1 + \exp(0)} \to \infty$$

This is a contradiction since $|z_v| \to \infty$. So the assumption deg$P = 2$ was wrong. □

3.4 Limit sets and the case $q_{\bar{X}} = 0$

Let $(\bar{X}, D)$ be the log surface and assume $f : \mathbb{C} \to X$ is a Brody curve which is not algebraically degenerate with $f(\infty)$ its limit set. We recall (section 2.2) that for a rational function $\bar{\Phi} : \bar{X} \to \mathbb{P}_1$ which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$, we always have $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree at most 2.

Lemma 18 *Assume there exists a rational function $\bar{\Phi} : \bar{X} - \to \mathbb{P}_1$ which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$ such that $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree (exactly) 2. Then we have $f(\infty) = \bigcup_{j=1}^{l} D_j$, where $D_1, ..., D_l$ is a subset of the set of the irreducible components $D_1, ..., D_k$ of D. Moreover, for* **any** *such rational function $\bar{\Phi}$, we have*
$\bar{\Phi}(D_j \setminus I_{\bar{\Phi}}) \equiv 0$ or $\equiv \infty$ *for $j = 1, ..., l$, where $I_{\bar{\Phi}}$ denotes the set of points of indeterminacy of $\bar{\Phi}$.*

Proof Let $\bar{\Phi} : \bar{X} \to \mathbb{P}_1$ be a rational function which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$ such that $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree (exactly) 2, and let $I_{\bar{\Phi}}$ be the set of points of indeterminacy of $\bar{\Phi}$. We first prove

$$\bar{\Phi}(f(\infty) \setminus I_{\bar{\Phi}}) \subset \{0, \infty\} \subset \mathbb{P}_1 . \qquad (16)$$

By a linear transformation $z \mapsto az+b$ in $\mathbb{C}$ we may assume that $P(z) = z^2 + c$. Then, by a multiplicative transformation $w \mapsto \exp(c) \cdot w$, we may assume $P(z) = z^2$. Assume that (16) does not hold. Then there exists a point $p \in f(\infty) \setminus I_{\bar{\Phi}}$ such that $\bar{\Phi}(p) \in \mathbb{C}^*$. Let $U(p) \subset \bar{X}$ be a neighborhood such that its topological closure (with respect to the usual topology) $\bar{U}(p)$ does not contain any points of $I_{\bar{\Phi}}$. Note that $\bar{\Phi}$ is a holomorphic function in a neighborhood of $\bar{U}(p) \subset \bar{X}$. Since $p \in f(\infty)$, there exists a sequence $(z_v) = (x_v + iy_v)$, $v \in \mathbb{N}$, such that $|z_v| \to \infty$ and $f(z_v) \to p$. Without loss of generality, we may assume that $f(z_v) \in U(p)$ for all $v \in \mathbb{N}$. Then we have

$$\exp(x_v^2 - y_v^2) = |\exp(x_v^2 - y_v^2) \cdot \exp(2ix_v^2 y_v^2)| = |\exp(z_v^2)| \qquad (17)$$

$$= |\Phi \circ f(z_v)| \to |\bar{\Phi}(p)| = C_1 > 0$$

Since f is a Brody curve, its derivative is uniformly bounded on $\mathbb{C}$, and since $\bar{U}(p) \subset \bar{X}$ is compact, the derivative of $\bar{\Phi}|_{\bar{U}(p)}$ is bounded too. Hence, there exists a constant $C_2 > 0$ such that $|\Phi \circ f|_{FS} \leq C_2$ on $f^{-1}(\bar{U}(p))$. So in particular

$$C_2 \geq |\Phi \circ f(z_v)|_{FS} = \frac{|2z_v| \cdot \exp(z_v^2)|}{1 + |\exp(z_v^2)|^2} = \frac{2|z_v| \cdot \exp(x_v^2 - y_v^2)}{1 + (\exp(x_v^2 - y_v^2))^2} \to \infty \qquad (18)$$

by (17) and as $|z_v| \to \infty$. This contradiction proves (16).

Now assume that there exists a rational function $\bar{\Phi} : \bar{X} \to \mathbb{P}_1$ which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$ such that $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree (exactly) 2. By (16) we have

$$\left(f(\infty) \setminus I_{\bar{\phi}}\right) \subset \bar{\Phi}^{-1}(\{0, \infty\}) \subset D \subset \bar{X}.$$

Now the Lemma follows from Proposition 16. □

For the rest of this subsection, we assume $\bar{q}_X = 2$ and $q_{\bar{X}} = 0$. Let $\bar{\Phi}_i = (\bar{\alpha}_X)_i : \bar{X} - \to \mathbb{P}_1$, $i = 1, 2$, be the two components of the quasi-Albanese map (see section 2.1).

Lemma 19 *Let* $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in M(2 \times 2, \mathbb{Z})$ *be a nonsingular matrix (not necessarily invertible over* $\mathbb{Z}$*). Then the map*

$$(\bar{\Psi}_1, \bar{\Psi}_2) = (\bar{\Phi}_1^{m_{11}} \bar{\Phi}_2^{m_{12}}, \bar{\Phi}_1^{m_{21}} \bar{\Phi}_2^{m_{22}}) : \bar{X} - \to (\mathbb{P}_1)^2$$

is a dominant rational map and restricts to a dominant morphism

$$(\Psi_1, \Psi_2) : X \to (\mathbb{C}^*)^2; x \mapsto \left(\exp\left(\int_{x_0}^x m_{11}\omega_1 + m_{12}\omega_2\right), \exp\left(\int_{x_0}^x m_{21}\omega_1 + m_{22}\omega_2\right)\right). \qquad (19)$$

(We remark that the map (Ψ_1, Ψ_2) is not a quasi-Albanese map, but it factors through the quasi-Albanese map by a finite etale map.)

Proof It suffices to prove that the morphism $(\Psi_1, \Psi_2) : X \to (\mathbb{C}^*)^2$ is dominant, the rest is clear from the properties of the quasi-Albanese map. Let ω_1, ω_2 be the two linearly independant log forms corresponding to Φ_1, Φ_2. By construction $(\Psi_1, \Psi_2) : X \to (\mathbb{C}^*)^2$ is of the form

$$(\Psi_1, \Psi_2)(x) = \left(\exp\left(\int_{x_0}^x m_{11}\omega_1 + m_{12}\omega_2\right), \exp\left(\int_{x_0}^x m_{21}\omega_1 + m_{22}\omega_2\right)\right) \qquad (20)$$

where x_0 is a fixed point in X. It suffices to prove that the lift of this map to the universal cover,

$$X \to \mathbb{C}^2;\ x \mapsto \left(m_{11}\left(\int_{x_0}^x \omega_1\right) + m_{12}\left(\int_{x_0}^x \omega_2\right), m_{21}\left(\int_{x_0}^x \omega_1\right) + m_{22}\left(\int_{x_0}^x \omega_2\right)\right)$$

has rank two in some points. But this is true since M is non singular and since the lift to the universal cover of the quasi-Albanese map, this means

$$X \to \mathbb{C}^2;\; x \mapsto \left(\int_{x_0}^{x} \omega_1, \int_{x_0}^{x} \omega_2 \right)$$

has this property. $\square$

By Lemma 15, we have $\Psi_i \circ f = \exp(Q_i(z))$ with $\deg Q_i \leq 2$. Then if one of the Q_i has degree 0, it follows immediately that f is algebraically degenerate (since $\Psi_i \circ f$ is constant then). Our aim is now to choose the nonsingular matrix M such that $\deg Q_i \leq 1$ for $i = 1, 2$. Then if one of the Q_i has degree 0, one is done, and if both polynomials Q_i have degree 1, Theorem 2 follows from Proposition 17.

By Lemma 15 we have $\Phi_i \circ f = \exp(P_i(z))$ with $\deg P_i \leq 2$ for the two components of the quasi-Albanese map. If $\deg P_i \leq 1$ for $i = 1, 2$, we put $M = I_2$, the identity matrix, and the proof is finished. Otherwise we may assume without loss of generality that $\deg P_2 = 2$. Then by Lemma 18, we have $f(\infty) = \bigcup_{j=1}^{l} D_j$. Consider the submatrix of the residues of ω_1, ω_2 with respect to the first l divisors forming $f(\infty)$

$$\begin{array}{ccccc} & D_1 & D_2 & \dots & D_l \\ \omega_1 & a_{11} & a_{12} & \dots & a_{1l} \\ \omega_2 & a_{21} & a_{22} & \dots & a_{2l} \end{array} \tag{21}$$

If this matrix has rank 2, there exists a nonsingular matrix M with coefficients in $\mathbb{Z}$ such that after passing from the log forms ω_1, ω_2 corresponding to the map (Φ_1, Φ_2), to the forms $m_{11}\omega_1 + m_{12}\omega_2, m_{21}\omega_1 + m_{22}\omega_2$ corresponding to (Ψ_1, Ψ_2), the matrix of residues (21) has at least one residue $= 0$ in every line.

If the matrix of residues (21) of (Φ_1, Φ_2) has rank 1, we can choose the nonsingular matrix M with $m_{11} \neq 0$, $m_{21} = 0$ and $m_{22} = 1$ such that after passing from the log forms ω_1, ω_2 corresponding to the map (Φ_1, Φ_2), to the forms $m_{11}\omega_1 + m_{12}\omega_2, m_{21}\omega_1 + m_{22}\omega_2$ corresponding to (Ψ_1, Ψ_2) the matrix of residues (21) has all residues $= 0$ in the first line. Here, there is the difficult case where the resulting matrix does not have any zero in the second line.

If the matrix of residues (21) of (Φ_1, Φ_2) has rank 0, we just take $M = I_2$ the identity matrix.

In all cases, except the difficult one, there exist some residues $= 0$ in every line of the residue matrix for (Ψ_1, Ψ_2). So by Lemma 18 and equation (12), we have $\deg Q_i \leq 1$ for $i = 1, 2$, and we are done.

So we are left with the only (difficult) case that the matrix of residues of (Ψ_1, Ψ_2) with respect to the log forms $\tilde{\omega}_1 := m_{11}\omega_1 + m_{12}\omega_2$ and ω_2 looks as follows (with $a_{2j} \neq 0$ for all $j = 1, ..., l$)

$$\begin{array}{cccccc} & D_1 & D_2 & ... & D_l \\ \tilde{\omega}_1 & 0 & 0 & ... & 0 \\ \omega_2 & a_{21} & a_{22} & ... & a_{2l} \end{array} \qquad (22)$$

and that $\Psi_i \circ f = \exp(Q_i(z))$ with $\deg Q_1 \leq 1$ and $\deg Q_1 = 2$. In this case, we have to use the explicit geometry of the entire curve f with respect to the map (Ψ_1, Ψ_2) in a similar way as we did in the proof for the case $q_{\bar{X}} = 1$ above.

Lemma 20
a) The rational function $\bar{\Psi}_1 : \bar{X} - \to \mathbb{P}_1$ is a morphism in a neighborhood of $f(\infty) = \bigcup_{j=1}^{l} D_j \subset \bar{X}$.
b) There exists at most one irreducible component of $f(\infty)$, say D_1, such that $\bar{\Psi}_1(D_1) = \mathbb{P}_1$. If it exists, there exists exactly one point $x_0 \in D_1$ and exactly one point $x_\infty \in D_1$ such that $\bar{\Psi}_1(x_0) = 0$ and $\bar{\Psi}_1(x_\infty) = \infty$ (in particular, for all points $x \neq x_0, x_\infty$, we have $\bar{\Psi}_1(x) \in \mathbb{C}^*$). Furthermore, x_0 respectively x_∞ are intersection points of D_1 with components D_j with $j \geq l+1$ (meaning components not belonging to $f(\infty)$) with residues $a_{1j} > 0$ respectively $a_{1j} < 0$.

Proof For a): This follow immediately from Proposition 13 since by (22), all the residues of all components of $f(\infty)$ are $= 0$.
For b): Since $\bar{\Psi}_1$ is a morphism around $f(\infty)$, the image of every irreducible component is a (closed) subvariety in $\mathbb{P}_1$, and hence a single point or all of $\mathbb{P}_1$. Since the residue of each component of $f(\infty)$ is 0, it follows by (12) that the image of each component, being a single point, has to lie in $\mathbb{C}^*$. By the same token, we also get that any point $x \in f(\infty)$ which is mapped to 0 respectively to ∞ by $\bar{\Psi}_1$ has to be an intersection point with a component of D having residue > 0 respectively < 0, and which cannot be one of the components of $f(\infty)$. Since D is a normal crossing divisor, no three irreducible components meet in one point, and so no point mapping to 0 or to ∞ can lie on the intersection of different irreducible components of $f(\infty)$. We will now prove that there can exist at most one point $x = x_0 \in f(\infty)$ with $\bar{\Psi}_1(x) = 0$ and at most one point $x = x_\infty \in f(\infty)$ with $\bar{\Psi}_1(x) = \infty$. From this it follows immediately that there can be at most one component of $f(\infty)$ mapping onto $\mathbb{P}_1$.

Let $\{(x_0)_1, ..., (x_0)_m\} = \{x \in f(\infty) : \bar{\Psi}_1(x) = 0\}$. Take a neighborhood $U(f(\infty))$ such that $\bar{\Psi}_1$ is still a morphism in a neighborhood of the closure $\bar{U}(f(\infty))$ of $U(f(\infty))$ (this is possible since there is only a finite number of points of indeterminacy of $\bar{\Psi}_1$). Take small neighborhoods $U_p = U_p((x_0)_p)$, $p = 1, ..., m$, such that their closures are still contained in $U(f(\infty))$.

Assume $m \geq 2$. For $p = 1, 2$, take sequences $(x_v^{(p)})$, $v \in \mathbb{N}$, such that $|x_v^{(p)}| \to \infty$ and $f(x_v^{(p)}) \to (x_0)_p$. Without loss of generality, we may assume that $f(x_v^{(p)}) \in U_p$ for all $v \in \mathbb{N}$. Let $[x_v^{(1)}, x_v^2] \subset \mathbb{C}$ be the (linear) segment between $x_v^{(1)}$ and $x_v^{(2)}$ in $\mathbb{C}$. Then there exists a point

$$y_v \in ([x_v^{(1)}, x_v^2] \cap f^{-1}(U(f(\infty)))) \setminus \bigcup_{p=1}^{m} f^{-1}(U_p) \ :$$

The image of the segment $[x_v^{(1)}, x_v^2]$ under f joins the two points $f(x_v^{(1)}) \in U_1$ and $f(x_v^{(2)}) \in U_2$. Since the two neighborhoods are disjoint, it has to cross the boundary. Let y_v be the inverse image of such a crossing point.

By (22) we have $\bar{\Psi}_1 \circ f(z) = \exp(az+b)$ with $a, b \in \mathbb{C}$. Since $f(x_v^{(p)}) \to (x_0)_p$ we get $\bar{\Psi}_1 \circ f(x_v^{(p)}) \to \bar{\Psi}_1((x_0)_p) = 0$, $p = 1, 2$. So $\exp(ax_v^{(p)} + b) \to 0$, meaning $\mathrm{Re}(ax_v^{(p)} + b) \to -\infty$, $p = 1, 2$. Now $y_v \in [x_v^{(1)}, x_v^2]$, so there exist λ_v with $0 \leq \lambda_v \leq 1$ such that $y_v = \lambda_v x_v^{(1)} + (1 - \lambda_v) x_v^{(2)}$. Then we have

$$\begin{aligned} \mathrm{Re}(ay_v + b) &= \mathrm{Re}(\lambda_v(ax_v^1 + b) + (1 - \lambda_v)(ax_v^2 + b)) \\ &= \lambda_v \mathrm{Re}(ax_v^1 + b) + (1 - \lambda_v)\mathrm{Re}(ax_v^2 + b) \to -\infty \end{aligned} \tag{23}$$

since $\lambda_v, 1 - \lambda_v \geq 0$. In particular $|y_v| \to \infty$.

Consider the sequence $f(y_v)$, $v \in \mathbb{N}$. After passing to a subsequence, we get $f(y_v) \to P \in \bar{U}(f(\infty))$. By construction, we have $P \in f(\infty)$, and by (23) we get

$$\bar{\Psi}_1(P) = \lim_{v \to \infty} \bar{\Psi}_1(f(y_v)) = \lim_{v \to \infty} \exp(ay_v + b) = 0 \,. \tag{24}$$

But $y_v \in U(f(\infty)) \setminus \bigcup_{p=1}^{m} U_p((x_0)_p)$, so $P \neq (x_0)_v$, $v = 1, ..., m$. This is a contradiction, since $\{(x_0)_1, ..., (x_0)_m\} = \{x \in f(\infty) : \bar{\Psi}_1(x) = 0\} \ni P$. So our assumption $m \geq 2$ was wrong, and there exists at most one point $x_0 \in f(\infty)$ such that $\bar{\Psi}_1(x_0) = 0$.

The proof for x_∞ is exactly the same, just change 0 and ∞ in the proof above, and in (23), change $-\infty$ to $+\infty$. □

Without loss of generality we may assume $x_0 \in D_1 \cap D_{l+1}$. The matrix of residues (22) extends to

$$\begin{array}{cccccc} & D_1 & D_2 & ... & D_l & D_{l+1} \\ \tilde{\omega}_1 & 0 & 0 & ... & 0 & a_{1,l+1} \\ \omega_2 & a_{21} & a_{22} & ... & a_{2l} & a_{2,l+1} \end{array} \tag{25}$$

with $a_{1,l+1} > 0$. We put

$$\tilde{\omega}_2 := a_{1,l+1}\omega_2 - a_{2,l+1}\omega_1$$

Then the new matrix of residues becomes

$$\begin{array}{cccccc} & D_1 & D_2 & ... & D_l & D_{l+1} \\ \tilde{\omega}_1 & 0 & 0 & ... & 0 & a_{1,l+1} \\ \tilde{\omega}_2 & a_{21} & a_{22} & ... & a_{2l} & 0 \end{array} \tag{26}$$

Now we finish up the proof by looking at $\bar{\Psi}_2$. By Proposition 13, we get that $x_0 \in D_1 \cap D_{l+1}$ cannot be a point of indeterminacy of $\bar{\Psi}_2$, since the corresponding product of residues is zero. Hence, the finite number of points of indeterminacy of $\bar{\Psi}_2$ can be distributed to three sets and we define

$$I_{\bar{\Psi}_2} = (I_{\bar{\Psi}_2} \cap \{x_\infty\}) \cup (I_{\bar{\Psi}_2} \cap (f(\infty) \setminus \{x_\infty\}) \cup (I_{\bar{\Psi}_2} \setminus f(\infty)) = A_\infty \cup A \cup B. \quad (27)$$

Let $U(A_\infty)$, $U(A)$ and $U(B)$ be small neighborhoods of the finite sets A_∞, A and B (where we put the empty set as a neighborhood of an empty set). In particular, assume that $f(\infty) \cap \bar{U}(B) = \emptyset$.

We consider the inverse image of each of the three neighborhoods $U(A_\infty)$, $U(A)$ and $U(B)$ under $f : \mathbb{C} \to X$ separately below.

We claim that $f^{-1}(U(B)) \subset\subset \mathbb{C}$ is relatively compact. Assume not, then there exists a sequence (z_v), $v \in \mathbb{N}$ with $z_v \in f^{-1}(U(B))$ for all $v \in \mathbb{N}$ with $|z_v| \to \infty$. By the compactness of $\bar{X}$ we can pass to a subsequence such that $f(z_v) \to P$. Then by definition $P \in f(\infty) \cap \bar{U}(B)$, which is a contradiction.

Next we claim that $f^{-1}(U(A))$ is contained in the image under a linear transformation of $\mathbb{C}$ of an infinite union (obtained by all translations by the semi-lattice) of small neighborhoods of the same shape (for any fixed c_p) of the points $\log(c_p) + 2\pi i\mathbb{Z}$ for a finite number of points $c_1, ..., c_m \in \mathbb{C}^*$. Let $A = \{x_1, ..., x_m\}$, and put $\bar{\Psi}_1(x_p) = c_p$. Then by definition of A we have $c_p \in \mathbb{C}^*$ for $p = 1, ..., m$. Take small neighborhoods of the points c_p, $p = 1, ..., m$ such that the points $0, \infty \in \mathbb{P}_1$ are not in their closure and that they contain the image of $U(A)$ under $\bar{\Psi}_1$ (recall that $\bar{\Psi}_1$ is a morphism around $f(\infty)$). Then the inverse image of the union of these small neighborhoods under the map $\bar{\Psi}_1 \circ f(z) = \exp(az + b) : \mathbb{C} \to \mathbb{P}_1$ contains $f^{-1}(U(A))$ and is of the desired form.

Finally we claim that the inverse image of $f^{-1}(U(A_\infty))$ is contained in the image of a linear transformation of the half plane $\mathrm{Re}(z) > R$ for a big real number $R \in \mathbb{R}^+$. As above, since $\bar{\Psi}_1$ is a morphism around $f(\infty)$, $U(A_\infty)$ is mapped into a small neighborhood of $\infty \in \mathbb{P}_1$. Its inverse image under the map $\bar{\Psi}_1 \circ f(z) = \exp(az + b) : \mathbb{C} \to \mathbb{P}_1$ is contained in a subset of $\mathbb{C}$ of the desired form.

Now we finish up the proof below as in the case $q_{\bar{X}} = 1$. On the closed set

$$\mathbb{C} \setminus f^{-1}(U(A_\infty) \cup U(A) \cup U(B)),$$

the derivative of the map $\bar{\Psi}_2 \circ f$ is uniformly bounded by a constant C. Assume now that $\bar{\Psi}_2 \circ f(z) = \exp(Q_2(z))$ with $\deg Q_2 = 2$. We get, by a linear coordinate change in $\mathbb{C}$ and a multiplicative transformation in $\mathbb{C}^* \subset \mathbb{P}_1$, that

$$Q_2(z) = z^2 .$$

By the special forms of the sets $f^{-1}(U(A_\infty)$, $f^{-1}(U(A))$ and $f^{-1}(U(B))$ as described above, and since we can increase the R by shrinking $U(A_\infty)$ such that the inverse image of $f^{-1}(U(A_\infty))$ contained in the image of a linear transformation of the half plane $\mathrm{Re}(z) > R$ does not contain the origin with respect to the new coordinates in $\mathbb{C}$, we get that there exists a sequence on the diagonal

$$(z_v = x_v + ix_v)_{v\to\infty} \subset \mathbb{C} \setminus f^{-1}(U(A_\infty) \cup U(A) \cup U(B)) \text{ with } |x_v| \to \infty.$$

We have

$$|(\Phi \circ f)'|_{FS}(z) = \frac{|2z| \exp(x^2 - y^2)}{1 + \exp\left(2(x^2 - y^2)\right)}$$

and since $(\Phi \circ f)'$ is bounded on $\mathbb{C} \setminus f^{-1}(U(A_\infty) \cup U(A) \cup U(B))$,

$$C \geq |(\Phi \circ f)'|_{FS}(z_v) = \frac{|2z_v| \exp(0)}{1 + \exp(0)} \to \infty.$$

This is a contradiction since $|z_v| \to \infty$. So the assumption $\deg Q_2 = 2$ was wrong, meaning that in all possible cases we have $\Psi_i \circ f = \exp(Q_i(z))$ with $\deg Q_i \leq 1$, which finishes the proof by Proposition 17. $\square$

4 Appendix

We give a brief argument for Proposition 6.

By the standard formulas for (log) irregularities and (log) canonical bundles of products of (log) varieties (see for example Iitaka '76 [14]) and for divisors in $\mathbb{P}_2$, we get immediately all the properties of the (log) varieties concerned.

The fact that f is always a Brody curve is an easy calculation with respect to the product metric of the standard hermitian metrics on the components or on $\mathbb{P}_2$.

It remains to prove that the curve $f : \mathbb{C} \to X \subset \bar{X}$ obtained by the curve

$$\hat{f} : \mathbb{C} \to \mathbb{C}^2; t \to (t, at + b) \text{ with } a \in \mathbb{R} \setminus \mathbb{Q} \tag{28}$$

on the universal cover $\mathbb{C}^2 \to X$ is not algebraically degenerate. Since $(\mathbb{P}_1)^2$ is birational to $\mathbb{P}_2$, it suffices to consider the proof for the case $(\mathbb{P}_1)^2$. For this we use the following well known number theoretic result.

Lemma 21 *If* $a \in \mathbb{R} \setminus \mathbb{Q}$*, then* $\#\{a\mathbb{Z} \text{ modulo } \mathbb{Z}\} = \infty$
(in fact this set is dense in the interval $[0, 1]$*).* $\square$

First we observe that by a change of the linear coordinates in $\mathbb{C}^2$ of the form $(z_1, z_2) \to (z_1 + c_1, z_2 + c_2)$, which does not affect the structure of the curve $\hat{f}$ in (28), we may assume that the lattice of the elliptic curve E is given by $\mathbb{Z} + c\mathbb{Z}$ and the semi-lattice of $\mathbb{C}^*$ is given by $2\pi i\mathbb{Z}$.

Assume now that there exists a proper algebraic subset $A \subset X = X_1 \times X_2$ such that $f(\mathbb{C}) \subset A$. For points $x_1 \subset X_1$, denote $A_{x_1} = (\mathrm{pr}_1)^{-1}(x_1) \cap A$ the fiber of A over the point x_1 under the projection $\mathrm{pr}_1 : X_1 \times X_2 \to X_1$ to the first factor. Then we know that for generic $x_1 \in X_1$, A_{x_1} is a finite set. The contradiction to the existence of A will be obtained by showing that for every point $x_1 \in X_1$, the set $\mathrm{pr}^{-1}(x_1) \cap f(\mathbb{C})$ is an infinite set.

Let $f = (f_1, f_2)$ and $t_0 \in \mathbb{C}$ such that $x_1 = f_1(t_0)$ and such that A_{x_1} is a finite set. Then $f_1(t_0 + \mathbb{Z} + c\mathbb{Z}) = x_1$ in the case $X_1 = E$, respectively $f_1(t_0 + 2\pi i\mathbb{Z}) = x_1$ in the case $X_1 = \mathbb{C}^*$.

In the case $X = E \times E$, we get since $\mathrm{Im}(c) \neq 0$ that
$\#\{\mathrm{pr}^{-1}(x_1) \cap f(\mathbb{C})\} = \#\{at_0 + b + a\mathbb{Z} + ac\mathbb{Z} \text{ modulo } \mathbb{Z} + c\mathbb{Z}\}$
$\leq \#\{a\mathbb{Z} \text{ modulo } \mathbb{Z} + c\mathbb{Z}\} = \#\{a\mathbb{Z} \text{ modulo } \mathbb{Z}\} = \infty$.

In the case $X = E \times \mathbb{C}^*$, we get
$\#\{\mathrm{pr}^{-1}(x_1) \cap f(\mathbb{C})\} = \#\{at_0 + b + a\mathbb{Z} + ac\mathbb{Z} \text{ modulo } 2\pi i\mathbb{Z}\}$
$\leq \#\{a\mathbb{Z} \text{ modulo } 2\pi i\mathbb{Z}\} = \#\{a\mathbb{Z}\} = \infty$.

In the case $X = \mathbb{C}^* \times \mathbb{C}^*$, we get
$\#\{\mathrm{pr}^{-1}(x_1) \cap f(\mathbb{C})\} = \#\{at_0 + b + a2\pi i\mathbb{Z} \text{ modulo } 2\pi i\mathbb{Z}\}$
$= \#\{a\mathbb{Z} \text{ modulo } \mathbb{Z}\} = \infty$.

So in all three possible cases, we get the desired contradiction against the existence of the proper algebraic subset $A \subset X$ with $f(\mathbb{C}) \subset A$. $\square$

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