Generalized Albanese and its dual

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Abstract

Let X be a projective variety over an algebraically closed field k of characteristic 0. We consider categories of rational maps from X to commutative algebraic groups, and ask for objects satisfying the universal mapping property. A necessary and sufficient condition for the existence of such universal objects is given, as well as their explicit construction, using duality theory of generalized 1-motives.

An important application is the Albanese of a singular projective variety, which was constructed by Esnault, Srinivas and Viehweg as a universal regular quotient of a relative Chow group of 0-cycles of degree 0 modulo rational equivalence. We obtain functorial descriptions of the universal regular quotient and its dual 1-motive.

Contents

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0 Introduction

For a projective variety X over an algebraically closed field k a generalized Albanese variety $\mathrm{Alb}\,(X)$ is constructed by Esnault, Srinivas and Viehweg in [ESV] as a universal regular quotient of the relative Chow-group $\mathrm{CH}_0\,(X)^0$ of Levine-Weibel [LW] of 0-cycles of degree 0 modulo rational equivalence. This is a smooth connected commutative algebraic group, universal for rational maps from X to smooth commutative algebraic groups G factoring through a homomorphism of groups $\mathrm{CH}_0\,(X)^0 \longrightarrow G(k)$. It is not in general an abelian variety if X is singular. Therefore it cannot be dualized in the same way as an abelian variety.

Laumon built up in [L] a duality theory of generalized 1-motives in characteristic 0, which are homomorphisms $[\mathcal{F} \to G]$ from a commutative torsion-free formal group $\mathcal{F}$ to a connected commutative algebraic group G. The universal regular quotient $\mathrm{Alb}\,(X)$ can be interpreted as a generalized 1-motive by setting $\mathcal{F} = 0$ and $G = \mathrm{Alb}\,(X)$. The motivation for this work was to find the functor which is represented by the dual 1-motive, in the situation where the base field k is algebraically closed and of characteristic 0. The duality gives an independent proof (alternative to the ones in [ESV]) as well as an explicit construction of the universal regular quotient in this situation (cf. Subsection 3.6). This forms one of the two main results of the present article:

Theorem 0.1 *Let X be a projective variety over an algebraically closed field k of characteristic 0, and $\widetilde{X} \longrightarrow X$ a projective resolution of singularities. Then the universal regular quotient* $\mathrm{Alb}\,(X)$ *exists and there is a subfunctor* $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ *of the relative Cartier divisors on $\widetilde{X}$ (cf. Subsection 3.4, Definition 3.24) such that the dual of* $\mathrm{Alb}\,(X)$ *(in the sense of 1-motives) represents the functor*

$$\underline{\mathrm{Div}}^0_{\widetilde{X}/X} \longrightarrow \underline{\mathrm{Pic}}^0_{\widetilde{X}}$$

i.e. the natural transformation of functors which assigns to a relative Cartier divisor the class of its associated line bundle. $\underline{\mathrm{Pic}}^0_{\widetilde{X}}$ is represented by an abelian variety and $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ by a formal group.

The other main result (Theorem 0.3) is a more general statement about the existence and construction of universal objects of categories of rational maps (the notion of *category of rational maps* is introduced in Definition 2.23, but the name is suggestive). This concept does not only contain the universal regular quotient, but also the generalized Jacobian of Rosenlicht-Serre [S3, Chapter V] as well as the generalized Albanese of Faltings-Wüstholz [FW] as special cases of such universal objects.

We only deal with a base field of characteristic 0, although the universal regular quotient exists in any characteristic. A first reason for this is that Laumon's 1-motives are only defined in characteristic 0. In order to match the case of arbitrary characteristic, one first needs to define a new category of 1-motives (in any characteristic) which contains smooth connected commutative algebraic groups as a subcategory. A commutative torsion-free formal group in characteristic 0 is completely determined by its k-valued points and its Lie-algebra (cf. Corollary 1.7), the first form a free abelian group of finite rank, the latter is a finite dimensional k-vector space. This allows to give an explicit and transparent description, which might not be possible in positive characteristic.

0.1 Leitfaden

In the following we give a short summary of each section.

Section 1 provides some basic facts about generalized 1-motives, which are used in the rest of the paper. A connected commutative algebraic group G is an extension of an abelian variety A by a linear group L. Then the dual 1-motive of $[0 \to G]$ is given by $[L^\vee \to A^\vee]$, where $L^\vee = \underline{\mathrm{Hom}}\,(L, \mathbb{G}_{\mathrm{m}})$ is the Cartier-dual of L and $A^\vee = \mathrm{Pic}^0_A = \underline{\mathrm{Ext}}\,(A, \mathbb{G}_{\mathrm{m}})$ is the dual abelian variety, and the homomorphism between them is the connecting homomorphism in the long exact cohomology sequence obtained from $0 \to L \to G \to A \to 0$ by applying the functor $\underline{\mathrm{Hom}}\,(\_\,, \mathbb{G}_{\mathrm{m}})$.

Section 2 states the universal factorization problem with respect to a category **Mr** of rational maps from a regular projective variety Y to connected commutative algebraic groups (cf. Definition 2.37):

Definition 0.2 *A rational map* $(u : Y \dashrightarrow \mathcal{U}) \in \mathbf{Mr}$ *is called* universal *for* **Mr** *if for all objects* $(\varphi : Y \dashrightarrow G) \in \mathbf{Mr}$ *there is a unique homomorphism of algebraic groups* $h : \mathcal{U} \longrightarrow G$ *such that* $\varphi = h \circ u$ *up to translation.*

An essential ingredient for the construction of such universal objects is the functor of relative Cartier divisors $\underline{\mathrm{Div}}_Y$ on Y, which assigns to an affine scheme T a family of Cartier divisors on Y, parameterized by T. This functor admits a natural transformation $\mathrm{cl} : \underline{\mathrm{Div}}_Y \longrightarrow \underline{\mathrm{Pic}}_Y$ to the Picard functor $\underline{\mathrm{Pic}}_Y$, which maps a relative divisor to its class. Then $\underline{\mathrm{Div}}^0_Y := \mathrm{cl}^{-1}\,\underline{\mathrm{Pic}}^0_Y$ is the functor of families of Cartier divisors whose associated line bundles are algebraically equivalent to the trivial bundle.

We give a necessary and sufficient condition for the existence of a universal object for a category of rational maps **Mr** which contains the category **Mav** of morphisms from Y to abelian varieties and satisfies a certain stability condition ($\diamondsuit$), see Subsection 2.3. Localization of **Mr** at the system of injective homomorphisms does not change the universal object; denote this localization by $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$. We observe that a rational map $\varphi : Y \dashrightarrow G$, where G is an extension of an abelian variety by a linear group L, induces a natural transformation $L^\vee \longrightarrow \underline{\mathrm{Div}}^0_Y$. If $\mathcal{F}$ is a formal group that is a subfunctor of $\underline{\mathrm{Div}}^0_Y$, then $\mathbf{Mr}_{\mathcal{F}}$ denotes the category of rational maps for which the image of this induced transformation lies in $\mathcal{F}$. We show (cf. Theorem 2.39):

Theorem 0.3 *For a category* **Mr** *containing* **Mav** *and satisfying* ($\diamondsuit$) *there exists a universal object* $\mathrm{Alb}_{\mathbf{Mr}}\,(Y)$ *if and only if there is a formal group* $\mathcal{F} \subset \underline{\mathrm{Div}}^0_Y$ *such that* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ *is equivalent to* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}_{\mathcal{F}}$.

The universal object $\mathrm{Alb}_{\mathcal{F}}\,(Y)$ of $\mathbf{Mr}_{\mathcal{F}}$ is an extension of the classical Albanese $\mathrm{Alb}\,(Y)$, which is the universal object of **Mav**, by the linear group $\mathcal{F}^\vee$, the Cartier-dual of $\mathcal{F}$. The dual 1-motive of $[0 \longrightarrow \mathrm{Alb}_{\mathcal{F}}\,(Y)]$ is hence given by $\left[\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y\right]$, which is the homomorphism induced by the natural transformation $\mathrm{cl} : \underline{\mathrm{Div}}^0_Y \longrightarrow \underline{\mathrm{Pic}}^0_Y$.

The universal regular quotient $\mathrm{Alb}\,(X)$ of a (singular) projective variety X is by definition the universal object for the category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ of rational maps factoring through rational equivalence. More precisely, the objects of $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ are rational maps $\varphi : X \dashrightarrow G$ whose associated map on zero-cycles of degree zero $\mathrm{Z}_0\,(U)^0 \longrightarrow G(k)$, $\sum n_i\, p_i \longmapsto \sum n_i\, \varphi(p_i)$ (here U is the open set on which φ is defined) factors through a homomorphism of groups $\mathrm{CH}_0(X)^0 \longrightarrow G(k)$, where $\mathrm{CH}_0(X)$ denotes the relative Chow group of 0-cycles $\mathrm{CH}_0(X, X_{\mathrm{sing}})$ in the sense of [LW]. Such a rational map is regular on the regular locus of X and may also be considered as a rational map from $\widetilde{X}$ to G, where $\pi : \widetilde{X} \longrightarrow X$ is a projective resolution of singularities. In particular, if X is nonsingular, the universal regular quotient coincides with the classical Albanese. The category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ contains $\mathbf{Mav}$ and satisfies ($\diamondsuit$). Therefore our problem reduces to finding the subfunctor of $\underline{\mathrm{Div}}^0_{\widetilde{X}}$ which is represented by a formal group $\mathcal{F}$ such that $\mathbf{Mr}_{\mathcal{F}}$ equivalent to $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$.

Section 3 answers the question for the formal group $\mathcal{F}$ which characterizes the category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$. This is a subfunctor of $\underline{\mathrm{Div}}^0_{\widetilde{X}}$ which measures the difference between $\widetilde{X}$ and X.

If $\pi : Y \longrightarrow X$ is a proper birational morphism of varieties, the *push-forward* of cycles gives a homomorphism $\pi_* : \mathrm{WDiv}(Y) \longrightarrow \mathrm{WDiv}(X)$ from the group of Weil divisors on Y to the group of those on X. For a curve C we introduce the k-vector space $\mathrm{LDiv}(C)$ of formal infinitesimal divisors, which generalizes infinitesimal deformations of the zero divisor. There exist non-trivial infinitesimal deformations of Cartier divisors, but not of Weil divisors, since prime Weil divisors are always reduced. Formal infinitesimal divisors also admit a *push-forward* $\pi_* : \mathrm{LDiv}(Z) \longrightarrow \mathrm{LDiv}(C)$ for finite morphisms $\pi : Z \longrightarrow C$ of curves, e.g. for the normalization, which is a resolution of singularities. There exist natural homomorphisms $\mathrm{weil} : \underline{\mathrm{Div}}_Y(k) \longrightarrow \mathrm{WDiv}(Y)$ and $\mathrm{fml} : \mathrm{Lie}(\underline{\mathrm{Div}}_Z) \longrightarrow \mathrm{LDiv}(Z)$.

For a curve C, a natural candidate for the formal group we are looking for is determined by the following conditions:

$$\begin{aligned}
\underline{\mathrm{Div}}^0_{\widetilde{C}/C}(k) &= \ker\left(\underline{\mathrm{Div}}^0_{\widetilde{C}}(k) \xrightarrow{\mathrm{weil}} \mathrm{WDiv}\left(\widetilde{C}\right) \xrightarrow{\pi_*} \mathrm{WDiv}(C)\right) \\
\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}/C}\right) &= \ker\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}}\right) \xrightarrow{\mathrm{fml}} \mathrm{LDiv}\left(\widetilde{C}\right) \xrightarrow{\pi_*} \mathrm{LDiv}(C)\right).
\end{aligned}$$

For a higher dimensional variety X, the definition is derived from the one for curves as follows. A morphism of varieties $V \longrightarrow Y$ induces a natural transformation: the *pull-back* of relative Cartier divisors $\_ \cdot V : \underline{\mathrm{Dec}}_{Y,V} \longrightarrow \underline{\mathrm{Div}}_V$, where $\underline{\mathrm{Dec}}_{Y,V}$ is the subfunctor of $\underline{\mathrm{Div}}_Y$ consisting of those relative Cartier divisors on Y which do not contain any component of $\mathrm{im}(V \longrightarrow Y)$. Then we let $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ be the formal subgroup of $\underline{\mathrm{Div}}^0_{\widetilde{X}}$ characterized by the conditions

$$\begin{aligned}
\underline{\mathrm{Div}}^0_{\widetilde{X}/X}(k) &= \bigcap_C \left(_ \cdot \widetilde{C}\right)^{-1} \underline{\mathrm{Div}}^0_{\widetilde{C}/C}(k) \\
\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right) &= \bigcap_C \left(_ \cdot \widetilde{C}\right)^{-1} \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}/C}\right)
\end{aligned}$$

where the intersection ranges over all Cartier curves in X relative to the singular locus of X in the sense of [LW]. Actually, it is not necessary to consider all Cartier curves, the functor $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ can be computed from one single general curve. The verification of the equivalence between $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$ is done using *local symbols*, for which [S3] is a good reference.

This gives an independent proof of the existence (alternative to the ones in [ESV]) as well as an explicit construction of the universal regular quotient over an algebraically closed base field of characteristic 0 (cf. Subsection 3.6).

The universal regular quotient for semi-abelian varieties, i.e. the universal object for rational maps to semi-abelian varieties factoring through rational equivalence (which is a quotient of our universal regular quotient), is a classical 1-motive in the sense of Deligne [Dl, Définition (10.1.2)]. The question for the dual 1-motive of this object was already answered by Barbieri-Viale and Srinivas in [BS].

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0.2 Notations and Conventions

k is a fixed algebraically closed field of characteristic 0. A variety is a reduced scheme of finite type over k. A curve is a variety of dimension 1. Algebraic groups and formal groups are always commutative and over k. We write $\mathbb{G}_{\mathrm{a}}$ for $\mathbb{G}_{\mathrm{a},k} = \operatorname{Spec} k[t]$ and $\mathbb{G}_{\mathrm{m}}$ for $\mathbb{G}_{\mathrm{m},k} = \operatorname{Spec} k\left[t, t^{-1}\right]$. The letter $\mathbb{G}$ stands for a linear algebraic group which is either $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$.

If Y is a scheme, then $y \in Y$ means that y is a point in the Zariski topological space of Y. The set of irreducible components of Y is denoted by $\mathrm{Cp}(Y)$.

If M is a module, then $\operatorname{Sym} M$ is the symmetric algebra of M. If A is a ring, then K_A denotes the total quotient ring of A. If Y is a scheme, then $\mathcal{K}_Y$ denotes the sheaf of total quotient rings of $\mathcal{O}_Y$. The group of units of a ring R is denoted by R^*.

If $\sigma : Y \longrightarrow X$ is a morphism of schemes, then $\sigma^\# : \mathcal{O}_X \longrightarrow \sigma_* \mathcal{O}_Y$ denotes the associated homomorphism of structure sheaves.

We think of $\operatorname{Ext}^1(A, B)$ as the space of extensions of A by B and therefore denote it by $\operatorname{Ext}(A, B)$.

The dual of an object O in its respective category is denoted by $O^\vee$, whereas $\widehat{O}$ is the completion of O. For example, if V is a k-vector space, then $V^\vee = \operatorname{Hom}_{\mathrm{k}}(V, k)$ is the dual k-vector space; if G is a linear algebraic group or a formal group, then $G^\vee = \underline{\operatorname{Hom}}_{\mathcal{A}b/\mathrm{k}}(G, \mathbb{G}_{\mathrm{m}})$ is the Cartier-dual; if A is an abelian variety, then $A^\vee = \operatorname{Pic}^0 A$ is the dual abelian variety.

1 1-Motives

The aim of this section is to summarize some foundational material about generalized 1-motives (following [L, Sections 4 and 5]), as far as necessary for the purpose of these notes.

Throughout the whole work the base field k is algebraically closed and of characteristic 0.

1.1 Algebraic Groups and Formal Groups

Here we recall some basic facts about algebraic groups and the notion of a formal group. References for formal groups and Cartier duality are e.g. [SGA3, $\mathrm{VII_B}$], [Dm, Chapter II] and [Fo, Chapitre I].

Algebraic Groups

An *algebraic group* is a commutative group-object in the category of separated schemes of finite type over k. As $\mathrm{char}(k) = 0$, an algebraic group is always smooth (see [M2, Chapter III, No. 11, p. 101]).

Theorem 1.1 (Chevalley) *A smooth connected algebraic group G admits a* canonical decomposition

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$$

where L is a connected linear algebraic group and A is an abelian variety.

(See [Ro, Section 5, Theorem 16, p. 439] or [B, Theorem 3.2, p. 97] or [C].)

Theorem 1.2 *A connected linear algebraic group L splits canonically into a direct product of a torus T and a unipotent group $\mathbb{V}$:*

$$L = T \times_k \mathbb{V} .$$

A torus over an algebraically closed field is the direct product of several copies of the multiplicative group $\mathbb{G}_{\mathrm{m}}$. For $\mathrm{char}(k) = 0$ a unipotent group is always vectorial, i.e. is the direct product of several copies of the additive group $\mathbb{G}_{\mathrm{a}}$.

(See [SGA3, Exposé XVII, 7.2.1] and [L, (4.1)].)

Formal Groups

A *formal scheme* over k is a functor from the category of k-algebras $\mathbf{Alg}/k$ to the category of sets $\mathbf{Set}$ which is the inductive limit of a directed inductive set of finite k-schemes: $\mathcal{F}$ is a formal scheme if there exists a directed projective system (A_i) of finite dimensional k-algebras and an isomorphism of functors $\mathcal{F} \cong \varinjlim \mathrm{Spf}\, A_i$, where $\mathrm{Spf}\, A_i : \mathbf{Alg}/k \longrightarrow \mathbf{Set}$ is the functor $R \longmapsto \mathrm{Hom}_{\text{k-Alg}}(A_i, R)$. Equivalently, there is a profinite k-algebra $\mathcal{A}$, i.e. $\mathcal{A}$ is the projective limit (as a topological ring) of discrete quotients which are finite dimensional k-algebras, and an isomorphism of functors $\mathcal{F} \cong \mathrm{Spf}\,\mathcal{A}$, where $\mathrm{Spf}\,\mathcal{A}$ is the functor which assigns to a k-algebra R the set of continuous homomorphisms of k-algebras from $\mathcal{A}$ to the discrete ring R. (Cf. [Dm, Chapter I, No. 6].)

A *formal group* $\mathcal{G}$ is a commutative group-object in the category of formal schemes over k, such that $\mathcal{G}(k)$ is an abelian group of finite type and $\mathfrak{m}_{\mathcal{G},0}/\mathfrak{m}_{\mathcal{G},0}^2$ is a finite dimensional k-vector space, where $\mathfrak{m}_{\mathcal{G},0}$ is the maximal ideal of the local ring $\mathcal{O}_{\mathcal{G},0}$. If $\mathrm{char}(k) = 0$, a formal group $\mathcal{G}$ is always equi-dimensional and formally smooth, i.e. there is a natural number $d \geq 0$ such that $\mathcal{O}_{\mathcal{G},0} \cong k[[x_1, \ldots, x_d]]$ (cf. [L, (4.2)]).

Theorem 1.3 *A formal group $\mathcal{G}$ admits a canonical decomposition*

$$\mathcal{G} \cong \mathcal{G}_{\text{ét}} \times \mathcal{G}_{\text{inf}}$$

where $\mathcal{G}_{\text{ét}}$ is étale over k and $\mathcal{G}_{\text{inf}}$ is the component of the identity (called infinitesimal *formal group).*

(See [Fo, Chapitre I, 7.2] or [L, (4.2.1)].)

Theorem 1.4 *An étale formal group $\mathcal{G}_{\text{ét}}$ admits a canonical decomposition*

$$0 \longrightarrow \mathcal{G}_{\text{ét}}^{\text{tor}} \longrightarrow \mathcal{G}_{\text{ét}} \longrightarrow \mathcal{G}_{\text{ét}}^{\text{lib}} \longrightarrow 0$$

where $\mathcal{G}_{\text{ét}}^{\text{tor}}$ is the largest sub-group scheme whose underlying k-scheme is finite and étale, and $\mathcal{G}_{\text{ét}}^{\text{lib}}(k)$ is a free abelian group of finite rank.

(See [L, (4.2.1)].)

Theorem 1.5 *For $\operatorname{char}(k) = 0$, the Lie-functor gives an equivalence between the following categories:*

$$\{\textit{infinitesimal formal groups}/k\} \longleftrightarrow \{\textit{finite dim. vector spaces}/k\}$$

(Cf. [SGA3, VII_{B}, 3.3.2.].)

Remark 1.6 *Theorem 1.5 says that for an infinitesimal formal group $\mathcal{G}_{\text{inf}}$, there is a finite dimensional k-vector space V, namely $V = \operatorname{Lie}(\mathcal{G}_{\text{inf}})$, such that $\mathcal{G}_{\text{inf}} \cong \operatorname{Spf}\left(\widehat{\operatorname{Sym} V^\vee}\right)$, where $\widehat{\operatorname{Sym} V^\vee}$ is the completion of the symmetric algebra $\operatorname{Sym} V^\vee$ w.r.t. the ideal generated by $V^\vee$. Therefore the R-valued points of $\mathcal{G}_{\text{inf}}$ are given by $\mathcal{G}_{\text{inf}}(R) = \operatorname{Hom}^{\text{cont}}_{\text{k-Alg}}\left(\widehat{\operatorname{Sym}\left(\operatorname{Lie}(\mathcal{G}_{\text{inf}})^\vee\right)}, R\right) = \operatorname{Lie}(\mathcal{G}_{\text{inf}}) \otimes_k \operatorname{Nil}(R)$.*

From these structure theorems we obtain

Corollary 1.7 *A formal group $\mathcal{G}$ in characteristic 0 is uniquely determined by its k-valued points $\mathcal{G}(k)$ and its Lie-algebra $\operatorname{Lie}(\mathcal{G})$.*

Sheaves of Abelian Groups

The category of algebraic groups and the category of formal groups can be viewed as full subcategories of the category of sheaves of abelian groups:

Definition 1.8 *Let*

Ab	*category of abelian groups,*
Alg$/k$	*category of k-algebras,*
Aff$/k$	*category of affine k-schemes,*
Sch$/k$	*category of k-schemes,*
FSch$/k$	*category of formal k-schemes.*

Aff$/k$ *is anti-equivalent to* **Alg**$/k$*. Let* **Aff**$/k$ *(resp.* **Alg**$/k$*) be equipped with the topology fppf. Let*

$\mathcal{S}et/k$	*category of sheaves of sets over* **Aff**$/k$,
$\mathcal{A}b/k$	*category of sheaves of abelian groups over* **Aff**$/k$,
$\mathcal{G}a/k$	*category of algebraic groups over k,*
$\mathcal{G}f/k$	*category of formal groups over k.*

Interpreting a k-scheme X as a sheaf over $\mathbf{Aff}/k$ given by

$$S \longmapsto X(S) = \mathrm{Mor}_k(S, X)$$

or equivalently over $\mathbf{Alg}/k$

$$R \longmapsto X(R) = \mathrm{Mor}_k(\mathrm{Spec}\, R, X)$$

makes $\mathbf{Sch}/k$ a full subcategory of $\mathcal{S}et/k$ and $\mathcal{G}a/k$ a full subcategory of $\mathcal{A}b/k$.

In the same manner $\mathbf{FSch}/k$ becomes a full subcategory of $\mathcal{S}et/k$ and $\mathcal{G}f/k$ a full subcategory of $\mathcal{A}b/k$: A formal k-scheme $\mathcal{Y} = \mathrm{Spf}\, \mathcal{A}$, where $\mathcal{A}$ is a profinite k-algebra, is viewed as the sheaf over $\mathbf{Aff}/k$ given by

$$R \longmapsto \mathcal{Y}(R) = \underline{\mathrm{Spf}\, \mathcal{A}}(R) = \mathrm{Hom}^{\mathrm{cont}}_{\text{k-Alg}}(\mathcal{A}, R)$$

which assigns to a k-algebra R with discrete topology the set of continuous homomorphisms of k-algebras from $\mathcal{A}$ to R.

The categories $\mathcal{G}a/k$ and $\mathcal{G}f/k$ are abelian (see [L, (4.1.1) and (4.2.1)]). Kernel and cokernel of a homomorphism in $\mathcal{G}a/k$ (resp. $\mathcal{G}f/k$) coincide with the ones in $\mathcal{A}b/k$, and an exact sequence $0 \to K \to G \to C \to 0$ in $\mathcal{A}b/k$, where K and C are objects of $\mathcal{G}a/k$ (resp. $\mathcal{G}f/k$), implies that G is also an object of $\mathcal{G}a/k$ (resp. $\mathcal{G}f/k$).

1.2 Definition of 1-Motive

In the following by a 1-motive always a *generalized 1-motive* in the sense of Laumon [L, Définition (5.1.1)] is meant:

Definition 1.9 *A* 1-motive *is a complex concentrated in degrees* -1 *and* 0 *in the category of sheaves of abelian groups of the form* $M = [\mathcal{F} \to G]$*, where* $\mathcal{F}$ *is a torsion-free formal group over* k *and* G *a connected algebraic group over* k*.*

1.3 Cartier-Duality

Let G be an algebraic or a formal group and let $\underline{\mathrm{Hom}}_{\mathcal{A}b/\mathrm{k}}(G, \mathbb{G}_{\mathrm{m}})$ be the sheaf of abelian groups over $\mathbf{Alg}/k$ associated to the functor

$$R \longmapsto \mathrm{Hom}_{\mathcal{A}b/R}(G_R, \mathbb{G}_{\mathrm{m},R})$$

which assigns to a k-algebra R the set of homomorphisms of sheaves of abelian groups over R from G_R to $\mathbb{G}_{\mathrm{m},R}$. If G is a linear algebraic group (resp. formal group), this functor is represented by a formal group (resp. linear algebraic group) $G^\vee$, called the *Cartier-dual* of G.

The Cartier-duality is an anti-equivalence between the category of linear algebraic groups and the category of formal groups. The functors $L \longmapsto L^\vee$ and $\mathcal{F} \longmapsto \mathcal{F}^\vee$ are quasi-inverse to each other. (See [SGA3, $\mathrm{VII_B}$, 2.2.2.])

The Cartier-dual of a torus $T \cong (\mathbb{G}_{\mathrm{m}})^t$ is a lattice of the same rank: $T^\vee \cong \mathbb{Z}^t$, i.e. a torsion-free étale formal group (cf. [L, (5.2)]).

Let V be a finite dimensional k-vector space. The Cartier-dual of the vectorial group $\mathbb{V} = \mathrm{Spec}(\mathrm{Sym}\, V^\vee)$ associated to V is the infinitesimal formal group $\mathbb{V}^\vee = \mathrm{Spf}\left(\widehat{\mathrm{Sym}\, V}\right)$, i.e. the completion w.r.t. 0 of the vectorial group associated to the dual k-vector space $V^\vee$ (cf. [L, (5.2)]).

1.4 Duality of 1-Motives

The dual of an abelian variety A is given by $A^\vee = \mathrm{Pic}^0 A$. Unfortunately, there is no analogue duality theory for algebraic groups in general. Instead, we embed the category of connected algebraic groups into the category of 1-motives by sending a connected algebraic group G to the 1-motive $[0 \to G]$. The category of 1-motives admits a duality theory.

Theorem 1.10 *Let L be a connected linear algebraic group, A an abelian variety and $A^\vee$ the dual abelian variety. There is a bijection*

$$\mathrm{Ext}_{\mathcal{A}b/k}(A, L) \simeq \mathrm{Hom}_{\mathcal{A}b/k}(L^\vee, A^\vee) \ .$$

Proof. See [L, Lemme (5.2.1)] for a complete proof.
The bijection $\Phi : \mathrm{Ext}(A, L) \longrightarrow \mathrm{Hom}(L^\vee, A^\vee)$ is constructed as follows: Given an extension G of A by L. By Corollary 1.7 it suffices to determine the homomorphism $\Phi(G) : L^\vee \longrightarrow A^\vee$ on the k-valued points and on the Lie-algebra of $L^\vee$. Let $L = T \times \mathbb{V}$ be the canonical splitting of L into a direct product of a torus T and a vectorial group $\mathbb{V}$ (Theorem 1.2). Now

$$L^\vee(k) = T^\vee(k) = \mathrm{Hom}_{\mathcal{A}b/\mathrm{k}}(T, \mathbb{G}_\mathrm{m})$$
$$\mathrm{Lie}(L^\vee) = \mathrm{Lie}(\mathbb{V}^\vee) = \mathrm{Hom}_\mathrm{k}(\mathrm{Lie}(\mathbb{V}), k) = \mathrm{Hom}_{\mathcal{A}b/\mathrm{k}}(\mathbb{V}, \mathbb{G}_\mathrm{a})$$

i.e. $\chi \in L^\vee(k)$ gives rise to a homomorphism $L \longrightarrow T \longrightarrow \mathbb{G}_\mathrm{m}$ and $\chi \in \mathrm{Lie}(L^\vee)$ to a homomorphism $L \longrightarrow \mathbb{V} \longrightarrow \mathbb{G}_\mathrm{a}$. Then the image of χ under $\Phi(G)$ is the push-out $\chi_* G \in \left\{ \begin{array}{l} \mathrm{Ext}(A, \mathbb{G}_\mathrm{m}) = \mathrm{Pic}^0_A(k) = A^\vee(k) \\ \mathrm{Ext}(A, \mathbb{G}_\mathrm{a}) = \mathrm{Lie}\left(\mathrm{Pic}^0_A\right) = \mathrm{Lie}(A^\vee) \end{array} \right.$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{G} & \longrightarrow & \chi_* G & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

where $\mathbb{G} = \mathbb{G}_\mathrm{m}$ or $\mathbb{G} = \mathbb{G}_\mathrm{a}$. ∎

A consequence of Theorem 1.10 is the duality of 1-motives:
Let $M = \left[\mathcal{F} \xrightarrow{\mu} G\right]$ be a 1-motive, and $0 \to L \to G \to A \to 0$ the canonical decomposition of G (Theorem 1.1). By Theorem 1.10 the composition $\bar{\mu} : \mathcal{F} \longrightarrow G \longrightarrow A$ defines an extension $0 \to \mathcal{F}^\vee \to G^{\bar{\mu}} \to A^\vee \to 0$ and the extension G defines a homomorphism of sheaves of abelian groups $\bar{\mu}^G : L^\vee \longrightarrow A^\vee$. Then μ determines uniquely a factorization $\mu^G : L^\vee \longrightarrow G^{\bar{\mu}}$ of $\bar{\mu}^G$, according to [L, Proposition (5.2.2)], hence gives rise to

Definition 1.11 *The dual 1-motive of $M = \left[\mathcal{F} \xrightarrow{\mu} G\right]$ with $G \in \mathrm{Ext}(A, L)$ is the 1-motive $M^\vee = \left[L^\vee \xrightarrow{\mu^G} G^{\bar{\mu}}\right]$.*

The double dual $M^{\vee\vee}$ of a 1-motive M is canonically isomorphic to M (see [L, (5.2.4)]).

2 Universal Factorization Problem

Let X be a projective variety over k (an algebraically closed field of characteristic 0). The universal factorization problem may be outlined as follows: one is looking for a "universal object" $\mathcal{U}$ and a rational map $u : X \dashrightarrow \mathcal{U}$ such that for every rational map $\varphi : X \dashrightarrow G$ to an algebraic group G there is a unique homomorphism $h : \mathcal{U} \longrightarrow G$ such that $\varphi = h \circ u$ up to translation.

The universal object $\mathcal{U}$, if it exists, is not in general an algebraic group. For this aim a certain finiteness condition on the rational maps is needed. In this section we work out a criterion, for which categories **Mr** of rational maps from X to algebraic groups one can find an algebraic group $\mathrm{Alb}_{\mathbf{Mr}}(X)$ satisfying the universal mapping property, and in this case we give a construction of $\mathrm{Alb}_{\mathbf{Mr}}(X)$. The way of procedure was inspired by Serre's exposé [S2], where the case of semi-abelian varieties (see Examples 2.33 and 2.45) is treated.

2.1 Relative Cartier Divisors

For the construction of universal objects as above we are concerned with the functor of families of Cartier divisors. This functor admits a natural transformation to the Picard functor, which describes families of line bundles, i.e. families of classes of Cartier divisors. References for *effective* relative Cartier divisors are [M1, Lecture 10] and [BLR, Section 8.2]. Since the relative Cartier divisors we are concerned with are not necessarily effective, we give a short overview on this subject.

Definition 2.1 *Let $F : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ be a covariant functor, R a k-algebra. Let $R_{\mathrm{nil}} := k + \mathrm{Nil}(R)$ be the induced subring of R and $\rho : R_{\mathrm{nil}} \longrightarrow k$, $\mathrm{Nil}(R) \ni n \longmapsto 0$ the augmentation. Define*

$$\begin{aligned} \mathrm{Inf}(F)(R) &= \ker\Big(F(\rho) : F(R_{\mathrm{nil}}) \longrightarrow F(k)\Big) \\ \mathrm{Lie}(F) &= \mathrm{Inf}(F)(k[\varepsilon]) \ . \end{aligned}$$

Notation 2.2 *If $F : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ is a covariant functor, then we will use the expression $\alpha \in F$ in order to state that either $\alpha \in F(k)$ or $\alpha \in \mathrm{Lie}(F)$.*

Functor of Relative Cartier Divisors

Let X and Y be noetherian schemes of finite type over k. A *Cartier divisor* on X is by definition a global section of the sheaf $\mathcal{K}_X^* / \mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings on X, and the star * denotes the unit groups.

$$\mathrm{Div}(X) = \Gamma(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$$

is the *group of Cartier divisors.*

Notation 2.3 *If R is a k-algebra, the scheme $Y \times_k \mathrm{Spec}\, R$ is often denoted by $Y \otimes R$.*

Definition 2.4 *Let A be an R-algebra. The set*

$$\mathrm{S}_{A/R} = \left\{ f \in A \;\middle|\; \begin{array}{l} f \textit{ not a zero divisor,} \\ A/fA \textit{ is flat over } R \end{array} \right\}$$

is a multiplicative system in A. Then the localization of A at $\mathrm{S}_{A/R}$

$$\mathrm{K}_{A/R} = \mathrm{S}_{A/R}^{-1} A$$

is called the total quotient ring of A relative to R.
Let $X \overset{\tau}{\longrightarrow} T$ be a scheme over T. The sheaf $\mathcal{K}_{X/T}$ associated to the presheaf

$$U \longmapsto \mathrm{K}_{\mathcal{O}_X(U)/(\tau^{-1}\mathcal{O}_T)(U)} = \mathrm{S}^{-1}_{\mathcal{O}_X(U)/(\tau^{-1}\mathcal{O}_T)(U)} \mathcal{O}_X(U)$$

for open $U \subset X$, is called the sheaf of total quotient rings of X relative to T.

Remark 2.5 *Let A be a finitely generated flat R-algebra, where R is a noetherian ring, $f \in A$ a non zero divisor. Notice that A/fA is flat over R if and only if for all homomorphisms $R \longrightarrow S$ the image of f in $A \otimes_R S$ is a non zero divisor. Equivalently, f is not contained in any associated prime ideal of $A \otimes_R k(p)$ for all $p \in \operatorname{Spec} R$ (cf. [M1, Lecture 10]).*

Proposition 2.6 *For a k-algebra R let*

$$\underline{\mathrm{Div}}_Y(R) = \Gamma\left(Y \otimes R, \mathcal{K}^*_{Y\otimes R/R}\Big/\mathcal{O}^*_{Y\otimes R}\right) .$$

Then the assignment $\quad R \longmapsto \underline{\mathrm{Div}}_Y(R) \quad$ *defines a covariant functor*

$$\underline{\mathrm{Div}}_Y : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

from the category of k-algebras to the category of abelian groups.

Remark 2.7 *For each k-algebra R we have*

$$\underline{\mathrm{Div}}_Y(R) = \left\{ \begin{array}{c} \textit{Cartier divisors } \mathcal{D} \textit{ on } Y \times_k \operatorname{Spec} R \\ \textit{which define Cartier divisors } \mathcal{D}_p \textit{ on } Y \times \{p\} \\ \forall p \in \operatorname{Spec} R \end{array} \right\}$$

and for a homomorphism $h : R \longrightarrow S$ in $\mathbf{Alg}/k$ the induced homomorphism $\underline{\mathrm{Div}}_Y(h) : \underline{\mathrm{Div}}_Y(R) \longrightarrow \underline{\mathrm{Div}}_Y(S)$ in $\mathbf{Ab}$ is the pull-back of Cartier divisors on $Y \times_k \operatorname{Spec} R$ to those on $Y \times_k \operatorname{Spec} S$.

Proposition 2.8 *Let R be a k-algebra, $R_{\mathrm{nil}} = k + \mathrm{Nil}(R)$ the induced ring. There is a canonical isomorphism of abelian groups*

$$\mathrm{Inf}(\underline{\mathrm{Div}}_Y)(R) = \mathrm{Lie}(\underline{\mathrm{Div}}_Y) \otimes_k \mathrm{Nil}(R)$$

where $\mathrm{Lie}(\underline{\mathrm{Div}}_Y) = \Gamma(Y, \mathcal{K}_Y / \mathcal{O}_Y)$.

Proof. $R_{\mathrm{nil}} = k[\mathrm{Nil}(R)]$ is a local ring with only prime ideal $\mathrm{Nil}(R) \in \operatorname{Spec} R_{\mathrm{nil}}$, and it consists of zero divisors only. Therefore it holds $\mathcal{K}^*_{Y\otimes R_{\mathrm{nil}}/R_{\mathrm{nil}}} = \mathcal{K}^*_{Y\otimes R_{\mathrm{nil}}} = \mathcal{K}_Y[\mathrm{Nil}(R)]^* = \mathcal{K}^*_Y + \mathcal{K}_Y \otimes_k \mathrm{Nil}(R)$. Hence we obtain an exact sequence

$$1 \longrightarrow \frac{\mathcal{O}^*_Y + \mathcal{K}_Y \otimes_k \mathrm{Nil}(R)}{\mathcal{O}^*_Y + \mathcal{O}_Y \otimes_k \mathrm{Nil}(R)} \longrightarrow \frac{\mathcal{K}^*_{Y\otimes R_{\mathrm{nil}}}}{\mathcal{O}^*_{Y\otimes R_{\mathrm{nil}}}} \longrightarrow \frac{\mathcal{K}^*_Y}{\mathcal{O}^*_Y} \longrightarrow 1 .$$

Now we have canonical isomorphisms

$$\frac{\mathcal{O}_Y^* + \mathcal{K}_Y \otimes_k \mathrm{Nil}\,(R)}{\mathcal{O}_Y^* + \mathcal{O}_Y \otimes_k \mathrm{Nil}\,(R)} \cong \frac{1 + \mathcal{K}_Y \otimes_k \mathrm{Nil}\,(R)}{1 + \mathcal{O}_Y \otimes_k \mathrm{Nil}\,(R)} \cong \frac{\mathcal{K}_Y \otimes_k \mathrm{Nil}\,(R)}{\mathcal{O}_Y \otimes_k \mathrm{Nil}\,(R)} \cong \frac{\mathcal{K}_Y}{\mathcal{O}_Y} \otimes_k \mathrm{Nil}\,(R)$$

where the second isomorphism is given by $\exp^{-1}$. Applying the global section functor $\Gamma\,(Y, \_)$ yields

$$0 \longrightarrow \Gamma\left(\frac{\mathcal{K}_Y}{\mathcal{O}_Y} \otimes_k \mathrm{Nil}\,(R)\right) \longrightarrow \Gamma\left(\frac{\mathcal{K}^*_{Y\otimes R_{\mathrm{nil}}}}{\mathcal{O}^*_{Y\otimes R_{\mathrm{nil}}}}\right) \longrightarrow \Gamma\left(\frac{\mathcal{K}_Y^*}{\mathcal{O}_Y^*}\right) .$$

Here $\Gamma\left(\mathcal{K}^*_{Y\otimes R_{\mathrm{nil}}} / \mathcal{O}^*_{Y\otimes R_{\mathrm{nil}}}\right) = \underline{\mathrm{Div}}_Y\,(R_{\mathrm{nil}})$ and $\Gamma\,(\mathcal{K}_Y^* / \mathcal{O}_Y^*) = \underline{\mathrm{Div}}_Y\,(k)$, therefore

$$\mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(R) = \Gamma\,(\mathcal{K}_Y / \mathcal{O}_Y) \otimes_k \mathrm{Nil}\,(R) .$$

In particular for $R = k[\varepsilon]$, as $\mathrm{Nil}\,(k[\varepsilon]) = \varepsilon k \cong k$, we have

$$\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) = \mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(k[\varepsilon]) \cong \Gamma\,(\mathcal{K}_Y / \mathcal{O}_Y)$$

and hence $\mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(R) \cong \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) \otimes_k \mathrm{Nil}\,(R)$. ∎

Definition 2.9 *If* $D \in \underline{\mathrm{Div}}_Y\,(k)$*, then* $\mathrm{Supp}\,(D)$ *denotes the locus of zeros and poles of local sections* $(f_\alpha)_\alpha$ *of* $\mathcal{K}_Y^*$ *representing* $D \in \Gamma\,(\mathcal{K}_Y^* / \mathcal{O}_Y^*)$.
If $\delta \in \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)$*, then* $\mathrm{Supp}\,(\delta)$ *denotes the locus of poles of local sections* $(g_\alpha)_\alpha$ *of* $\mathcal{K}_Y$ *representing* $\delta \in \Gamma\,(\mathcal{K}_Y / \mathcal{O}_Y)$.

Definition 2.10 *Let* $\mathcal{F}$ *be a subfunctor of* $\underline{\mathrm{Div}}_Y^0$ *which is a formal group. Then* $\mathrm{Supp}(\mathcal{F})$ *is defined to be the union of* $\mathrm{Supp}(D)$ *for* $D \in \mathcal{F}(k)$ *and* $D \in \mathrm{Lie}(\mathcal{F})$.

$\mathrm{Supp}\,(\mathcal{F})$ is a closed subscheme of codimension 1 in Y, since $\mathcal{F}\,(k)$ and $\mathrm{Lie}\,(\mathcal{F})$ are both finitely generated.

Definition 2.11 *For a morphism* $\sigma : V \longrightarrow Y$ *of varieties define* $\underline{\mathrm{Dec}}_{Y,V}$ *to be the subfunctor of* $\underline{\mathrm{Div}}_Y$ *consisting of families of those Cartier divisors that do not contain any component of the image* $\sigma(V)$.

Proposition 2.12 *Let* $\sigma : V \longrightarrow Y$ *be a morphism of varieties. Then the pull-back of Cartier divisors* σ^* *induces a natural transformation of functors*

$$\_ \cdot V : \underline{\mathrm{Dec}}_{Y,V} \longrightarrow \underline{\mathrm{Div}}_V .$$

Proposition 2.13 *Let* $\mathcal{F}$ *be a formal group. Then each pair* (a, l) *of a homomorphism of abelian groups* $a : \mathcal{F}(k) \longrightarrow \underline{\mathrm{Div}}_Y(k)$ *and a* k*-linear map* $l : \mathrm{Lie}\,(\mathcal{F}) \longrightarrow \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)$ *determines uniquely a natural transformation* $\tau : \mathcal{F} \longrightarrow \underline{\mathrm{Div}}_Y$ *with* $\tau(k) = a$ *and* $\mathrm{Lie}(\tau) = l$*. Moreover, the image of* $\mathcal{F}$ *in* $\underline{\mathrm{Div}}_Y$ *is also a formal group.*

Proof. We construct a natural transformation with the required property by giving homomorphisms $\tau(R) : \mathcal{F}(R) \longrightarrow \underline{\mathrm{Div}}_Y(R)$, $R \in \mathbf{Alg}/k$. As $\mathcal{F}$ and $\underline{\mathrm{Div}}_Y$ both commute with direct products, we may reduce to k-algebras R with $\mathrm{Spec}\,R$ connected. In this case $\mathcal{F}(R) = \mathcal{F}(k) \times \left(\mathrm{Lie}\,(\mathcal{F}) \otimes_k \mathrm{Nil}(R)\right)$, cf. Theorem 1.3 and Remark 1.6. Then (a, l) yields a homomorphism

$$h : \mathcal{F}(k) \times \left(\mathrm{Lie}\,(\mathcal{F}) \otimes_k \mathrm{Nil}(R)\right) \longrightarrow \underline{\mathrm{Div}}_Y(k) \times \left(\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) \otimes_k \mathrm{Nil}(R)\right) .$$

We have functoriality maps $\underline{\mathrm{Div}}_Y(k) \to \underline{\mathrm{Div}}_Y(R)$ and $\underline{\mathrm{Div}}_Y(R_{\mathrm{nil}}) \to \underline{\mathrm{Div}}_Y(R)$, and $\mathrm{Lie}(\underline{\mathrm{Div}}_Y) \otimes_k \mathrm{Nil}(R) \cong \mathrm{Inf}(\underline{\mathrm{Div}}_Y)(R) \subset \underline{\mathrm{Div}}_Y(R_{\mathrm{nil}})$. Then composition with h gives $\tau(R)$. It is straightforward that the homomorphisms $\tau(R)$ yield a natural transformation. The definition of $\mathrm{Inf}(\_)$ implies that $\underline{\mathrm{Div}}_Y(k) \times \mathrm{Inf}(\underline{\mathrm{Div}}_Y)(R) \longrightarrow \underline{\mathrm{Div}}_Y(R)$ is injective. Then the second assertion follows from the structure of formal groups, cf. Corollary 1.7. ■

Picard Functor

Although the Picard functor is an established object in algebraic geometry, we give a summary of facts that we need in the following. References for the Picard functor are e.g. [BLR, Chapter 8], [M1, Lectures 19-21] or [K].

Let Y be a projective scheme over k. The isomorphism classes of line bundles on a k-scheme X form a group $\mathrm{Pic}(X)$, the *(absolute) Picard group* of X, which is given by $\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^*)$. The *(relative) Picard functor* $\underline{\mathrm{Pic}}_Y$ from the category of k-algebras to the category of abelian groups is defined by

$$\underline{\mathrm{Pic}}_Y(R) = \mathrm{Pic}(Y \times_k \mathrm{Spec}\, R) / \mathrm{Pic}(\mathrm{Spec}\, R)$$

for each k-algebra R. In other words, $\underline{\mathrm{Pic}}_Y(R)$ is the abelian group of line bundles on $Y \otimes R$ modulo line bundles that arise as a pull-back of a line bundle on $\mathrm{Spec}\, R$.

Similarly to the proof of Proposition 2.8 one shows

Proposition 2.14 *Let R be a k-algebra, $R_{\mathrm{nil}} = k + \mathrm{Nil}(R)$ the induced ring. There is a canonical isomorphism of abelian groups*

$$\mathrm{Inf}(\underline{\mathrm{Pic}}_Y)(R) = \mathrm{Lie}(\underline{\mathrm{Pic}}_Y) \otimes_k \mathrm{Nil}(R)$$

where $\mathrm{Lie}(\underline{\mathrm{Pic}}_Y) = \mathrm{H}^1(Y, \mathcal{O}_Y)$.

Since a scheme over an algebraically closed field k admits a section, the fppf-sheaf associated to the Picard functor on Y coincides with the relative Picard functor $\underline{\mathrm{Pic}}_Y$, see [BLR, Section 8.1 Proposition 4].

Theorem 2.15 *Let Y be an integral projective k-scheme of finite type. Then the Picard functor $\underline{\mathrm{Pic}}_Y$ is represented by a k-group scheme* Pic_Y*, which is called the* Picard scheme of Y.

Proof. [FGA, No. 232, Theorem 2] or [BLR, Section 8.2, Theorem 1] or [K, Theorem 4.8, Theorem 4.18.1]. ■

Definition 2.16 *Let M, N be line bundles on Y. Then M is said to be* algebraically equivalent *to N, if there exists a connected k-scheme C, a line bundle $\mathcal{L}$ on $Y \times_k C$ and closed points $p, q \in C$ such that $\mathcal{L}|_{Y\times\{p\}} = M$ and $\mathcal{L}|_{Y\times\{q\}} = N$.*

If $\underline{\mathrm{Pic}}_Y$ is represented by a scheme Pic_Y, then a line bundle L on Y is algebraically equivalent to $\mathcal{O}_Y$ if and only if L lies in the connected component Pic_Y^0 of the identity of Pic_Y. Then Pic_Y^0 represents the functor $\underline{\mathrm{Pic}}_Y^0$ which assigns to $R \in \mathbf{Alg}/k$ the abelian group of line bundles $\mathcal{L}$ on $Y \otimes R$ with $\mathcal{L}|_{Y\times\{t\}}$ algebraically equivalent to $\mathcal{O}_Y$ for each $t \in \mathrm{Spec}\, R$, modulo line bundles that come from $\mathrm{Spec}\, R$.

Theorem 2.17 *Let Y be an integral projective k-scheme of finite type. Then $\underline{\mathrm{Pic}}^0_Y$ is represented by a quasi-projective k-group scheme Pic^0_Y. If Y is also normal, then Pic^0_Y is projective.*

Proof. Follows from Theorem 2.15 and [K, Theorem 5.4]. ∎

A normal projective variety Y over k is the disjoint union of its irreducible components (see [Mm, Chapter 3, §9, Remark p. 64]). Applying Theorem 2.17 to each irreducible component Z of Y yields that $\underline{\mathrm{Pic}}^0_Y$ is represented by the product of the connected projective k-group schemes Pic^0_Z. In characteristic 0 a connected projective k-group scheme is an abelian variety. We obtain

Corollary 2.18 *Let Y be a normal projective variety over k. Then $\underline{\mathrm{Pic}}^0_Y$ is represented by an abelian variety Pic^0_Y.*

Transformation $\underline{\mathrm{Div}}_Y \longrightarrow \underline{\mathrm{Pic}}_Y$

Let Y be a k-scheme and R be a k-algebra. Consider the exact sequence $\mathrm{Seq}\,(R)$:

$$1 \longrightarrow \mathcal{O}^*_{Y\otimes R} \longrightarrow \mathcal{K}^*_{Y\otimes R} \longrightarrow \mathcal{K}^*_{Y\otimes R}/\,\mathcal{O}^*_{Y\otimes R} \longrightarrow 1\ .$$

In the corresponding long exact sequence $\mathrm{H}^\bullet\,\mathrm{Seq}\,(R)$:

$$\longrightarrow \mathrm{H}^0\left(\mathcal{K}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^0\left(\mathcal{K}^*_{Y\otimes R}/\,\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{K}^*_{Y\otimes R}\right) \longrightarrow$$

the connecting homomorphism $\delta^0\,(R) : \mathrm{H}^0\left(\mathcal{K}^*_{Y\otimes R}/\,\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{O}^*_{Y\otimes R}\right)$ gives a natural transformation $\mathrm{Div}\,(Y\otimes R) \longrightarrow \mathrm{Pic}\,(Y\otimes R)$. Composing this transformation with the injection $\underline{\mathrm{Div}}_Y\,(R) \hookrightarrow \mathrm{Div}\,(Y\otimes R)$ and the projection $\mathrm{Pic}\,(Y\otimes R) \twoheadrightarrow \underline{\mathrm{Pic}}_Y\,(R)$ yields a natural transformation

$$\mathrm{cl} : \underline{\mathrm{Div}}_Y \longrightarrow \underline{\mathrm{Pic}}_Y\ .$$

Definition 2.19 *Let $\underline{\mathrm{Div}}^0_Y$ be the subfunctor of $\underline{\mathrm{Div}}_Y$ defined by $\underline{\mathrm{Div}}^0_Y\,(R) = \mathrm{cl}^{-1}\left(\underline{\mathrm{Pic}}^0_Y\,(R)\right)$ for each k-algebra R.*

2.2 Categories of Rational Maps to Algebraic Groups

Let Y be a regular projective variety over k (an algebraically closed field of characteristic 0). Algebraic groups are always assumed to be connected, unless stated otherwise.

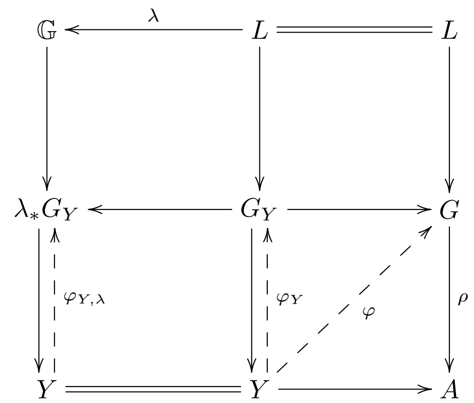
Notation 2.20 *$\mathbb{G}$ stands for one of the groups $\mathbb{G}_\mathrm{m}$ or $\mathbb{G}_\mathrm{a}$.*

Lemma 2.21 *Let P be a principal $\mathbb{G}$-bundle over Y. Then a local section $\sigma : U \subset Y \longrightarrow P$ determines uniquely a divisor $\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\sigma) \in \underline{\mathrm{Div}}_Y(k)$ on Y, if $\mathbb{G} = \mathbb{G}_\mathrm{m}$, or an infinitesimal deformation $\mathrm{div}_{\mathbb{G}_\mathrm{a}}(\sigma) \in \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)$ of the zero divisor on Y, if $\mathbb{G} = \mathbb{G}_\mathrm{a}$.*

Proof. For $V = \begin{cases} k & \text{if } \mathbb{G} = \mathbb{G}_\mathrm{m} \\ k[\varepsilon] & \text{if } \mathbb{G} = \mathbb{G}_\mathrm{a} \end{cases}$ let $\lambda : \mathbb{G}(k) \longrightarrow \mathrm{GL}\,(V)$ be the representation of $\mathbb{G}$ given by $l \longmapsto \begin{cases} l & \text{if } \mathbb{G} = \mathbb{G}_\mathrm{m} \\ 1+\varepsilon l & \text{if } \mathbb{G} = \mathbb{G}_\mathrm{a} \end{cases}$. Let $\mathcal{V} = P \amalg_{[\mathbb{G},\lambda]} V$

be the vector-bundle associated to P of fibre-type V. Denote by $s \in \Gamma(U, \mathcal{V})$ the image of σ under the map $P \longrightarrow \mathcal{V}$ induced by λ on the fibres. There is an effective divisor H, supported on $Y \setminus U$, such that the local section $s \in \Gamma(U, \mathcal{V})$ extends to a global section of $\mathcal{V}(H) = \mathcal{V} \otimes_{\mathcal{O}} \mathcal{O}_Y(H)$. Local trivializations $\mathcal{V}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha} \otimes_k V$ induce local isomorphisms $\Phi_\alpha : \mathcal{V}(H)|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}(H)|_{U_\alpha} \otimes_k V$ of the twisted bundles. Then the local sections $\Phi_\alpha(s) \in \Gamma(U_\alpha, \mathcal{O}_Y(H) \otimes_k V)$ yield a divisor $\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\sigma)$ on Y if $V = k$, and a divisor $\operatorname{div}_{\mathbb{G}_{\mathrm{a}}}(\sigma)$ on $Y[\varepsilon] = Y \times_k \operatorname{Spec} k[\varepsilon]$ if $V = k[\varepsilon]$. The description of $\underline{\mathrm{Div}}_Y$ from Remark 2.7 shows that $\operatorname{div}_{\mathbb{G}}(\sigma) \in \underline{\mathrm{Div}}_Y(V)$. In the second case, the image of $\lambda : \mathbb{G}_{\mathrm{a}}(k) \longrightarrow \mathrm{GL}(k[\varepsilon])$ lies in $1 + \varepsilon k$. This implies that the restriction of $\operatorname{div}_{\mathbb{G}_{\mathrm{a}}}(\sigma)$ to Y is the zero divisor, thus $\operatorname{div}_{\mathbb{G}_{\mathrm{a}}}(\sigma) \in \operatorname{Lie}(\underline{\mathrm{Div}}_Y)$. ■

Let $\varphi : Y \dashrightarrow G$ be a rational map to an algebraic group G with canonical decomposition $0 \to L \to G \xrightarrow{\rho} A \to 0$. Since a rational map to an abelian variety is defined at every regular point (see [La, Chapter II, §1, Theorem 2]), the composition $Y \overset{\varphi}{\dashrightarrow} G \xrightarrow{\rho} A$ extends to a morphism $\overline{\varphi} : Y \longrightarrow A$. Let $G_Y = G \times_A Y$ be the fibre-product of G and Y over A. The graph-morphism $\varphi_Y : U \subset Y \longrightarrow G \times_A Y, \quad y \longmapsto (\varphi(y), y)$ of φ is a section of the L-bundle G_Y over Y. Each $\lambda \in L^\vee$ (see Notation 2.2, where we consider $L^\vee$ as a functor on $\mathbf{Alg}/k$) gives rise to a homomorphism $\lambda : L \longrightarrow \mathbb{G}$ (cf. proof of Theorem 1.10). Then the composition of φ_Y with the push-out of G_Y via λ gives a local section $\varphi_{Y,\lambda} : U \subset Y \longrightarrow \lambda_* G_Y$ of the $\mathbb{G}$-bundle $\lambda_* G_Y$ over Y:



Lemma 2.21 says that the local section $\varphi_{Y,\lambda}$ determines a unique divisor or deformation $\operatorname{div}_{\mathbb{G}}(\varphi_{Y,\lambda}) \in \begin{cases} \Gamma(\mathcal{K}_Y^* / \mathcal{O}_Y^*) & \text{if } \mathbb{G} = \mathbb{G}_{\mathrm{m}} \\ \Gamma\left(\mathcal{K}_{Y[\varepsilon]}^* \middle/ \mathcal{O}_{Y[\varepsilon]}^*\right) & \text{if } \mathbb{G} = \mathbb{G}_{\mathrm{a}} \end{cases}$. Now the bundle $\lambda_* G_Y$ comes from an extension of algebraic groups, and $\underline{\mathrm{Ext}}_{\mathcal{A}b/\mathrm{k}}(A, \mathbb{G}_{\mathrm{m}}) \cong \underline{\mathrm{Pic}}_A^0$, hence it is an element of $\underline{\mathrm{Pic}}_Y^0(k)$ if $\mathbb{G} = \mathbb{G}_{\mathrm{m}}$, or of $\operatorname{Lie}\left(\underline{\mathrm{Pic}}_Y^0\right)$ if $\mathbb{G} = \mathbb{G}_{\mathrm{a}}$. Therefore $\operatorname{div}_{\mathbb{G}}(\varphi_{Y,\lambda})$ is a divisor in $\underline{\mathrm{Div}}_Y^0(k)$ if $\lambda \in L(k)$, or an element of $\operatorname{Lie}\left(\underline{\mathrm{Div}}_Y^0\right)$ if $\lambda \in \operatorname{Lie}(L)$.

Proposition 2.22 *Let $G \in \operatorname{Ext}(A, L)$ be an algebraic group and $\varphi : Y \dashrightarrow G$ a rational map. Then φ induces a natural transformation of functors $\tau_\varphi : L^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$.*

Proof. The construction above yields a homomorphism of abelian groups $L^\vee(k) \to \underline{\mathrm{Div}}_Y^0(k)$, $\lambda \mapsto \operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\varphi_{Y,\lambda})$ and a k-linear map $\operatorname{Lie}(L) \to \operatorname{Lie}\left(\underline{\mathrm{Div}}_Y^0\right)$,

$\lambda \mapsto \operatorname{div}_{\mathbb{G}_a}(\varphi_{Y,\lambda})$. Then by Proposition 2.13 this extends to a natural transformation $L^\vee \longrightarrow \underline{\mathrm{Div}}^0_Y$. ■

Definition 2.23 *A category* $\mathbf{Mr}$ *is called a* category of rational maps from Y to algebraic groups, *if objects and morphisms of* $\mathbf{Mr}$ *satisfy the following conditions: The objects of* $\mathbf{Mr}$ *are rational maps* $\varphi : Y \dashrightarrow G$, *where* G *is an algebraic group, such that* $\varphi(U)$ *generates a* connected *algebraic subgroup of* G *for any open set* $U \subset Y$ *on which* φ *is defined. The morphisms of* $\mathbf{Mr}$ *between two objects* $\varphi : Y \dashrightarrow G$ *and* $\psi : Y \dashrightarrow H$ *are given by the set of all homomorphisms of algebraic groups* $h : G \longrightarrow H$ *such that* $h \circ \varphi = \psi$, *i.e. the following diagram commutes:*

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ G & \xrightarrow{\quad h \quad} & H \end{array}$$

Remark 2.24 *Let* $\varphi : Y \dashrightarrow G$ *and* $\psi : Y \dashrightarrow H$ *be two rational maps from* Y *to algebraic groups. Then Definition 2.23 implies that for any category* $\mathbf{Mr}$ *of rational maps from* Y *to algebraic groups containing* φ *and* ψ *as objects the set of morphisms* $\mathrm{Hom}_{\mathbf{Mr}}(\varphi, \psi)$ *is the same. Therefore two categories* $\mathbf{Mr}$ *and* $\mathbf{Mr}'$ *of rational maps from* Y *to algebraic groups are equivalent if every object of* $\mathbf{Mr}$ *is isomorphic to an object of* $\mathbf{Mr}'$.

Definition 2.25 *We denote by* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ *the localization of a category* $\mathbf{Mr}$ *of rational maps from* Y *to algebraic groups at the system of injective homomorphisms among the morphisms of* $\mathbf{Mr}$.

Remark 2.26 *In* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ *we may assume for an object* $\varphi : Y \dashrightarrow G$ *that* G *is generated by* φ: *The inclusion* $\langle \operatorname{im} \varphi \rangle \hookrightarrow G$ *of the subgroup* $\langle \operatorname{im} \varphi \rangle$ *generated by* φ *is an injective homomorphism, hence* $\varphi : Y \dashrightarrow G$ *is isomorphic to* $\varphi : Y \dashrightarrow \langle \operatorname{im} \varphi \rangle$, *if* $(\varphi : Y \dashrightarrow \langle \operatorname{im} \varphi \rangle) \in \mathbf{Mr}$.

Definition 2.27 *The category of rational maps from* Y *to abelian varieties is denoted by* $\mathbf{Mav}$.

Remark 2.28 *The objects of* $\mathbf{Mav}$ *are in fact* morphisms *from* Y, *since a rational map from a regular variety* Y *to an abelian variety* A *extends to a morphism from* Y *to* A.

Definition 2.29 *Let* $\mathcal{F}$ *be a subfunctor of* $\underline{\mathrm{Div}}^0_Y$ *which is a formal group. Then* $\mathbf{Mr}_{\mathcal{F}}$ *denotes the category of those rational maps* $\varphi : Y \dashrightarrow G$ *from* Y *to algebraic groups for which the images of the natural transformations* $\tau_\varphi : L^\vee \longrightarrow \underline{\mathrm{Div}}^0_Y$ *(Proposition 2.22) lie in* $\mathcal{F}$, *i.e. which induce homomorphisms of formal groups* $L^\vee \longrightarrow \mathcal{F}$, *where* $0 \to L \to G \to A \to 0$ *is the canonical decomposition of* G.

$$\mathbf{Mr}_{\mathcal{F}} = \{\varphi : Y \dashrightarrow G \mid \operatorname{im} \tau_\varphi \subset \mathcal{F}\}$$

Example 2.30 *For the category* $\mathbf{Mr}_0$ *associated to the trivial formal group* $\{0\}$ *the localization* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}_0$ *(Definition 2.25) is equivalent to the localization* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mav}$ *of the category of morphisms from* Y *to abelian varieties:*

Let $\varphi : Y \dashrightarrow G$ be a rational map to an algebraic group $G \in \mathrm{Ext}(A, L)$, and assume φ generates a connected subgroup of G. Then the following conditions are equivalent:

(i) The transformation $\tau_\varphi : L^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$ induced by φ has image $\{0\}$.

(ii) The section φ_Y to the L-bundle G_Y over Y extends to a global section.

In this case φ is defined on the whole of Y, i.e. is a morphism from Y to G. Since Y is complete, $\mathrm{im}\,\varphi$ is a complete subvariety of G. Then the subgroup $\langle \mathrm{im}\,\varphi \rangle$ of G is also complete (cf. [ESV, Lemma 1.10 (ii)]) and connected, hence an abelian variety.

Definition 2.31 *Let X be a (singular) projective variety. A morphism of varieties $\pi : \widetilde{X} \longrightarrow X$ is called a* resolution of singularities *for X, if $\widetilde{X}$ is nonsingular and π is a proper birational morphism which is an isomorphism over the nonsingular points of X.*

Example 2.32 *Let X be a singular projective variety and $Y = \widetilde{X}$, where $\pi : \widetilde{X} \longrightarrow X$ is a projective resolution of singularities. Denote by $\mathbf{Mr}_0^X$ the category of rational maps $\varphi : X \dashrightarrow G$ whose associated map on 0-cycles of degree 0 $\mathrm{Z}_0(U)^0 \longrightarrow G(k)$ (where U is the open set on which φ is defined) factors through the homological Chow group of 0-cycles of degree 0 modulo rational equivalence $\mathrm{A}_0(X)^0$. The definition of $\mathrm{A}_0(X)$ (see [Fu, Section 1.3, p.10]) implies that such a rational map is necessarily defined on the whole of X, i.e. is a morphism $\varphi : X \longrightarrow G$. Then the composition $\varphi \circ \pi : Y \longrightarrow G$ is an object of $\mathbf{Mr}_0$. Thus $\mathbf{Mr}_0^X$ is a subcategory of the category $\mathbf{Mr}_0$ from Example 2.30.*

Example 2.33 *Let D be a reduced effective divisor on Y. Let $\mathbf{Mr}^D$ be the category of rational maps from Y to semi-abelian varieties (i.e. extensions of an abelian variety by a torus) which are regular away from D.*

Let $\mathcal{F}_D$ be the formal group whose étale part is given by divisors in $\underline{\mathrm{Div}}_Y^0(k)$ with support in $\mathrm{Supp}\,(D)$ and whose infinitesimal part is trivial.

Then $\mathbf{Mr}^D$ is equivalent to $\mathbf{Mr}_{\mathcal{F}_D}$: For a rational map $\varphi : Y \dashrightarrow G$ the induced sections $\varphi_{Y,\lambda}$ determine divisors in $\underline{\mathrm{Div}}_Y^0(k)$ supported on $\mathrm{Supp}\,(D)$ for all $\lambda \in L^\vee$ (where L is the largest linear subgroup of G) if and only if L is a torus, i.e. it consists of several copies of $\mathbb{G}_\mathrm{m}$ only, and φ is regular on $Y \setminus \mathrm{Supp}\,(D)$.

Example 2.34 *Let $Y = C$ be a smooth projective curve, $\mathfrak{d} = \sum_i n_i\, p_i$ with $p_i \in C$, n_i integers ≥ 1, an effective divisor on C and let v_p be the valuation attached to the point $p \in C$.*

Let $\mathbf{Mr}^{\mathfrak{d}}$ be the category of those rational maps $\varphi : Y \dashrightarrow G$ such that for all $f \in \mathcal{K}_C$ it holds:

$$\mathrm{v}_{p_i}(1 - f) \geq n_i \quad \forall i \quad \Longrightarrow \quad \varphi\,(\mathrm{div}\,(f)) = 0\,.$$

Let $\mathcal{F}_{\mathfrak{d}}$ be the formal group defined by

$$\begin{aligned} \mathcal{F}_{\mathfrak{d}}\,(k) &= \left\{ \sum_i l_i\, p_i \;\middle|\; l_i \in \mathbb{Z},\ \sum_i l_i = 0 \right\} \\ \mathrm{Lie}\,(\mathcal{F}_{\mathfrak{d}}) &= \Gamma\left(\mathcal{O}_C \left(\sum_i (n_i - 1)\, p_i \right) \Big/ \mathcal{O}_C \right)\,. \end{aligned}$$

Then $\mathbf{Mr}^{\mathfrak{d}}$ *is equivalent to* $\mathbf{Mr}_{\mathcal{F}_{\mathfrak{d}}}$. *By constructing a singular curve associated to the modulus* $\mathfrak{d}$ *(see [S3, Chapter IV, No. 4]), this turns out to be a special case of Example 2.36 (cf. Lemma 3.21 for the computation of* $\mathrm{Lie}\,(\mathcal{F}_{\mathfrak{d}})$*).*

Example 2.35 *Let* Y *be a smooth projective variety over* $\mathbb{C}$, D *a divisor on* Y *with normal crossings, and let* $U = Y \setminus \mathrm{Supp}(D)$. *Let* $W \subset \Gamma\left(U, \Omega^1_U\right)^{\mathrm{d}=0}$ *be a finite dimensional* k*-vector space containing* $\Gamma\left(Y, \Omega^1_Y[\log D]\right)$. *Let* $\mathbf{Mr}^W$ *be the category of those morphisms* $\varphi : U \longrightarrow G$ *from* U *to algebraic groups for which* $\varphi^*\left(\mathrm{Lie}\,(G)^\vee\right) \subset W$, *where* $\mathrm{Lie}\,(G)^\vee = \Gamma\left(G, \Omega^1_G\right)^{\mathrm{const}}$ *is the* k*-vector space of translation invariant regular 1-forms on* G. *A 1-form* $\omega \in W$ *determines a deformation of the zero divisor* $\delta(\omega) \in \Gamma\left(\mathcal{K}_Y / \mathcal{O}_Y\right)$ *as follows: There are a covering* $\{V_i\}_i$ *of* Y *and regular functions* $f_i \in \mathcal{O}_Y(U \cap V_i)$ *such that* $\omega - \mathrm{d}f_i$ *is regular on* V_i *(see [FW, VI.4 proof of Lemma 7]). Define* $\delta(\omega) = [(f_i)_i] \in \Gamma\left(\mathcal{K}_Y / \mathcal{O}_Y\right)$.

Let $\mathcal{F}_W$ *be the formal group determined by*

$$\begin{aligned} \mathcal{F}_W\,(k) &= \left\{ D' \in \underline{\mathrm{Div}}^0_Y\,(k) \,\middle|\, \mathrm{Supp}(D') \subset \mathrm{Supp}(D) \right\} \\ \mathrm{Lie}\,(\mathcal{F}_W) &= \mathrm{im}\left(\delta : W \longrightarrow \Gamma\left(\mathcal{K}_Y / \mathcal{O}_Y\right)\right). \end{aligned}$$

Then $\mathbf{Mr}^W$ *is equivalent to* $\mathbf{Mr}_{\mathcal{F}_W}$. *This follows from the construction of the generalized Albanese variety of Faltings and Wüstholz (see [FW, VI.2. Satz 6]).*

Example 2.36 *Let* X *be a singular projective variety and* $Y = \widetilde{X}$, *where* $\pi : \widetilde{X} \longrightarrow X$ *is a projective resolution of singularities. A rational map* $\varphi : X \dashrightarrow G$ *which is regular on the regular locus* X_{reg} *of* X *can also be considered as a rational map from* Y *to* G. *Let* $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ *be the category of morphisms* $\varphi : X_{\mathrm{reg}} \longrightarrow G$ *which factor through a homomorphism of groups* $\mathrm{CH}_0\,(X)^0 \longrightarrow G(k)$, *see Definition 3.27 (cf. [ESV, Definition 1.14] for the notion of* regular homomorphism*). Let* $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ *be the formal group given by the kernel of the push-forward* π_* *(Proposition 3.24).*
Then $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ *is equivalent to* $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$. *This is the subject of Section 3.*

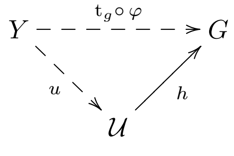
2.3 Universal Objects

Let Y be a regular projective variety over k (an algebraically closed field of characteristic 0).

Existence and Construction

Definition 2.37 *Let* $\mathbf{Mr}$ *be a category of rational maps from* Y *to algebraic groups. Then* $(u : Y \dashrightarrow \mathcal{U}) \in \mathbf{Mr}$ *is called a* universal object for $\mathbf{Mr}$ *if it has the universal mapping property in* $\mathbf{Mr}$*:*

for all $(\varphi : Y \dashrightarrow G) \in \mathbf{Mr}$ *there exists a unique homomorphism of algebraic groups* $h : \mathcal{U} \longrightarrow G$ *such that* $\varphi = h \circ u$ *up to translation, i.e. there is a constant* $g \in G(k)$ *such that the following diagram is commutative*



where $t_g : x \longmapsto x + g$ *is the translation by* g.

Remark 2.38 *Localization of a category* **Mr** *of rational maps from* Y *to algebraic groups at the system of injective homomorphisms does not change (the equivalence class of) the universal object. Therefore it is often convenient to pass to the localization* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ *(Definition 2.25).*

For the category **Mav** of morphisms from Y to abelian varieties (Definition 2.27) there exists a universal object: the *Albanese mapping* to the *Albanese variety*, which is denoted by $\mathrm{alb} : Y \longrightarrow \mathrm{Alb}\,(Y)$. This is a classical result (see e.g. [La], [Ms], [S1]).

In the following we consider categories **Mr** of rational maps from Y to algebraic groups satisfying the following conditions:

($\diamondsuit$ 1) **Mr** contains the category **Mav**.

($\diamondsuit$ 2) $(\varphi : Y \dashrightarrow G) \in \mathbf{Mr}$ if and only if
$\forall \lambda \in L^{\vee}$ the induced rational map $(\varphi_{\lambda} : Y \dashrightarrow \lambda_* G) \in \mathbf{Mr}$.

($\diamondsuit$ 3) If $\varphi : Y \dashrightarrow G$ is a rational map s.t. $\rho \circ \varphi$ factors through a homomorphism $\alpha : B \longrightarrow A$ of abelian varieties, then $(\varphi : Y \dashrightarrow G) \in \mathbf{Mr}$ if and only if $(\varphi^{\alpha} : Y \dashrightarrow \alpha^* G) \in \mathbf{Mr}$.

Here $0 \to L \to G \xrightarrow{\rho} A \to 0$ is the canonical decomposition of the algebraic group G.

Theorem 2.39 *Let* **Mr** *be a category of rational maps from* Y *to algebraic groups which satisfies* ($\diamondsuit$ $1-3$)*. Then for* **Mr** *there exists a universal object* $(u : Y \dashrightarrow \mathcal{U}) \in \mathbf{Mr}$ *if and only if there is a formal group* $\mathcal{F}$ *which is a subfunctor of* $\underline{\mathrm{Div}}_Y^0$ *such that* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ *is equivalent to* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}_{\mathcal{F}}$*, where* $\mathbf{Mr}_{\mathcal{F}}$ *is the category of rational maps which induce a homomorphism of formal groups to* $\mathcal{F}$ *(Definition 2.29).*

Proof. ($\Longleftarrow$) Assume that $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ is equivalent to $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}_{\mathcal{F}}$, where $\mathcal{F}$ is a formal group in $\underline{\mathrm{Div}}_Y^0$. The first step is the construction of an algebraic group $\mathcal{U}$ and a rational map $u : Y \dashrightarrow \mathcal{U}$. In a second step the universality of $u : Y \dashrightarrow \mathcal{U}$ for $\mathbf{Mr}_{\mathcal{F}}$ has to be shown.

Step 1: Construction of $u : Y \dashrightarrow \mathcal{U}$
Y is a regular projective variety over k, thus the functor $\underline{\mathrm{Pic}}_Y^0$ is represented by an abelian variety Pic_Y^0 (Corollary 2.18). Since $\mathcal{F} \subset \underline{\mathrm{Div}}_Y^0$, the formal group $\mathcal{F}$ is torsion-free. The natural transformation $\underline{\mathrm{Div}}_Y^0 \longrightarrow \underline{\mathrm{Pic}}_Y^0$ induces a 1-motive $M = \left[\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0\right]$. Let $M^{\vee}$ be the dual 1-motive of M. The formal group in degree -1 of $M^{\vee}$ is the Cartier-dual of the largest linear subgroup of Pic_Y^0, and this is zero, since an abelian variety does not contain any non-trivial linear subgroup. Then define $\mathcal{U}$ to be the algebraic group in degree 0 of $M^{\vee}$, i.e. $[0 \longrightarrow \mathcal{U}]$ is the dual 1-motive of $\left[\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0\right]$. The canonical decomposition $0 \to \mathcal{L} \to \mathcal{U} \to \mathcal{A} \to 0$ is the extension of $\left(\mathrm{Pic}_Y^0\right)^{\vee}$ by $\mathcal{F}^{\vee}$ induced by the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0$ (Theorem 1.10), where $\mathcal{L} = \mathcal{F}^{\vee}$ is the Cartier-dual of $\mathcal{F}$ and $\mathcal{A} = \left(\mathrm{Pic}_Y^0\right)^{\vee}$ is the dual abelian variety of Pic_Y^0, which is $\mathrm{Alb}\,(Y)$.

As $\mathcal{L}$ is a linear algebraic group, there is a canonical splitting $\mathcal{L} \cong T \times \mathbb{V}$ of $\mathcal{L}$ into the direct product of a torus T of rank t and a vectorial group $\mathbb{V}$ of dimension v (Theorem 1.2). The homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0$ is uniquely determined by the values on a basis Ω of the finite free $\mathbb{Z}$-module

$$\mathcal{F}\,(k) = \mathcal{L}^{\vee}\,(k) = T^{\vee}(k) = \mathrm{Hom}_{\mathcal{A}b/\mathrm{k}}\,(T, \mathbb{G}_{\mathrm{m}})$$

and on a basis Θ of the finite dimensional k-vector space

$$\mathrm{Lie}\,(\mathcal{F}) = \mathrm{Lie}\,(\mathcal{L}^\vee) = \mathrm{Lie}\,(\mathbb{V}^\vee) = \mathrm{Hom}_{\mathrm{k}}\,(\mathrm{Lie}(\mathbb{V}), k) = \mathrm{Hom}_{\mathcal{A}b/\mathrm{k}}\,(\mathbb{V}, \mathbb{G}_{\mathrm{a}})\ .$$

By duality, such a choice of bases corresponds to a decomposition

$$\mathcal{L} \xrightarrow{\sim} (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v\ ,$$

and induces a decomposition

$$\begin{array}{rcl} \mathrm{Ext}\,(\mathcal{A}, \mathcal{L}) & \xrightarrow{\sim} & \mathrm{Ext}\,(\mathcal{A}, \mathbb{G}_{\mathrm{m}})^t \times \mathrm{Ext}\,(\mathcal{A}, \mathbb{G}_{\mathrm{a}})^v \\ \mathcal{U} & \longmapsto & \prod_{\omega\in\Omega} \omega_*\mathcal{U} \times \prod_{\vartheta\in\Theta} \vartheta_*\mathcal{U}\ . \end{array}$$

Therefore the rational map $u : Y \dashrightarrow \mathcal{U}$ is uniquely determined by the following rational maps to push-outs of $\mathcal{U}$

$$\begin{array}{ll} u_\omega : Y \dashrightarrow \omega_*\mathcal{U} & \omega \in \Omega \\ u_\vartheta : Y \dashrightarrow \vartheta_*\mathcal{U} & \vartheta \in \Theta \end{array}$$

whenever Ω is a basis of $\mathcal{F}\,(k)$ and Θ a basis of $\mathrm{Lie}\,(\mathcal{F})$. We have isomorphisms

$$\begin{array}{rcl} \mathrm{Ext}\,(\mathcal{A}, \mathbb{G}_{\mathrm{m}}) \simeq \mathrm{Pic}^0_{\mathcal{A}}\,(k) & \xrightarrow{\sim} & \mathrm{Pic}^0_Y\,(k) \\ P & \longmapsto & P_Y = P \times_{\mathcal{A}} Y \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Ext}\,(\mathcal{A}, \mathbb{G}_{\mathrm{a}}) \simeq \mathrm{Lie}\left(\mathrm{Pic}^0_{\mathcal{A}}\right) & \xrightarrow{\sim} & \mathrm{Lie}\left(\mathrm{Pic}^0_Y\right) \\ T & \longmapsto & T_Y = T \times_{\mathcal{A}} Y\ . \end{array}$$

From the proof of Theorem 1.10 it follows that $(\omega_*\mathcal{U})_Y$ is just the image of $\omega \in \mathcal{F}\,(k) \subset \underline{\mathrm{Div}}^0_Y\,(k)$ under the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y$, which is the divisor-class $[\omega] \in \mathrm{Pic}^0_Y\,(k)$. Likewise from the proof of Theorem 1.10 follows that $(\vartheta_*\mathcal{U})_Y$ is the image of $\vartheta \in \mathrm{Lie}\,(\mathcal{F}) \subset \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right)$ under the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y$, which is the class of deformation $[\vartheta] \in \mathrm{Lie}\left(\mathrm{Pic}^0_Y\right)$. Then define the rational map $u : Y \dashrightarrow \mathcal{U}$ by the condition that for all $\omega \in \Omega$ the section

$$u_{Y,\omega} : Y \dashrightarrow \omega_*\mathcal{U}_Y = [\omega]$$

corresponds to the divisor $\omega \in \underline{\mathrm{Div}}^0_Y\,(k)$, and for all $\vartheta \in \Theta$ the section

$$u_{Y,\vartheta} : Y \dashrightarrow \vartheta_*\mathcal{U}_Y = [\vartheta]$$

corresponds to the deformation $\vartheta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right)$, in the sense of Lemma 2.21, i.e.

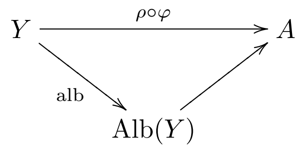
$$\begin{array}{ll} \mathrm{div}_{\mathbb{G}_{\mathrm{m}}}\,(u_{Y,\omega}) = \omega & \forall\omega \in \Omega \\ \mathrm{div}_{\mathbb{G}_{\mathrm{a}}}\,(u_{Y,\vartheta}) = \vartheta & \forall\vartheta \in \Theta\ . \end{array}$$

This determines u up to translation by a constant. The conditions $(\diamondsuit\ 1-3)$ guarantee that $(u : Y \dashrightarrow \mathcal{U}) \in \mathbf{Mr}$.

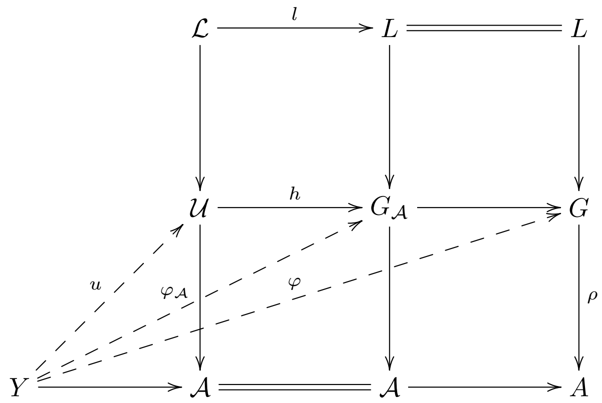
Step 2: Universality of $u : Y \dashrightarrow \mathcal{U}$

Let G be an algebraic group with canonical decomposition $0 \to L \to G \xrightarrow{\rho} A \to 0$

and $\varphi : Y \dashrightarrow G$ a rational map inducing a homomorphism of formal groups $l^\vee : L^\vee \longrightarrow \mathcal{F}$, $\lambda \longmapsto \mathrm{div}_{\mathbb{G}}(\varphi_{Y,\lambda})$ for $\lambda \in L^\vee$ (Proposition 2.22). Let $l : \mathcal{L} \longrightarrow L$ be the dual homomorphism of linear groups. The composition $Y \overset{\varphi}{\dashrightarrow} G \overset{\rho}{\longrightarrow} A$ extends to a morphism from Y to an abelian variety. Translating φ by a constant $g \in G(k)$, if necessary, we may hence assume that $\rho \circ \varphi$ factors through $\mathcal{A} = \mathrm{Alb}(Y)$:



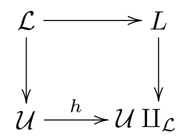
We are going to show that the following diagram commutes:



i.e. the task is to show that

(a) $G_{\mathcal{A}} = l_* \mathcal{U}$
(b) $\varphi_{\mathcal{A}} = h \circ u$ mod translation

where $G_{\mathcal{A}} = G \times_A \mathcal{A}$ is the fibre-product of G and $\mathcal{A}$ over A and $\varphi_{\mathcal{A}} : Y \dashrightarrow G_{\mathcal{A}}$ is the unique map obtained from $(\varphi, \mathrm{alb}) : Y \dashrightarrow G \times \mathcal{A}$ by the universal property of the fibre-product $G_{\mathcal{A}}$, and where h is the homomorphism obtained by the amalgamated sum



as by definition of the push-out we have $l_* \mathcal{U} = \mathcal{U} \amalg_{\mathcal{L}} L$.
For this purpose, by additivity of extensions, it is enough to show that for all $\lambda \in L^\vee$ it holds

(a') $\lambda_* G_{\mathcal{A}} = l^\vee(\lambda)_* \mathcal{U}$
(b') $\varphi_{\mathcal{A},\lambda} = u_{l^\vee(\lambda)}$ mod translation

where $l^\vee(\lambda) = \lambda \circ l$ and $l^\vee(\lambda)_* = (\lambda \circ l)_* = \lambda_* l_*$. Using the isomorphism $\mathrm{Pic}^0_{\mathcal{A}} \xrightarrow{\sim} \mathrm{Pic}^0_Y$, this is equivalent to showing that for all $\lambda \in L^\vee$ it holds

(a") $\lambda_* G_Y = l^\vee(\lambda)_* \mathcal{U}_Y$
(b") $\varphi_{Y,\lambda} = u_{Y,l^\vee(\lambda)}$ mod translation .

By construction of $u : Y \dashrightarrow \mathcal{U}$, we have for all $\lambda \in L^\vee$:

$$\operatorname{div}_{\mathbb{G}}\left(u_{Y,l^\vee(\lambda)}\right) = l^\vee(\lambda) = \operatorname{div}_{\mathbb{G}}\left(\varphi_{Y,\lambda}\right)$$

and hence

$$\begin{aligned} l^\vee(\lambda)_* \mathcal{U}_Y &= [l^\vee(\lambda)] \\ &= [\operatorname{div}_{\mathbb{G}}(\varphi_{Y,\lambda})] \;=\; \lambda_* G_Y \;. \end{aligned}$$

As $u : Y \longrightarrow \mathcal{U}$ generates $\mathcal{U}$, each $h' : \mathcal{U} \longrightarrow G_{\mathcal{A}}$ fulfilling $h' \circ u = \varphi_{\mathcal{A}}$ coincides with h. Hence h is unique.

($\Longrightarrow$) Assume that $u : Y \dashrightarrow \mathcal{U}$ is universal for **Mr**. Let $0 \to \mathcal{L} \to \mathcal{U} \to \mathcal{A} \to 0$ be the canonical decomposition of $\mathcal{U}$, and let $\mathcal{F}$ be the image of the induced transformation $\mathcal{L}^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$. For $\lambda \in \mathcal{L}^\vee$ the uniqueness of the homomorphism $h_\lambda : \mathcal{U} \longrightarrow \lambda_* \mathcal{U}$ fulfilling $u_\lambda = h_\lambda \circ u$ implies that the rational maps $u_\lambda : Y \dashrightarrow \lambda_* \mathcal{U}$ are non-isomorphic to each other for distinct $\lambda \in \mathcal{L}^\vee$. Hence $\operatorname{div}_{\mathbb{G}}(u_{Y,\lambda}) \neq \operatorname{div}_{\mathbb{G}}(u_{Y,\lambda'})$ for $\lambda \neq \lambda' \in \mathcal{L}^\vee$. Therefore $\mathcal{L}^\vee \longrightarrow \mathcal{F}$ is injective, hence an isomorphism.

Let $\varphi : Y \dashrightarrow G$ be an object of **Mr** and $0 \to L \to G \to A \to 0$ be the canonical decomposition of G. Translating φ by a constant $g \in G(k)$, if necessary, we may assume that $\varphi : Y \dashrightarrow G$ factorizes through a unique homomorphism $h : \mathcal{U} \longrightarrow G$. The restriction of h to $\mathcal{L}$ gives a homomorphism of linear groups $l : \mathcal{L} \longrightarrow L$. Then the dual homomorphism $l^\vee : L^\vee \longrightarrow \mathcal{F}$ yields a factorization of $L^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$ through $\mathcal{F}$. Thus **Mr** is a subcategory of $\mathbf{Mr}_{\mathcal{F}}$. Now the properties ($\diamondsuit$ $1-3$) guarantee that $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ contains the equivalence classes of all rational maps which induce a transformation to $\mathcal{F}$, hence $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}$ is equivalent to $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}_{\mathcal{F}}$. ■

Notation 2.40 *The universal object for a category* **Mr** *of rational maps from Y to algebraic groups, if it exists, is denoted by* $\mathrm{alb}_{\mathbf{Mr}} : Y \dashrightarrow \mathrm{Alb}_{\mathbf{Mr}}(Y)$.
If $\mathcal{F}$ is a formal group in $\underline{\mathrm{Div}}_Y^0$, then the universal object for $\mathbf{Mr}_{\mathcal{F}}$ is also denoted by $\mathrm{alb}_{\mathcal{F}} : Y \dashrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)$.

Remark 2.41 *In the proof of Theorem 2.39 we have seen that* $\mathrm{Alb}_{\mathcal{F}}(Y)$ *is an extension of the abelian variety* $\mathrm{Alb}(Y)$ *by the linear group $\mathcal{F}^\vee$, and the rational map* $(\mathrm{alb}_{\mathcal{F}} : Y \dashrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)) \in \mathbf{Mr}_{\mathcal{F}}$ *is characterized by the fact that the transformation* $\tau_{\mathrm{alb}_{\mathcal{F}}} : L^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$ *is the identity* $\mathrm{id} : \mathcal{F} \longrightarrow \mathcal{F}$.
More precisely, $[0 \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)]$ *is the dual 1-motive of* $\left[\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0\right]$.

Example 2.42 *The universal object* $\mathrm{alb} : Y \longrightarrow \mathrm{Alb}(Y)$ *for* **Mav** *from Definition 2.27 is the* classical Albanese mapping *and* $\mathrm{Alb}(Y)$ *the* classical Albanese variety *of a regular projective variety Y.*

Example 2.43 *The universal object* $\mathrm{alb}_0 : Y \dashrightarrow \mathrm{Alb}_0(Y)$ *for* $\mathbf{Mr}_0$ *from Example 2.30 coincides with the classical Albanese mapping to the classical Albanese variety by Theorem 2.39, since* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mr}_0$ *is equivalent to* $\mathrm{H}_{\mathrm{I}}^{-1}\,\mathbf{Mav}$.

Example 2.44 *The universal object for the category* $\mathbf{Mr}_0^X$ *from Example 2.32 is a quotient of the classical Albanese* $\mathrm{Alb}\left(\widetilde{X}\right)$ *of a projective resolution of singularities $\widetilde{X}$ for X, as* $\mathbf{Mr}_0^X$ *is a subcategory of* $\mathbf{Mr}_0$ *for $Y = \widetilde{X}$. It is the universal object for the category of morphisms from X to abelian varieties and coincides with the* universal morphism for the variety X and for the category of abelian varieties *in the sense of [S1].*

Example 2.45 *The universal object* $\mathrm{alb}_{\mathcal{F}_D} : Y \dashrightarrow \mathrm{Alb}_{\mathcal{F}_D}(Y)$ *for* $\mathbf{Mr}_{\mathcal{F}_D}$ *from Example 2.33 is the* generalized Albanese of Serre *(see [S2]).*

Example 2.46 *The universal object* $\mathrm{alb}_{\mathcal{F}_\mathfrak{d}} : C \dashrightarrow \mathrm{Alb}_{\mathcal{F}_\mathfrak{d}}(Y)$ *for* $\mathbf{Mr}_{\mathcal{F}_\mathfrak{d}}$ *from Example 2.34 is Rosenlicht's* generalized Jacobian $J_\mathfrak{d}$ *to the modulus* $\mathfrak{d}$ *(see [S3]).*

Example 2.47 *The universal object* $\mathrm{alb}_{\mathcal{F}_W} : Y \dashrightarrow \mathrm{Alb}_{\mathcal{F}_W}(Y)$ *for* $\mathbf{Mr}_{\mathcal{F}_W}$ *from Example 2.35 is the* generalized Albanese of Faltings/Wüstholz *(see [FW, VI.2.])*

Example 2.48 *The universal object* $\mathrm{alb}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}} : X_{\mathrm{reg}} \dashrightarrow \mathrm{Alb}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}(\widetilde{X})$ *from Example 2.36 is the* universal regular quotient *of the Chow group of points* $\mathrm{CH}_0(X)^0$ *(see [ESV]). In the following we will simply denote it by* $\mathrm{Alb}(X)$*. This is consistent, since in the case that* X *is regular it coincides with the classical Albanese variety.*

Remark 2.49 *Also the generalized Albanese of Serre (Example 2.45) and the generalized Jacobian (Example 2.46) can be interpreted as special cases of the universal regular quotient (Example 2.48) by constructing an appropriate singular variety* X*.*

Functoriality

The Question is whether a morphism of regular projective varieties induces a homomorphism of algebraic groups between universal objects.

Proposition 2.50 *Let* $\sigma : V \longrightarrow Y$ *be a morphism of regular projective varieties. Let* $\mathbf{Vr}$ *and* $\mathbf{Yr}$ *be categories of rational maps from* V *and* Y *respectively to algebraic groups, and suppose there exist universal objects* $\mathrm{Alb}_{\mathbf{Vr}}(V)$ *and* $\mathrm{Alb}_{\mathbf{Yr}}(Y)$ *for* $\mathbf{Vr}$ *and* $\mathbf{Yr}$ *respectively. The universal property of* $\mathrm{Alb}_{\mathbf{Vr}}(V)$ *yields:*
If the composition $\mathrm{alb}_{\mathbf{Yr}} \circ \sigma : V \dashrightarrow \mathrm{Alb}_{\mathbf{Yr}}(Y)$ *is an object of* $\mathbf{Vr}$*, then* σ *induces a homomorphism of algebraic groups*

$$\mathrm{Alb}^{\mathbf{Yr}}_{\mathbf{Vr}}(\sigma) : \mathrm{Alb}_{\mathbf{Vr}}(V) \longrightarrow \mathrm{Alb}_{\mathbf{Yr}}(Y) .$$

Theorem 2.39 allows to give a more explicit description:

Proposition 2.51 *Let* $\sigma : V \longrightarrow Y$ *be a morphism of regular projective varieties. Let* $\mathcal{F} \subset \underline{\mathrm{Div}}^0_Y$ *be a formal group s.t.* $\mathrm{Supp}(\mathcal{F})$ *does not contain any component of* $\sigma(V)$*. Let* $\_ \cdot V : \underline{\mathrm{Dec}}_{Y,V} \longrightarrow \underline{\mathrm{Div}}_V$ *denote the pull-back of Cartier divisors from* Y *to* V *(Definition 2.11, Proposition 2.12).*
For each formal group $\mathcal{G} \subset \underline{\mathrm{Div}}^0_V$ *satisfying* $\mathcal{G} \supset \mathcal{F} \cdot V$*, the pull-back of relative Cartier divisors and of line bundles induces a transformation of 1-motives*

$$\begin{bmatrix} \mathcal{G} \\ \downarrow \\ \underline{\mathrm{Pic}}^0_V \end{bmatrix} \longleftarrow \begin{bmatrix} \mathcal{F} \\ \downarrow \\ \underline{\mathrm{Pic}}^0_Y \end{bmatrix}$$

Remembering the construction of the universal objects (Remark 2.41), dualization of 1-motives translates Proposition 2.51 into the following reformulation of Proposition 2.50:

Proposition 2.52 *Let $\sigma : V \longrightarrow Y$ be a morphism of regular projective varieties. Let $\mathcal{F} \subset \underline{\mathrm{Div}}_Y^0$ be a formal group s.t.* $\mathrm{Supp}(\mathcal{F})$ *does not contain any component of $\sigma(V)$. Then σ induces a homomorphism of algebraic groups*

$$\mathrm{Alb}_{\mathcal{G}}^{\mathcal{F}}(\sigma) : \mathrm{Alb}_{\mathcal{G}}(V) \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)$$

for each formal group $\mathcal{G} \subset \underline{\mathrm{Div}}_V^0$ satisfying $\mathcal{G} \supset \mathcal{F} \cdot V$.

3 Rational Maps Factoring through $\mathrm{CH}_0(X)^0$

Throughout this section let X be a projective variety over k (an algebraically closed field of characteristic 0) and $\pi : Y \longrightarrow X$ a projective resolution of singularities. Let $U \subset Y$ be an open dense subset of Y where π is an isomorphism. U is identified with its image in X, and we suppose $U \subset X_{\mathrm{reg}}$. We consider the category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ of morphisms $\varphi : U \longrightarrow G$ from U to algebraic groups G factoring through $\mathrm{CH}_0(X)^0$ (Definition 3.27), where we assume algebraic groups G always to be connected, unless stated otherwise.

The goal of this section is to show that the category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ is equivalent to the category $\mathbf{Mr}_{\underline{\mathrm{Div}}_{Y/X}^0}$ of rational maps which induce a transformation of formal groups to $\underline{\mathrm{Div}}_{Y/X}^0$, which is defined as the kernel in $\underline{\mathrm{Div}}_Y^0$ of a kind of push-forward π_* of relative Cartier divisors (see Propositions 3.23 and 3.24).

3.1 Chow Group of Points

In this subsection the Chow group $\mathrm{CH}_0(X)^0$ of 0-cycles of degree 0 modulo rational equivalence is presented, quite similar as in [LW], see also [ESV], [BiS].

Definition 3.1 *A* Cartier curve *in X, relative to $X \setminus U$, is a curve $C \subset X$ satisfying*

(a) C is pure of dimension 1.
(b) No component of C is contained in $X \setminus U$.
(c) If $p \in C \setminus U$, the ideal of C in $\mathcal{O}_{X,p}$ is generated by a regular sequence.

Definition 3.2 *Let C be a Cartier curve in X relative to $X \setminus U$,* $\mathrm{Cp}(C)$ *the set of irreducible components of C and γ_Z the generic points of $Z \in \mathrm{Cp}(C)$. Let $\mathcal{O}_{C,\Theta}$ be the semilocal ring on C at $\Theta = (C \setminus U) \cup \{\gamma_Z \mid Z \in \mathrm{Cp}(C)\}$. Define*

$$\mathrm{K}(C,U)^* = \mathcal{O}_{C,\Theta}^* \,.$$

Definition 3.3 *Let C be a Cartier curve in X relative to $X \setminus U$ and $\nu : \widetilde{C} \longrightarrow C$ its normalization. For $f \in \mathrm{K}(C,U)^*$ and $p \in C$ let*

$$\mathrm{ord}_p(f) = \sum_{\widetilde{p} \to p} \mathrm{v}_{\widetilde{p}}\left(\widetilde{f}\right)$$

where $\widetilde{f} := \nu^{\#} f \in \mathcal{K}_{\widetilde{C}}$ and $\mathrm{v}_{\widetilde{p}}$ is the discrete valuation attached to the point $\widetilde{p} \in \widetilde{C}$ above $p \in C$ (cf. [Fu, Example A.3.1]).
Define the divisor of f to be

$$\mathrm{div}(f)_C = \sum_{p \in C} \mathrm{ord}_p(f)\ [p] \,.$$

Definition 3.4 *Let* $\mathrm{Z}_0(U)$ *be the group of 0-cycles on* U*, set*

$$\mathfrak{R}_0(X,U) = \left\{ (C,f) \;\middle|\; \begin{array}{c} C \text{ is a Cartier curve in } X \text{ relative to } X \setminus U \\ \text{and } f \in \mathrm{K}(C,U)^* \end{array} \right\}$$

and let $\mathrm{R}_0(X,U)$ *be the subgroup of* $\mathrm{Z}_0(U)$ *generated by the elements* $\operatorname{div}(f)_C$ *with* $(C,f) \in \mathfrak{R}_0(X,U)$*. Then define*

$$\mathrm{CH}_0(X) = \mathrm{Z}_0(U)/\,\mathrm{R}_0(X,U)\ .$$

Let $\mathrm{CH}_0(X)^0$ *be the subgroup of* $\mathrm{CH}_0(X)$ *of cycles* ζ *with* $\deg \zeta|_W = 0$ *for all irreducible components* $W \in \mathrm{Cp}(U)$ *of* U*.*

Remark 3.5 *The definition of* $\mathrm{CH}_0(X)$ *and* $\mathrm{CH}_0(X)^0$ *is independent of the choice of the dense open subscheme* $U \subset X_{\mathrm{reg}}$ *(see [ESV, Corollary 1.4]).*

Remark 3.6 *Note that by our terminology a curve is always reduced, in particular a Cartier curve. In the literature, e.g. [ESV], [LW], a slightly different definition of* Cartier curve *seems to be common, which allows non-reduced Cartier curves. Actually this does not change the groups* $\mathrm{CH}_0(X)$ *and* $\mathrm{CH}_0(X)^0$*, see [ESV, Lemma 1.3] for more explanation.*

3.2 Local Symbols

The description of rational maps factoring through $\mathrm{CH}_0(X)^0$ requires the notion of a *local symbol* as in [S3, Chapter III, §1].

Let C be a smooth projective curve over k. The composition law of an unspecified algebraic group G is written additively in this subsection.

Definition 3.7 *For an effective divisor* $\mathfrak{d} = \sum n_p\, p$ *on* C*, a subset* $S \subset C$ *and rational functions* $f, g \in \mathcal{K}_C$ *define*

$$\begin{aligned} f \equiv g \mod \mathfrak{d} \text{ at } S \quad &:\Longleftrightarrow \quad \mathrm{v}_p(f-g) \geq n_p \qquad \forall p \in S \cap \operatorname{Supp}(\mathfrak{d})\ , \\ f \equiv g \mod \mathfrak{d} \quad &:\Longleftrightarrow \quad f \equiv g \mod \mathfrak{d} \text{ at } C\ . \end{aligned}$$

where v_p *is the valuation attached to the point* $p \in C$*.*

Let $\psi : C \dashrightarrow G$ be a rational map from C to an algebraic group G which is regular away from a finite subset S. The morphism $\psi : C \setminus S \longrightarrow G$ extends to a homomorphism from the group of 0-cycles $\mathrm{Z}_0(C \setminus S)$ to G by setting $\psi\left(\sum l_i\, c_i\right) := \sum l_i\, \psi(c_i)$ for $c_i \in C \setminus S$, $l_i \in \mathbb{Z}$.

Definition 3.8 *An effective divisor* $\mathfrak{d}$ *on* C *is said to be a* modulus *for* ψ *if* $\psi(\operatorname{div}(f)) = 0$ *for all* $f \in \mathcal{K}_C$ *with* $f \equiv 1 \mod \mathfrak{d}$*.*

Theorem 3.9 *Let* $\psi : C \dashrightarrow G$ *be a rational map from* C *to an algebraic group* G *and* S *the finite subset of* C *where* ψ *is not regular. Then* ψ *has a modulus supported on* S*.*

This theorem is proven in [S3, Chapter III, §2], using the following concept:

Definition 3.10 *Let $\mathfrak{d}$ be an effective divisor supported on $S \subset C$ and $\psi : C \dashrightarrow G$ a rational function from C to an algebraic group G, regular away from S. A* local symbol *associated to ψ and $\mathfrak{d}$ is a function*

$$(\psi, \_)_{\_} : \mathcal{K}_C^* \times C \longrightarrow G$$

which assigns to $f \in \mathcal{K}_C^*$ and $p \in C$ an element $(\psi, f)_p \in G$, satisfying the following conditions:

(a) $(\psi, fg)_p = (\psi, f)_p + (\psi, g)_p$,

(b) $(\psi, f)_c = \mathrm{v}_c(f)\ \psi(c)$ *if* $c \in C \setminus S$,

(c) $(\psi, f)_s = 0$ *if* $s \in S$ *and* $f \equiv 1 \mod \mathfrak{d}$ *at* s ,

(d) $\sum_{p \in C} (\psi, f)_p = 0$.

Proposition 3.11 *The rational map ψ has a modulus $\mathfrak{d}$ if and only if there exists a local symbol associated to ψ and $\mathfrak{d}$, and this symbol is then unique.*

Proof. [S3, Chapter III, No. 1, Proposition 1]. ∎

Theorem 3.9 in combination with Proposition 3.11 states for each rational map $\psi : C \dashrightarrow G$ the existence of a modulus $\mathfrak{d}$ for ψ and of a unique local symbol $(\psi, \_)_{\_}$ associated to ψ and $\mathfrak{d}$.

From the definitions it is clear that if $\mathfrak{d}$ is a modulus for ψ then $\mathfrak{e}$ is also for all $\mathfrak{e} \geq \mathfrak{d}$. Likewise a local symbol $(\psi, \_)_{\_}$ associated to ψ and $\mathfrak{d}$ is also associated to ψ and $\mathfrak{e}$ for all $\mathfrak{e} \geq \mathfrak{d}$.

Suppose we are given two moduli $\mathfrak{d}$ and $\mathfrak{d}'$ for ψ, and hence two local symbols $(\psi, \_)_{\_}$ and $(\psi, \_)'_{\_}$ associated to $\mathfrak{d}$ and $\mathfrak{d}'$ respectively. Then both local symbols are also associated to $\mathfrak{e} := \mathfrak{d} + \mathfrak{d}'$. The uniqueness of the local symbol associated to ψ and $\mathfrak{e}$ implies that $(\psi, \_)_{\_}$ and $(\psi, \_)'_{\_}$ coincide. It is therefore morally justified to speak about *the local symbol associated to ψ* (without mentioning a modulus), cf. [S3, Chapter III, No. 1, Remark of Proposition 1].

Corollary 3.12 *For each rational map $\psi : C \dashrightarrow G$ from C to an algebraic group G there exists a unique associated local symbol $(\psi, \_)_{\_} : \mathcal{K}_C^* \times C \longrightarrow G$. If $\mathfrak{d}$ is a modulus for ψ supported on S, then this local symbol is given by*

$$\begin{aligned} (\psi, f)_c &= \mathrm{v}_c(f)\ \psi(c) && \forall c \in C \setminus S \\ (\psi, f)_s &= -\sum_{c \notin S} \mathrm{v}_c(f_s)\ \psi(c) && \forall s \in S \end{aligned}$$

where $f_s \in \mathcal{K}_C^*$ is a rational function with $f_s \equiv 1 \mod \mathfrak{d}$ at z for all $z \in S \setminus s$ and $f/f_s \equiv 1 \mod \mathfrak{d}$ at s.

The above formula is shown in [S3, Chapter III, No. 1], in the proof of Proposition 1.

Example 3.13 *In the case that G is the multiplicative group $\mathbb{G}_{\mathrm{m}}$, a rational map $\psi : C \dashrightarrow \mathbb{G}_{\mathrm{m}}$ can be identified with a rational function in $\mathcal{K}_C$, and S is the set of zeros and poles of ψ, i.e. $S = \mathrm{Supp}(\mathrm{div}(\psi))$. Then the local symbol associated to ψ is given by*

$$(\psi, f)_p = (-1)^{mn} \frac{\psi^m}{f^n}(p) \qquad \text{with } m = \mathrm{v}_p(f),\ n = \mathrm{v}_p(\psi) \ .$$

(See [S3, Chapter III, No. 4, Proposition 6].)

Example 3.14 *In the case that G is the additive group $\mathbb{G}_a$, a rational map $\psi : C \dashrightarrow \mathbb{G}_a$ can be identified with a rational function in $\mathcal{K}_C$, and S is the set of poles of ψ. Then the local symbol associated to ψ is given by*

$$(\psi, f)_p = \mathrm{Res}_p (\psi \, \mathrm{d}f/f) \ .$$

(See [S3, Chapter III, No. 3, Proposition 5].)

Proposition 3.15 *Let $\varphi, \psi : C \dashrightarrow G$ be two rational maps from C to an algebraic group G, with associated local symbols $(\varphi, \_)_{\_}$ and $(\psi, \_)_{\_}$. Then the local symbol $(\varphi + \psi, \_)_{\_}$ associated to the rational map $\varphi + \psi : C \dashrightarrow G$, $c \longmapsto \varphi(c) + \psi(c)$ is given by*

$$(\varphi + \psi, f)_p = (\varphi, f)_p + (\psi, f)_p \ .$$

Proof. Let $\mathfrak{d}_\varphi$ be a modulus for φ and $\mathfrak{d}_\psi$ one for ψ. Then both maps φ, ψ and the map $\varphi + \psi$ have $\mathfrak{d}_{\varphi+\psi} := \mathfrak{d}_\varphi + \mathfrak{d}_\psi$ as a modulus and both local symbols $(\varphi, \_)_{\_}$ and $(\psi, \_)_{\_}$ are associated to $\mathfrak{d}_{\varphi+\psi}$. Now the formula in Corollary 3.12 and the distributive law imply the assertion. ■

Lemma 3.16 *Let $\psi : C \dashrightarrow G$ be a rational map from C to an algebraic group G which is an L-bundle over an abelian variety A, i.e. $G \in \mathrm{Ext}(A, L)$, where L is a linear group. Let $p \in C$ be a point, $U \ni p$ a neighbourhood and $\Phi : U \times L \xrightarrow{\sim} G_U$, $(u, l) \longmapsto \phi(u) + l$ a local trivialization of the induced L-bundle $G_C = G \times_A C$ over C, i.e. $\phi : U \longrightarrow G_C$ a local section. Moreover let $[\psi]_\Phi : C \dashrightarrow L$, $c \longmapsto \psi(c) - \phi(c)$ be the rational map ψ considered in the local trivialization Φ. Then for each rational function $f \in \mathcal{O}^*_{C,p}$ it holds*

$$(\psi, f)_p = ([\psi]_\Phi, f)_p \ .$$

Proof. Proposition 3.15 yields

$$([\psi]_\Phi, f)_p = (\psi - \phi, f)_p = (\psi, f)_p - (\phi, f)_p \ .$$

ϕ is regular at p, therefore we have $(\phi, f)_p = \mathrm{v}_p(f) \cdot \phi(p)$. Since f is a unit at p, it holds $\mathrm{v}_p(f) = 0$. Thus $(\phi, f)_p = 0$ and hence $([\psi]_\Phi, f)_p = (\psi, f)_p$. ■

3.3 Formal Infinitesimal Divisors

For a k-scheme Y the functor of relative Cartier divisors $\underline{\mathrm{Div}}_Y$ admits a pull-back, but not a push-forward. Supposed Y is a normal scheme, the group of Cartier divisors $\mathrm{Div}(Y)$ on Y can be identified with the group of locally principal Weil divisors, and there is a push-forward of Weil divisors.

We are looking for a concept of infinitesimal divisors $\mathrm{LDiv}(Y)$ which admits a push-forward and a transformation $\mathrm{Lie}(\underline{\mathrm{Div}}_Y) \longrightarrow \mathrm{LDiv}(Y)$. In this subsection we consider the case that Y is a curve Z.

Functor of Formal Infinitesimal Divisors

Let Z be a curve over k.

Definition 3.17 *Define the k-vector space of* formal infinitesimal divisors *on Z by*

$$\mathrm{LDiv}(Z) = \bigoplus_{q \in Z(k)} \mathrm{Hom}_k^{\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Z,q}, k\right)$$

where $\mathrm{Hom}_k^{\mathrm{cont}}$ denotes the set of continuous k-linear maps. $\widehat{\mathfrak{m}}_{Z,q}$ carries the $\widehat{\mathfrak{m}}_{Z,q}$-adic topology, while k is endowed with the discrete topology.

Proposition 3.18 *Let $\pi : Z \longrightarrow C$ be a finite morphism of curves over k. Then π induces a* push-forward *of formal infinitesimal divisors*

$$\pi_* : \mathrm{LDiv}(Z) \longrightarrow \mathrm{LDiv}(C)$$

induced by the homomorphisms

$$\mathrm{Hom}_k^{\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Z,q}, k\right) \longrightarrow \mathrm{Hom}_k^{\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C,\pi(q)}, k\right), \quad h \longmapsto h \circ \widehat{\pi^{\#}}$$

where $q \in Z(k)$ and $\widehat{\pi^{\#}} : \widehat{\mathcal{O}}_{C,\pi(q)} \longrightarrow \widehat{\mathcal{O}}_{Z,q}$ is the homomorphism of completed structure sheaves associated to π.

Proposition 3.19 *Let Z be a normal curve over k. Then there is an isomorphism of k-vector spaces*

$$\mathrm{fml} : \mathrm{Lie}\left(\underline{\mathrm{Div}}_Z\right) \longrightarrow \mathrm{LDiv}(Z)\ .$$

Proof. We construct the isomorphism fml via factorization, i.e. give isomorphisms

$$\Gamma\left(\mathcal{K}_Z / \mathcal{O}_Z\right) \xrightarrow{\sim} \bigoplus_{q \in Z(k)} \mathcal{K}_{Z,q} / \mathcal{O}_{Z,q} \xrightarrow{\sim} \bigoplus_{q \in Z(k)} \mathrm{Hom}_k^{\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Z,q}, k\right)\ .$$

The first of these two maps is given by the natural k-linear map

$$\Gamma\left(\mathcal{K}_Z / \mathcal{O}_Z\right) \longrightarrow \bigoplus_{q \in Z(k)} \mathcal{K}_{Z,q} / \mathcal{O}_{Z,q}, \quad \delta \longmapsto \sum_{q \in Z(k)} [\delta]_q$$

which is an isomorphism by the Approximation Lemma (see [S4, Part One, Chapter I, §3]). As Z is normal, each local ring is regular. Since $\mathcal{K}_{Z,q} / \mathcal{O}_{Z,q} = \bigcup_{\nu > 0} t_q^{-\nu}\, \widehat{\mathcal{O}}_{Z,q} \Big/ \widehat{\mathcal{O}}_{Z,q}$ for a local parameter t_q of the maximal ideal $\mathfrak{m}_q \subset \mathcal{O}_{Z,q}$, Lemma 3.20 below yields a canonical isomorphism of k-vector spaces

$$\mathcal{K}_{Z,q} / \mathcal{O}_{Z,q} \xrightarrow{\sim} \mathrm{Hom}_k^{\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Z,q}, k\right), \quad [f] \longmapsto \mathrm{Res}_q\left(f \cdot \mathrm{d}\_\right)\ .$$

Then the isomorphism fml is obtained by composition. ∎

Lemma 3.20 *Let $(\mathcal{A}, \mathfrak{m})$ be a complete local k-algebra, endowed with the $\mathfrak{m}$-adic topology, while k carries the discrete topology. Let $l \in \mathrm{Hom}_k(\mathfrak{m}, k)$ be a k-linear map. Then the following conditions are equivalent:*

(i) l *is continuous,*
(ii) $\ker(l)$ *is open,*
(iii) $\ker(l) \supset \mathfrak{m}^\nu$ *for some* $\nu > 0$,
(iv) $l \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^\nu, k)$ *for some* $\nu > 0$.

If furthermore $\mathcal{A}$ is a discrete valuation ring, this is equivalent to

(v) $l = \mathrm{Res}\left(f \cdot \mathrm{d}\_\right) : g \longmapsto \mathrm{Res}\left(f \cdot \mathrm{d}g\right)$ *for some* $f \in t^{-\nu}\mathcal{A}/\mathcal{A}$, $\nu \geq 0$ *where t is a local parameter of $\mathfrak{m}$, $\mathcal{K} = \mathrm{Q}(\mathcal{A})$ the quotient field, $\mathrm{Res} : \Omega_{\mathcal{K}/k} \longrightarrow k$ the residue and $\mathrm{d} : \mathcal{A} \longrightarrow \Omega_{\mathcal{A}/k}$ the universal derivation.*

Proof. (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii) is folklore of rings with $\mathfrak{m}$-adic topology.
(iii)$\Longleftrightarrow$(iv) $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^\nu, k) = \ker\big(\mathrm{Hom}_k(\mathfrak{m}, k) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}^\nu, k)\big)$.
(iv)$\Longleftrightarrow$(v) If $\mathcal{A}$ is regular and $t \in \mathfrak{m}$ a local parameter, then we may identify $\mathcal{A} \cong k[[t]]$ and $\mathcal{K} \cong k((t))$. The residue over k is defined as

$$\mathrm{Res}: \ \Omega_{\mathcal{K}/k} \longrightarrow k, \qquad \sum_{\nu \gg -\infty} a_\nu t^\nu \, \mathrm{d}t \longmapsto a_{-1}$$

and the definition is independent of the choice of local parameter, (see [S3, Chapter II, No. 7, Proposition 5]).

$\mathrm{d}: \mathcal{A} \longrightarrow \Omega_{\mathcal{A}/k}$ and $\mathrm{Res}: \Omega_{\mathcal{K}/k} \longrightarrow k$ are both k-linear maps. Since $\mathrm{Res}(\omega) = 0$ for all $\omega \in \Omega_{\mathcal{A}/k}$, the expression $\mathrm{Res}(f\,\mathrm{d}g)$ is well defined for $g \in \mathfrak{m}/\mathfrak{m}^{\nu+1}$ and $f \in t^{-\nu}\mathcal{A}/\mathcal{A}$.

The pairing $\quad t^{-\nu}\mathcal{A}/\mathcal{A} \times \mathfrak{m}/\mathfrak{m}^{\nu+1} \longrightarrow k, \quad (f,g) \longmapsto \mathrm{Res}(f\,\mathrm{d}g) \quad$ is a perfect pairing, hence $t^{-\nu}\mathcal{A}/\mathcal{A} \xrightarrow{\sim} \mathrm{Hom}_k\big(\mathfrak{m}/\mathfrak{m}^{\nu+1}, k\big), \quad f \longmapsto \mathrm{Res}(f\,\mathrm{d}\_)$ is an isomorphism. ■

Lemma 3.21 *Let C be a projective curve over a field k, and let $\pi: Z \longrightarrow C$ be its normalization. Then the kernel of the composition $\pi_* \circ \mathrm{fml}$*

$$\ker\left(\mathrm{Lie}\,(\underline{\mathrm{Div}}_Z) \xrightarrow{\mathrm{fml}} \mathrm{LDiv}(Z) \xrightarrow{\pi_*} \mathrm{LDiv}(C)\right)$$

is a finite dimensional k-vector space. More precisely, if S denotes the inverse image in Z of the singular locus of C, for each $q \in S$ there is an integer $n_q \geq 0$ such that $\bigoplus_{q \to p} \widehat{\mathfrak{m}}_{Z,q}^{n_q+1} \subset \widehat{\mathfrak{m}}_{C,p}$. Then

$$\ker(\pi_* \circ \mathrm{fml}) \ \subset \ \Gamma\left(\mathcal{O}_Z\left(\sum_{q \in S} n_q\, q\right) \Big/ \mathcal{O}_Z\right)$$

Proof. Since the normalization is birational, the set S of k-rational points $q \in Z$ such that $(\mathcal{O}_{Z,q}, \mathfrak{m}_{Z,q}) \neq \big(\mathcal{O}_{C,\pi(q)}, \mathfrak{m}_{C,\pi(q)}\big)$ is finite. As $\pi_*\mathcal{O}_Z/\mathcal{O}_C = \prod_{p \in \pi(S)} \mathcal{O}_{Z,p}/\mathcal{O}_{C,p}$ is a coherent sheaf, $\mathcal{O}_{Z,p}/\mathcal{O}_{C,p}$ is finite dimensional for each $p \in \pi(S)$, hence compatible with completion. Thus $\left(\bigoplus_{q \to p} \widehat{\mathfrak{m}}_{Z,q}\right) \Big/ \widehat{\mathfrak{m}}_{C,p} \subset \widehat{\mathcal{O}}_{Z,p}/\widehat{\mathcal{O}}_{C,\pi(p)} = \mathcal{O}_{Z,p}/\mathcal{O}_{C,\pi(p)}$ is also finite dimensional. We obtain

$$\begin{aligned}
& \ker\Big(\mathrm{LDiv}(Z) \longrightarrow \mathrm{LDiv}(C)\Big) \\
= \ & \ker\left(\bigoplus_{q \in Z(k)} \mathrm{Hom}_k^{\mathrm{cont}}\Big(\widehat{\mathfrak{m}}_{Z,q}, k\Big) \longrightarrow \bigoplus_{p \in C(k)} \mathrm{Hom}_k^{\mathrm{cont}}\Big(\widehat{\mathfrak{m}}_{C,p}, k\Big)\right) \\
= \ & \bigoplus_{p \in C(k)} \ker\left(\bigoplus_{q \to p} \mathrm{Hom}_k^{\mathrm{cont}}\Big(\widehat{\mathfrak{m}}_{Z,q}, k\Big) \longrightarrow \mathrm{Hom}_k^{\mathrm{cont}}\Big(\widehat{\mathfrak{m}}_{C,p}, k\Big)\right) \\
= \ & \bigoplus_{p \in C(k)} \ker\left(\mathrm{Hom}_k^{\mathrm{cont}}\left(\bigoplus_{q \to p} \widehat{\mathfrak{m}}_{Z,q}, k\right) \longrightarrow \mathrm{Hom}_k^{\mathrm{cont}}\Big(\widehat{\mathfrak{m}}_{C,p}, k\Big)\right) \\
= \ & \bigoplus_{p \in \pi(S)} \mathrm{Hom}_k\left(\left(\bigoplus_{q \to p} \widehat{\mathfrak{m}}_{Z,q}\right) \Big/ \widehat{\mathfrak{m}}_{C,p} \ , \ k\right)
\end{aligned}$$

is finite dimensional. Since $\mathrm{fml} : \mathrm{Lie}\,(\underline{\mathrm{Div}}_Z) \longrightarrow \mathrm{LDiv}(Z)$ is injective by Proposition 3.19, it follows that $\ker(\pi_* \circ \mathrm{fml})$ is finite dimensional.

The finiteness of the dimension of $\left(\bigoplus_{q \to p} \widehat{\mathfrak{m}}_{Z,q}\right) \Big/ \widehat{\mathfrak{m}}_{C,p}$ implies that for each $q \in S$ there is an integer $n_q \geq 0$ such that $\bigoplus_{q \to p} \widehat{\mathfrak{m}}_{Z,q}^{n_q+1} \subset \widehat{\mathfrak{m}}_{C,p}$. Then

$$\ker\big(\mathrm{LDiv}(Z) \longrightarrow \mathrm{LDiv}(C)\big) \quad \subset \quad \bigoplus_{q \in S} \mathrm{Hom}_k \left(\widehat{\mathfrak{m}}_{Z,q} \Big/ (\widehat{\mathfrak{m}}_{Z,q})^{n_q+1}\,, k\right) .$$

If t_q is a local parameter of $\widehat{\mathfrak{m}}_{Z,q}$, Lemma 3.20 (iv)$\Longleftrightarrow$(v) yields

$$\mathrm{Hom}_k \left(\widehat{\mathfrak{m}}_{Z,q} \Big/ (\widehat{\mathfrak{m}}_{Z,q})^{n_q+1}\,, k\right) \quad \cong \quad \bigoplus_{q \in S} t_q^{-n_q} \widehat{\mathcal{O}}_{Z,q} / \widehat{\mathcal{O}}_{Z,q}$$

Then

$$\begin{aligned} \ker(\pi_* \circ \mathrm{fml}) \quad &\subset \quad \mathrm{fml}^{-1} \left(\bigoplus_{q \in S} t_q^{-n_q} \widehat{\mathcal{O}}_{Z,q} / \widehat{\mathcal{O}}_{Z,q} \right) \\ &= \quad \Gamma \left(\mathcal{O}_Z \left(\sum_{q \in S} n_q\, q \right) \Big/ \mathcal{O}_Z \right) . \end{aligned}$$

■

3.4 The Functor $\underline{\mathrm{Div}}^0_{Y/X}$

The idea about $\underline{\mathrm{Div}}^0_{Y/X}$ is to define a functor which admits a natural transformation to the Picard functor $\underline{\mathrm{Pic}}^0_Y$ and measures the difference between the schemes Y and X, where $\pi : Y \longrightarrow X$ is a projective resolution of singularities. Roughly speaking, $\underline{\mathrm{Div}}^0_{Y/X}$ is a subfunctor of $\underline{\mathrm{Div}}^0_Y$ which lies in the kernel of some kind of push-forward π_*.

Definition 3.22 *For a k-scheme Y denote by* $\mathrm{WDiv}(Y)$ *the abelian group of Weil divisors on Y. Write* $\mathrm{weil} : \mathrm{Div}(Y) \longrightarrow \mathrm{WDiv}(Y)$ *for the homomorphism which maps a Cartier divisor to its associated Weil divisor, as defined in [Fu, 2.1].*

Proposition 3.23 *Let C be a projective curve over k, and let $\pi : Z \longrightarrow C$ be its normalization. Then there is a subfunctor $\underline{\mathrm{Div}}^0_{Z/C}$ of $\underline{\mathrm{Div}}^0_Z$, represented by a formal group, characterized by the following conditions:*

$$\begin{aligned} \underline{\mathrm{Div}}^0_{Z/C}(k) \quad &= \quad \ker\left(\underline{\mathrm{Div}}^0_Z(k) \xrightarrow{\mathrm{weil}} \mathrm{WDiv}(Z) \xrightarrow{\pi_*} \mathrm{WDiv}(C)\right) \\ \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Z/C}\right) \quad &= \quad \ker\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Z\right) \xrightarrow{\mathrm{fml}} \mathrm{LDiv}(Z) \xrightarrow{\pi_*} \mathrm{LDiv}(C)\right) . \end{aligned}$$

Proof. A formal group in characteristic 0 is determined by its k-valued points and its Lie-algebra (Corollary 1.7). Then the conditions on $\underline{\mathrm{Div}}^0_{Z/C}$ determine uniquely a subfunctor of $\underline{\mathrm{Div}}_Z$ (cf. Proposition 2.13). Thus it suffices to show that $\underline{\mathrm{Div}}^0_{Z/C}(k)$ is a free abelian group of finite rank and $\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Z/C}\right)$ is a k-vector space of finite dimension. The latter assertion was proven in Lemma

3.21. For the first assertion note that the normalization $\pi : Z \longrightarrow C$ is an isomorphism on the regular locus of C. As Z is normal, weil : $\underline{\mathrm{Div}}_Z(k) \longrightarrow \mathrm{WDiv}(Z)$ is an isomorphism. Then $\underline{\mathrm{Div}}^0_{Z/C}(k)$ is contained in the free abelian group generated by the preimages of the singular points of C, of which there exist only finitely many. $\underline{\mathrm{Div}}^0_{Z/C}(k)$ being a subgroup of a finitely generated free abelian group is also free abelian of finite rank. ■

Proposition 3.24 *Let X be a projective variety over k, and let $\pi : Y \longrightarrow X$ be a projective resolution of singularities. Let $\mathcal{F} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ be the functor*

$$\mathcal{F} \;=\; \bigcap_C \Big(\_ \cdot \widetilde{C} \Big)^{-1} \underline{\mathrm{Div}}^0_{\widetilde{C}/C}$$

where C ranges over all Cartier curves in X relative to the singular locus X_{sing} (Definition 3.1), $\widetilde{C}$ is the normalization of C and $\_ \cdot \widetilde{C} : \underline{\mathrm{Dec}}_{Y,\widetilde{C}} \longrightarrow \underline{\mathrm{Div}}_{\widetilde{C}}$ the pull-back of relative Cartier divisors from Y to $\widetilde{C}$ (Definition 2.11, Proposition 2.12).

Then there is a subfunctor $\underline{\mathrm{Div}}^0_{Y/X}$ of $\underline{\mathrm{Div}}^0_Y$, represented by a formal group, characterized by the conditions $\underline{\mathrm{Div}}^0_{Y/X}(k) = \mathcal{F}(k)$ and $\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right) = \mathrm{Lie}\,(\mathcal{F})$.

Remark 3.25 *Let $\delta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right) = \Gamma\left(\mathcal{K}_Y / \mathcal{O}_Y\right)$ be a deformation of the zero divisor in Y. Then δ determines an effective divisor by the poles of its local sections. Hence for each generic point η of height 1 in Y, with associated discrete valuation v_η, the expression $\mathrm{v}_\eta(\delta)$ is well defined and $\mathrm{v}_\eta(\delta) \leq 0$. Thus we obtain a homomorphism $\mathrm{v}_\eta : \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right) \longrightarrow \mathbb{Z}$.*

Proof of Prop. 3.24. As in the proof of Proposition 3.23 it suffices to show that $\underline{\mathrm{Div}}^0_{Y/X}(k)$ is a free abelian group of finite rank and $\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ is a k-vector space of finite dimension.

Let $D \in \underline{\mathrm{Div}}_Y(k)$ be a non-trivial divisor on Y whose support is not contained in the inverse image $S_Y = S \times_X Y$ of the singular locus $S = X_{\mathrm{sing}}$ of X. Then $\pi\big(\mathrm{Supp}(D)\big)$ on X is not contained in S. Let $\mathcal{L}$ be a very ample line bundle on X, consider the space $|\mathcal{L}|^{d-1}$, where $d = \dim X$, of complete intersection curves $C = H_1 \cap \ldots \cap H_{d-1}$ with $H_i \in |\mathcal{L}| = \mathbb{P}\big(\mathrm{H}^0(X, \mathcal{L})\big)$ for $i = 1, \ldots, d-1$. For Cartier curves C in $|\mathcal{L}|^{d-1}$ the following properties are open and dense:

(a) $C_Y = C \times_X Y$ is regular.

(b) C intersects $\pi\big(\mathrm{Supp}(D)\big) \cap X_{\mathrm{reg}}$ properly.

(c) $D \cdot C_Y$ is a non-trivial divisor on $C_Y \cap X_{\mathrm{reg}}$.

(a) is a consequence of the Bertini theorems, (b) is due to the fact that $\mathcal{L}$ is very ample and (c) follows from (b) and the fact that $\mathrm{Supp}(D)$ is locally a prime divisor at almost every $q \in Y$. Therefore there exists a Cartier curve C in X satisfying the conditions (a)-(c). Then the normalization $\nu : \widetilde{C} \longrightarrow C$ coincides with $\pi|_{C_Y}$ and hence is an isomorphism on $C_Y \cap X_{\mathrm{reg}}$. Thus $\nu_*\big(D \cdot \widetilde{C}\big) \neq 0$. This implies $D \notin \underline{\mathrm{Div}}^0_{Y/X}(k)$. Hence $\underline{\mathrm{Div}}^0_{Y/X}(k)$ is a subgroup of the free abelian group generated by the irreducible components of S_Y of codimension 1. As S_Y has only finitely many components, this group has finite rank. So $\underline{\mathrm{Div}}^0_{Y/X}(k)$ is a subgroup of a free abelian group of finite rank, hence is also free abelian of finite rank.

Now let $\delta \in \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)$ be a deformation of the trivial divisor on Y. The same argument as above shows that if $\delta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$, then $\mathrm{Supp}(\delta) \subset S_Y$.

If C is a Cartier curve in X relative to X_{sing}, we denote by C^Y the proper transform of C, i.e. the closure of $\pi^{-1}(C \cap X_{\mathrm{reg}})$ in Y. As $\pi|_{C^Y} : C^Y \longrightarrow C$ is a birational morphism, the normalization $\nu : \widetilde{C} \longrightarrow C$ factors through a morphism $\mu : \widetilde{C} \longrightarrow C^Y$. Given $\eta \in S_Y^{\mathrm{ht}=1}$, where $S_Y^{\mathrm{ht}=1}$ denotes the set of generic points of S_Y of height 1 in Y, let Z be a curve in Y which intersects the prime divisor E_η associated to η properly in a point p. As Y is regular, $\mathcal{O}_{Y,\eta}$ is a discrete valuation ring. Let v_η be the valuation at η, and let v_q be the valuation attached to a point $q \in \widetilde{C}$ above $p \in C_Y$. Since $\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}/C}\right)$ is finite dimensional, there exists a number $n_q \in \mathbb{N}$ such that $\mathrm{v}_q(\gamma) \geq -n_q$ for all $\gamma \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}/C}\right)$. The bound $-n_q$ depends only on the singularity of C at q. More precisely, n_q satisfies $\mathfrak{m}_{\widetilde{C},q}^{n_q+1} \subset \mathfrak{m}_{C,p}$ (cf. Lemma 3.21). The number n_q is related to the dimension of the affine part of Pic_C, see [BLR, Section 9.2, proof of Proposition 9]. If $\mathcal{L}$ is a sufficiently ample line bundle on X (i.e. a sufficiently high power of an ample line bundle on X), one finds a family $T \subset |\mathcal{L}|^{d-1}$ of Cartier curves C whose proper transforms C^Y intersect E_η properly in points p_C such that the set $\{p_C \mid C \in T\}$ contains an open dense subset U_η of E_η. By upper semi-continuity of $\dim \mathrm{Pic}_C$ for the curves C in $|\mathcal{L}|^{d-1}$, we may assume that the sets $\mathrm{v}_{q_C}\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}/C}\right)\right)$ for $q_C \in \mu^{-1}(p_C)$ admit a common bound $-n_\eta$ for all $C \in T$. Then for each $\delta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ with $\eta \in \mathrm{Supp}(\delta)$ there is a curve $C \in T$ with $\mathrm{v}_{q_C}\left(\delta \cdot \widetilde{C}\right) = \mathrm{v}_\eta(\delta)$, since this is an open dense property among the curves that intersect E_η. By definition, $\delta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ implies that $\delta \cdot \widetilde{C} \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{C}/C}\right)$. Then $\mathrm{v}_\eta(\delta) = \mathrm{v}_{q_C}\left(\delta \cdot \widetilde{C}\right) \geq -n_\eta$. We obtain $\min\left(\mathrm{v}_\eta\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)\right)\right) \geq -n_\eta$, i.e. the orders of poles of deformations in $\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ are bounded. Hence for all $\eta \in S_Y^{\mathrm{ht}=1}$ there exist n_η such that

$$\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{Y/X}\right) \subset \Gamma\left(\mathcal{O}_Y\left(\sum_{\eta \in S_Y^{\mathrm{ht}=1}} n_\eta\, E_\eta\right) \Big/ \mathcal{O}_Y\right) .$$

As Y is projective, the k-vector space on the right hand side is finite dimensional. ■

Remark 3.26 *The construction of $\underline{\mathrm{Div}}_{Y/X}$ involves an intersection ranging over all Cartier curves in X, which makes this object hard to grasp. In fact, once a formal subgroup $\mathcal{E}$ of $\underline{\mathrm{Div}}^0_Y$ containing $\underline{\mathrm{Div}}^0_{Y/X}$ is found, $\underline{\mathrm{Div}}^0_{Y/X}$ can be computed from one single curve: Let (for example) $\mathcal{E}$ be the formal group defined by $\mathcal{E}(k) = \{D \in \underline{\mathrm{Div}}^0_Y \mid \mathrm{Supp}(D) \subset S_Y\}$, $\mathrm{Lie}(\mathcal{E}) = \Gamma\left(\mathcal{O}_Y\left(\sum_{\eta \in S_Y^{\mathrm{ht}=1}} n_\eta\, E_\eta\right)/\mathcal{O}_Y\right)$, as in the proof of Proposition 3.24. Then there exists a Cartier curve C in X relative to X_{sing} such that*

$$\underline{\mathrm{Div}}^0_{Y/X} = \left(\_ \cdot \widetilde{C}\right)\Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}^0_{\widetilde{C}/C} .$$

Indication of Proof. For each Cartier curve C in X, define a subfunctor $\mathcal{F}_C$ of $\mathcal{E}$ by $\mathcal{F}_C := \left(\_ \cdot \widetilde{C}\right)\big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}^0_{\widetilde{C}/C}$. Then $\mathcal{F}_C$ is a formal group for each C, and it holds $\underline{\mathrm{Div}}^0_{Y/X} = \bigcap_C \mathcal{F}_C$. For each sequence $\{C_\nu\}$ of Cartier curves the formal groups $\mathcal{E}_0 := \mathcal{E}$, $\mathcal{E}_{\nu+1} := \mathcal{E}_\nu \cap \mathcal{F}_{C_\nu}$ form a descending chain

$$\mathcal{E}_0 \supset \mathcal{E}_1 \supset \ldots \supset \mathcal{E}_\nu \supset \ldots \supset \bigcap_C \mathcal{F}_C$$

of formal subgroups of $\mathcal{E}$. It is obvious that if $rD \in \underline{\mathrm{Div}}^0_{Y/X}(k)$ for some $D \in \underline{\mathrm{Div}}_Y(k)$ and $r \in \mathbb{Z} \setminus \{0\}$, then $D \in \underline{\mathrm{Div}}^0_{Y/X}(k)$. Therefore $\underline{\mathrm{Div}}^0_{Y/X}(k)$ is generated by a subset of a set of generators of $\mathcal{E}(k)$, and this is a finitely generated free abelian group. Moreover, Lie $(\mathcal{E})$ is a finite dimensional k-vector space. Hence the sequence $\{\mathcal{E}_\nu\}$ becomes stationary. Thus there is a finite set of Cartier curves $C_1, \dots, C_r$ such that $\underline{\mathrm{Div}}^0_{Y/X} = \mathcal{F}_{C_1} \cap \dots \cap \mathcal{F}_{C_r}$. Then each Cartier curve C containing $C_1, \dots, C_r$ gives the desired Cartier curve. ■

3.5 The Category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$

We keep the notation fixed at the beginning of Section 3: X is a projective variety, $\pi : Y \longrightarrow X$ a projective resolution of singularities and $U \subset X_{\mathrm{reg}}$ a dense open subset.

Definition 3.27 $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ *is a category of rational maps from X to algebraic groups defined as follows: The objects of* $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ *are morphisms $\varphi : U \longrightarrow G$ whose associated map on zero-cycles of degree zero* $\mathrm{Z}_0(U)^0 \longrightarrow G(k)$, $\sum l_i\, p_i \longmapsto \sum l_i\, \varphi(p_i)$ *factors through a homomorphism of groups* $\mathrm{CH}_0(X)^0 \longrightarrow G(k)$. [2]
We refer to the objects of $\mathbf{Mr}^{\mathrm{CH}_0(x)^0}$ *as rational maps from X to algebraic groups* factoring through rational equivalence *or* factoring through $\mathrm{CH}_0(X)^0$.

Theorem 3.28 *The category* $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ *of morphisms from U to algebraic groups factoring through* $\mathrm{CH}_0(X)^0$ *is equivalent to the category* $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{Y/X}}$ *of rational maps from Y to algebraic groups which induce a transformation of formal groups to* $\underline{\mathrm{Div}}^0_{Y/X}$ *(see Proposition 3.24 for the Definition of* $\underline{\mathrm{Div}}^0_{Y/X}$*).*

Proof. First notice that a rational map from Y to an algebraic group which induces a transformation to $\underline{\mathrm{Div}}^0_{Y/X}$ is necessarily regular on U, since all $D \in \underline{\mathrm{Div}}^0_{Y/X}$ have support only on $Y \setminus U$. Then according to Definition 3.4 and Definition 2.29 the task is to show that for a morphism $\varphi : U \longrightarrow G$ from U to an algebraic group G with canonical decomposition $0 \to L \to G \to A \to 0$ the following conditions are equivalent:

(i) $\varphi(\mathrm{div}(f)_C) = 0 \qquad \forall (C, f) \in \mathfrak{R}_0(X, U)$

(ii) $\mathrm{div}_{\mathbb{G}}(\varphi_{Y,\lambda}) \in \underline{\mathrm{Div}}^0_{Y/X} \qquad \forall \lambda \in L^\vee$

where $\varphi_{Y,\lambda}$ is the induced section of the $\mathbb{G}$-bundle $\lambda_* G_Y$ over Y introduced in Subsection 2.2. The principal L-bundle G is a direct sum of $\mathbb{G}$-bundles $\lambda_* G$ over A, $\lambda \in L^\vee$; let $\varphi_\lambda : U \longrightarrow \lambda_* G$ be the induced morphisms. Then condition (i) is equivalent to

(i') $\varphi_\lambda(\mathrm{div}(f)_C) = 0 \qquad \forall \lambda \in L^\vee,\ \forall (C, f) \in \mathfrak{R}_0(X, U)$.

Hence it comes down to show that for all $\lambda \in L^\vee$ the following conditions are equivalent:

(j) $\varphi_\lambda(\mathrm{div}(f)_C) = 0 \qquad \forall (C, f) \in \mathfrak{R}_0(X, U)$,

(jj) $\mathrm{div}_{\mathbb{G}}(\varphi_{Y,\lambda}) \in \underline{\mathrm{Div}}^0_{Y/X}$.

This is the content of Lemma 3.29 below. ■

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[2] A category of rational maps to algebraic groups is defined already by its objects, according to Remark 2.24.

Lemma 3.29 *Let $\varphi_\lambda : U \longrightarrow G_\lambda$ be a morphism from U to an algebraic group $G_\lambda \in \mathrm{Ext}\,(A, \mathbb{G})$, i.e. G_λ is a $\mathbb{G}$-bundle over an abelian variety A. Then the following conditions are equivalent:*

(i) $\varphi_\lambda\,(\mathrm{div}\,(f)_C) = 0 \qquad \forall\,(C, f) \in \mathfrak{R}_0\,(X, U)$,

(ii) $\mathrm{div}_{\mathbb{G}}\,(\varphi_{Y,\lambda}) \in \underline{\mathrm{Div}}^0_{Y/X}$.

Proof. Let C be a Cartier curve in X relative to $X \setminus U$, and let $\nu : \widetilde{C} \longrightarrow C$ be its normalization. In the case $\mathbb{G} = \mathbb{G}_\mathrm{m}$ Lemma 3.30 and in the case $\mathbb{G} = \mathbb{G}_\mathrm{a}$ Lemma 3.31 assert that the following conditions are equivalent:

(j) $\varphi_\lambda|_C\,(\mathrm{div}\,(f)) = 0 \qquad \forall f \in \mathrm{K}\,(C, C \cap U)^*$,

(jj) $\nu_*\,(\mathrm{div}_{\mathbb{G}}\,(\varphi_\lambda|_C)) = 0$.

We have $\mathrm{div}_{\mathbb{G}}\,(\varphi_\lambda|_C) = \mathrm{div}_{\mathbb{G}}\,(\varphi_{Y,\lambda}) \cdot \widetilde{C}$, where $\_ \cdot \widetilde{C} : \underline{\mathrm{Dec}}_{Y,\widetilde{C}} \longrightarrow \underline{\mathrm{Div}}_{\widetilde{C}}$ is the pull-back of Cartier divisors from Y to $\widetilde{C}$ (Definition 2.11, Proposition 2.12). Using the equivalence (j)$\Longleftrightarrow$(jj) above, condition (i) is equivalent to

(i') $(\nu_C)_*\left(\mathrm{div}_{\mathbb{G}}\,(\varphi_{Y,\lambda}) \cdot \widetilde{C}\right) = 0 \qquad \forall$ Cartier curves C relative to $X \setminus U$.

Conditions (i') and (ii) are equivalent by definition of $\underline{\mathrm{Div}}^0_{Y/X}$ (Proposition 3.24), taking into account that $\mathrm{div}_{\mathbb{G}}\,(\varphi_{Y,\lambda}) \in \underline{\mathrm{Div}}^0_Y$ by Proposition 2.22. ∎

Lemma 3.30 *Let C be a projective curve and $\nu : \widetilde{C} \longrightarrow C$ its normalization. Let $\psi : C \dashrightarrow G_\mu$ be a rational map from C to an algebraic group $G_\mu \in \mathrm{Ext}\,(A, \mathbb{G}_\mathrm{m})$, i.e. G_μ is a $\mathbb{G}_\mathrm{m}$-bundle over an abelian variety A. Suppose that ψ is regular on a dense open subset $U_C \subset C_\mathrm{reg}$, which we identify with its preimage in $\widetilde{C}$. Then the following conditions are equivalent:*

(i) $\psi\,(\mathrm{div}\,(f)) = 0 \qquad \forall f \in \mathrm{K}\,(C, U_C)^*$,

(ii) $(f \circ \nu)\,(\mathrm{div}_{\mathbb{G}_\mathrm{m}}\,(\psi)) = 0 \qquad \forall f \in \mathrm{K}\,(C, U_C)^*$,

(iii) $\nu_*\,(\mathrm{div}_{\mathbb{G}_\mathrm{m}}\,(\psi)) = 0$.

Proof. (i)$\Longleftrightarrow$(ii) We show that for all $f \in \mathrm{K}\,(C, U_C)^*$ it holds

$$\psi\,(\mathrm{div}\,(f)) = (f \circ \nu)\,(\mathrm{div}_{\mathbb{G}_\mathrm{m}}\,(\psi)) \ .$$

Let $f \in \mathrm{K}\,(C, U_C)^*$. Write $\widetilde{f} := \nu^\# f = f \circ \nu$. Set $S := \widetilde{C} \setminus U_C$. For each $s \in S$ let $\Phi_s : U_s \times \mathbb{G}_\mathrm{m} \longrightarrow G_\mu$ be a local trivialization of the induced $\mathbb{G}_\mathrm{m}$-bundle over $\widetilde{C}$ in a neighbourhood $U_s \ni s$. Notice that $\mathrm{v}_p\,(\psi) := \mathrm{v}_p\left([\psi]_{\Phi_p}\right)$ is independent of the local trivialization. Since $f \in \mathrm{K}\,(C, U_C)^*$, we have $f \in \mathcal{O}^*_{C,s}$ for all $s \in S$ and hence $\mathrm{div}\,(f) \cap S = \varnothing$. Then by Lemma 3.32 it holds

$$\psi\,(\mathrm{div}\,(f)) = (\psi \circ \nu)\left(\mathrm{div}\left(\widetilde{f}\right)\right) = \prod_{c \notin S} \psi(c)^{\mathrm{v}_c(\widetilde{f})} \ .$$

The defining properties of a local symbol from Definition 3.10 imply

$$\prod_{c \notin S} \psi(c)^{\mathrm{v}_c(\widetilde{f})} = \prod_{c \notin S} \left(\psi, \widetilde{f}\right)_c = \prod_{s \in S} \left(\psi, \widetilde{f}\right)_s^{-1} \ .$$

According to Lemma 3.16 and the explicit description from Example 3.13 of local symbols for rational maps to $\mathbb{G}_\mathrm{m}$ this is equal to

$$\prod_{s \in S} \left(\psi, \widetilde{f}\right)_s^{-1} = \prod_{s \in S} \left([\psi]_{\Phi_s}, \widetilde{f}\right)_s^{-1} = \prod_{s \in S} \left(\widetilde{f}, [\psi]_{\Phi_s}\right)_s \ .$$

Finally, using again the defining properties of a local symbol from Definition 3.10, we obtain

$$\prod_{s\in S}\left(\widetilde{f},[\psi]_{\Phi_s}\right)_s = \prod_{p\notin \mathrm{Supp}\left(\mathrm{div}\left(\widetilde{f}\right)\right)} \widetilde{f}(p)^{\mathrm{v}_p(\psi)} = \widetilde{f}\left(\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\psi)\right)\ .$$

(ii)$\Longleftrightarrow$(iii) The implication (iii)$\Longrightarrow$(ii) is clear. For the converse direction first observe that the support of $\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\psi)$ lies necessarily in $\widetilde{C}\setminus U_C$, since ψ is regular on U_C. For each $s\in C\setminus U_C$ there is a rational function $f_s\in \mathrm{K}\left(C,U_C\right)^*$ such that $f(s)=t\in\mathbb{G}_\mathrm{m}\setminus\{1\}$ and $f(z)=1$ for all $z\in C\setminus(U_C\cup\{s\})$ by the approximation theorem. Then $(f_s\circ\nu)\left(\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\psi)\right)=0$ if and only if $\nu_*\left(\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\psi)\,|_{\nu^{-1}(s)}\right)=0$, where $\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\psi)\,|_{\nu^{-1}(s)}$ is the part of $\mathrm{div}_{\mathbb{G}_\mathrm{m}}(\psi)$ which has support on $\nu^{-1}(s)$. As this is true for all $s\in C\setminus U_C$, it shows the implication (ii)$\Longrightarrow$(iii). ■

Lemma 3.31 *Let C be a projective curve and $\nu:\widetilde{C}\longrightarrow C$ its normalization. Let $\psi:C\dashrightarrow G_\alpha$ be rational map from C to an algebraic group $G_\alpha\in$* $\mathrm{Ext}\left(A,\mathbb{G}_\mathrm{a}\right)$*, i.e. G_α is a $\mathbb{G}_\mathrm{a}$-bundle over an abelian variety A. Suppose that ψ is regular on a dense open subset $U_C\subset C_{\mathrm{reg}}$, which we identify with its preimage in $\widetilde{C}$. Then the following conditions are equivalent:*

(i) $\psi\left(\mathrm{div}\left(f\right)\right)=0 \qquad \forall f\in\mathrm{K}\left(C,U_C\right)^*\ ,$

(ii) $\sum_{q\to p}\mathrm{Res}_q\left(\psi\,\mathrm{d}g\right)=0 \qquad \forall g\in\widehat{\mathcal{O}}_{C,p},\ \forall p\in C\ ,$

(iii) $\nu_*\left(\mathrm{div}_{\mathbb{G}_\mathrm{a}}(\psi)\right)=0\ .$

Proof. (i)$\Longleftrightarrow$(ii) Let $f\in\mathrm{K}\left(C,U_C\right)^*$. We will identify f with $f':=\nu^{\#}f$. Set $S:=\widetilde{C}\setminus U_C$. For each $s\in S$ let $\Phi_s:U_s\times\mathbb{G}_\mathrm{a}\longrightarrow G_\alpha$ be a local trivialization of the induced $\mathbb{G}_\mathrm{a}$-bundle over $\widetilde{C}$ in a neighbourhood $U_s\ni s$. Notice that for each $\omega\in\Omega_{\widetilde{C}}$ which is regular at $q\in\widetilde{C}$ the expression $\mathrm{Res}_q\left(\psi\,\omega\right):=\mathrm{Res}_q\left([\psi]_{\Phi_q}\,\omega\right)$ is independent of the local trivialization. Then by Lemma 3.32 it holds

$$\psi\left(\mathrm{div}\left(f\right)\right)=\left(\psi\circ\nu\right)\left(\mathrm{div}\left(f'\right)\right)=\sum_{c\notin S}\mathrm{v}_c\left(f'\right)\ \psi(c)\ .$$

The defining properties of a local symbol from Definition 3.10 imply

$$\sum_{c\notin S}\mathrm{v}_c\left(f'\right)\ \psi(c)=\sum_{c\notin S}\left(\psi,f'\right)_c=-\sum_{s\in S}\left(\psi,f'\right)_s\ .$$

According to Lemma 3.16 and the explicit description from Example 3.14 of local symbols for rational maps to $\mathbb{G}_\mathrm{a}$ we obtain

$$-\sum_{s\in S}\left(\psi,f'\right)_s=-\sum_{s\in S}\left([\psi]_{\Phi_s},f'\right)_s=-\sum_{s\in S}\mathrm{Res}_s\left(\psi\,\mathrm{d}f'/f'\right)\ .$$

Now $\mathrm{d}f/f=\mathrm{d}\log f$. More precisely, if $f=\alpha(1+h)\in\mathcal{O}^*_{C,p}$ with $\alpha\in k^*$ and $h\in\mathfrak{m}_{C,p}$, then $\mathrm{d}f/f=\mathrm{d}(1+h)/(1+h)=\mathrm{d}\log(1+h)$ and $\log:1+\widehat{\mathfrak{m}}_{C,p}\xrightarrow{\sim}\widehat{\mathfrak{m}}_{C,p}$ is well-defined. Thus the implication (ii)$\Longrightarrow$(i) is clear.

For the converse, we first show that for each $p\in\nu(S)$, each $g\in\widehat{\mathcal{O}}_{C,p}$ and each effective divisor $\mathfrak{e}$ supported on S there is a rational function $f_p\in\mathrm{K}\left(C,U_C\right)^*$ such that $\mathrm{d}\log f_p\equiv\mathrm{d}g\mod\mathfrak{e}$ at $\nu^{-1}(p)$ and $f_p\equiv1\mod\mathfrak{e}$ at $s\in S\setminus\nu^{-1}(p)$.

Since $\mathrm{im}\left(\widehat{\mathcal{O}}_{C,p} \xrightarrow{\mathrm{d}} \Omega_{\widehat{\mathcal{O}}_{C,p}}\right) = \mathrm{im}\left(\widehat{\mathfrak{m}}_{C,p} \xrightarrow{\mathrm{d}} \Omega_{\widehat{\mathcal{O}}_{C,p}}\right)$, we may assume $g \in \widehat{\mathfrak{m}}_{C,p}$. According to the approximation theorem, for $\mathfrak{d} = \sum_{s\in S} n_s\, s$ there is $f_p \in \mathcal{K}_{\widetilde{C}}$ such that $f_p \equiv \exp g \mod \mathfrak{d}$ at $\nu^{-1}(p)$ and $f_p \equiv 1 \mod \mathfrak{d}$ at $s \in S\backslash\nu^{-1}(p)$. One sees that $f_p \in \mathrm{K}\left(C, U_C\right)^*$. In particular, there is $h \in \widehat{\mathfrak{m}}_{C,p}$ with $h \in \widehat{\mathfrak{m}}^{n_q}_{\widetilde{C},q}$ for each $q \to p$ such that $\exp g = f_p + h = f_p(1 + f_p^{-1}h)$. Then $g = \log f_p + \log(1 + f_p^{-1}h)$, where $\log(1 + f_p^{-1}h) \in \widehat{\mathfrak{m}}^{n_q}_{\widetilde{C},q}$, since $f_p^{-1}h \in \widehat{\mathfrak{m}}^{n_q}_{\widetilde{C},q}$ and $\log(1 + \widehat{\mathfrak{m}}^{n}_{\widetilde{C},q}) = \widehat{\mathfrak{m}}^{n}_{\widetilde{C},q}$ for $n \geq 1$. This yields $\mathrm{d}\log f_p \equiv \mathrm{d}g \mod \mathfrak{e}$ at $\nu^{-1}(p)$ if $\mathfrak{e} = \sum_{s\in S} m_s\, s$ and $\mathfrak{d} = \sum_{s\in S}(m_s + 1)\, s$.

Choosing $\mathfrak{e} = \sum_{s\in S} m_s\, s$ large enough, i.e. m_s larger than the pole order of ψ at s, yields that $\mathrm{Res}_s\left(\psi\, \mathrm{d}f_p/f_p\right) = 0$ for all $s \in S \setminus \nu^{-1}(p)$, as $\mathrm{d}f_p/f_p$ has a zero of order $\geq m_s - 1$ at $s \in S \setminus \nu^{-1}(p)$. Hence $\psi\left(\mathrm{div}\left(f_p\right)\right) = 0$ if and only if $\sum_{q\to p} \mathrm{Res}_p\left(\psi\, \mathrm{d}f_p/f_p\right) = \sum_{q\to p} \mathrm{Res}_p\left(\psi\, \mathrm{d}g\right) = 0$. It remains to remark that $\mathrm{Res}_c\left(\psi\, \mathrm{d}h\right) = 0$ for all $h \in \widehat{\mathcal{O}}_{\widetilde{C},c} \supset \widehat{\mathcal{O}}_{C,\nu(c)}$, $c \in U_C$, since ψ and $\mathrm{d}h$ are both regular at c.

(ii)$\Longleftrightarrow$(iii) Let $q \in \widetilde{C}$. Then $\sum_{q\to p} \mathrm{Res}_p\left(\psi\, \mathrm{d}g\right) = 0$ for all $g \in \widehat{\mathcal{O}}_{C,p}$ is equivalent to the condition that the image $\sum_{q\to p}\left[\mathrm{div}_{\mathbb{G}_\mathrm{a}}\left(\psi\right)\right]_q$ of $\mathrm{div}_{\mathbb{G}_\mathrm{a}}\left(\psi\right)$ in $\bigoplus_{q\to p} \mathrm{Hom}^{\mathrm{cont}}_{k(q)}\left(\widehat{\mathfrak{m}}_{\widetilde{C},q}, k(q)\right)$ vanishes on $\widehat{\mathfrak{m}}_{C,p}$, by construction (Proposition 3.19), which says $0 = \sum_{q\to p}\left[\mathrm{div}_{\mathbb{G}_\mathrm{a}}\left(\psi\right)\right]_q \circ \widehat{\nu^{\#}} \in \bigoplus_{q\to p} \mathrm{Hom}^{\mathrm{cont}}_{k(q)}\left(\widehat{\mathfrak{m}}_{C,\nu(q)}, k(q)\right)$. This is true for all $p \in C$ if and only if $\nu_*\left(\mathrm{div}_{\mathbb{G}_\mathrm{a}}\left(\psi\right)\right) = 0$ by definition of the push-forward for formal infinitesimal divisors (Proposition 3.18). ■

Lemma 3.32 *Let C be a Cartier curve in X relative to $X \setminus U$ and $\nu : \widetilde{C} \longrightarrow C$ its normalization. If $\psi : C \cap U \longrightarrow G$ is a morphism from $C \cap U$ to an algebraic group G, then for each $f \in \mathrm{K}\left(C, C \cap U\right)^*$ it holds*

$$\psi\left(\mathrm{div}\left(f\right)_C\right) = \left(\psi \circ \nu\right)\left(\mathrm{div}\left(\nu^{\#} f\right)_{\widetilde{C}}\right) .$$

Proof. Follows immediately from Definition 3.3. ■

3.6 Universal Regular Quotient

The results obtained up to now provide the necessary foundations for a description of the universal regular quotient and its dual, which was the initial intention of this work.

Existence and Construction

The universal regular quotient $\mathrm{Alb}\left(X\right)$ of a (singular) projective variety X is by definition (see [ESV]) the universal object for the category $\mathbf{Mr}^{\mathrm{CH}_0(X)^0}$ of morphisms from $U \subset X_{\mathrm{reg}}$ factoring through $\mathrm{CH}_0\left(X\right)^0$ (Definition 3.27). In Theorem 3.28 we have seen that this category is equivalent to the category $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{Y/X}}$ of rational maps from a projective resolution of singularities Y for X to algebraic groups which induce a transformation to the formal group $\underline{\mathrm{Div}}^0_{Y/X}$. Now Theorem 2.39 implies the existence of a universal object $\mathrm{Alb}_{\underline{\mathrm{Div}}^0_{Y/X}}(Y)$ for this category, which was constructed (Remark 2.41) as the dual 1-motive of $\left[\underline{\mathrm{Div}}^0_{Y/X} \longrightarrow \underline{\mathrm{Pic}}^0_Y\right]$. As $\mathrm{Alb}\left(X\right) = \mathrm{Alb}_{\underline{\mathrm{Div}}^0_{Y/X}}(Y)$, this gives the existence and an explicit construction of the universal regular quotient, as well as a description of its dual. The proof of Theorem 0.1 is thus complete.

Functoriality

Let X, V be projective varieties whose normalizations $\widetilde{X}$, $\widetilde{V}$ are regular. We analyze whether a morphism $\sigma : V \longrightarrow X$ induces a homomorphism of algebraic groups $\mathrm{Alb}\,(V) \longrightarrow \mathrm{Alb}\,(X)$.

As the functoriality of the universal objects $\mathrm{Alb}_{\mathcal{F}}(Y)$, where Y is regular and $\mathcal{F} \subset \underline{\mathrm{Div}}^0_Y$ is a formal group, has already been treated in Proposition 2.52, we will reduce the problem to this case. Therefore it obliges to show under which assumptions the following conditions hold:

(α) A morphism $\sigma : V \longrightarrow X$ induces a morphism $\widetilde{\sigma} : \widetilde{V} \longrightarrow \widetilde{X}$.

(β) The pull-back of relative Cartier divisors maps $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ to $\underline{\mathrm{Div}}^0_{\widetilde{V}/V}$.

For this purpose we introduce the following notion, analogue to Definition 3.1 (keeping the notation fixed at the beginning of this Section 3):

Definition 3.33 *A* Cartier subvariety in X relative to $X \setminus U$ *is a subvariety $V \subset X$ satisfying*

(a) V is equi-dimensional.

(b) No component of V is contained in $X \setminus U$.

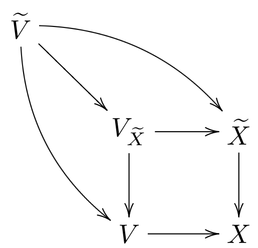
(c) If $p \in V \setminus U$, the ideal of V in $\mathcal{O}_{X,p}$ is generated by a regular sequence.

Remark 3.34 *A Cartier subvariety V in X relative to $X \setminus U$* in codimension one *needs not to be a Cartier divisor on the whole of X. Point (c) of Definition 3.33 implies that V is a locally principal divisor in a neighbourhood of $X \setminus U$.*

Proposition 3.35 *Let $V \subset X$ be a Cartier subvariety relative to $X \setminus U$. Then the pull-back of relative Cartier divisors and of line bundles induces a transformation of 1-motives*

$$\left[\begin{array}{c} \underline{\mathrm{Div}}^0_{\widetilde{V}/V} \\ \downarrow \\ \underline{\mathrm{Pic}}^0_{\widetilde{V}} \end{array}\right] \longleftarrow \left[\begin{array}{c} \underline{\mathrm{Div}}^0_{\widetilde{X}/X} \\ \downarrow \\ \underline{\mathrm{Pic}}^0_{\widetilde{X}} \end{array}\right] .$$

Proof. It suffices to verify the conditions (α) and (β) mentioned above. As no irreducible component of V is contained in $X \setminus U$ by condition (b) of Definition 3.33, the base change $V \times_X \widetilde{X} =: V_{\widetilde{X}} \longrightarrow V$ of $\widetilde{X} \longrightarrow X$ is biregular for each irreducible component of $V_{\widetilde{X}}$, and there is exactly one irreducible component of $V_{\widetilde{X}}$ lying over each irreducible component of V. Thus the normalization $\widetilde{V} \longrightarrow V$ factors through $V_{\widetilde{X}} \longrightarrow V$. We obtain a commutative diagram



The morphism $\widetilde{V} \longrightarrow \widetilde{X}$ induces a pull-back of families of line bundles $\underline{\mathrm{Pic}}^0_{\widetilde{X}} \longrightarrow \underline{\mathrm{Pic}}^0_{\widetilde{V}}$ and a pull-back of relative Cartier divisors $\underline{\mathrm{Div}}^0_{\widetilde{X}/X} \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{V}}$, since no component of V is contained in $\mathrm{Supp}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right)$. A Cartier curve in V relative to $U_V = V \times_X U$ is also a Cartier curve in X relative to $X \setminus U$. Therefore the definition of $\underline{\mathrm{Div}}^0_{\widetilde{V}/V}$ (Proposition 3.24) implies that the image of $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ under pull-back $\_ \cdot \widetilde{V}$ lies actually in $\underline{\mathrm{Div}}^0_{\widetilde{V}/V}$. This gives a commutative diagram of natural transformations of functors

$$\begin{array}{ccc} \underline{\mathrm{Div}}^0_{\widetilde{V}/V} & \longleftarrow & \underline{\mathrm{Div}}^0_{\widetilde{X}/X} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Pic}}^0_{\widetilde{V}} & \longleftarrow & \underline{\mathrm{Pic}}^0_{\widetilde{X}} \end{array}$$

■

Dualization of 1-motives yields the following functoriality of the universal regular quotient:

Proposition 3.36 *Let $\iota : V \subset X$ be a Cartier subvariety relative to $X \setminus U$. Then ι induces a homomorphism of algebraic groups*

$$\mathrm{Alb}(\iota) : \mathrm{Alb}\,(V) \longrightarrow \mathrm{Alb}\,(X) \ .$$

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