

HIGHER COHOMOLOGY OF DIVISORS ON A PROJECTIVE VARIETY

TOMMASO DE FERNEX, ALEX KÜRONYA, AND ROBERT LAZARSFELD

INTRODUCTION

The purpose of this paper is to study the growth of higher cohomology of line bundles on a projective variety.

Let X be an irreducible complex projective variety of dimension $d \geq 1$, and let L be a Cartier divisor on X . It is elementary and well-known that the dimensions of the cohomology groups $H^i(X, \mathcal{O}_X(mL))$ grow at most like m^d , i.e.

$$\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(mL)) = O(m^d) \quad \text{for all } i \geq 0$$

(cf. [9, 1.2.33]). It is natural to ask when one of these actually has maximal growth, i.e. when $h^i(X, \mathcal{O}_X(mL)) \geq Cm^d$ for some positive constant $C > 0$ and arbitrarily large m . For $i = 0$ this happens by definition exactly when L is *big*, and the geometry of big classes is fairly well understood. Here we focus on the question of when one or more of the higher cohomology groups grows maximally.

If L is ample, or merely nef, then of course $h^i(X, \mathcal{O}_X(mL)) = o(m^d)$ for $i > 0$. In general the converse is false: for instance, it can happen that $H^i(X, \mathcal{O}_X(mL)) = 0$ for $m > 0$ and all i even if L is not nef (or, for that matter, pseudoeffective). However our main result shows that if one considers also small perturbations of the divisor in question, then in fact the maximal growth of higher cohomology characterizes non-ample divisors:

Theorem A. *Fix any very ample divisor A on X . If L is not ample, then for sufficiently small rational numbers $t > 0$, at least one of the higher cohomology groups of suitable multiples of $L - tA$ has maximal growth. More precisely, there is an index $i > 0$ such that for any sufficiently small $t > 0$,*

$$\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(m(L - tA))) \geq C \cdot m^d$$

for some constant $C = C(L, A, t) > 0$ and arbitrarily large values of m clearing the denominator of t .

In other words, a divisor L is ample if and only if

$$h^i(X, \mathcal{O}_X(m(L - tA))) = o(m^d) \quad \text{when } i > 0$$

for all small t and suitably divisible m . We remark that the essential content of the theorem is the maximal growth of higher cohomology when L is big but not ample.

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One can get a more picturesque statement by introducing asymptotic invariants of line bundles. As above let X be an irreducible complex projective variety of dimension d , and let L be a Cartier divisor on X . Define

$$\widehat{h}^i(X, L) \stackrel{\text{def}}{=} \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(mL))}{m^d/d!}.$$

The definition extends in the natural manner to \mathbb{Q} -divisors. When $i = 0$ this is the *volume* $\text{vol}_X(L)$ of L , which has been the focus of considerable attention in recent years ([6], [3], [2], [9]). The higher cohomology functions were introduced and studied by the second author in [8]. It was established there that $\widehat{h}^i(L)$ depends only on the numerical equivalence class of a \mathbb{Q} -divisor L , and that it uniquely determines a continuous function

$$\widehat{h}^i = \widehat{h}^i(X, \cdot): N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

on the real Néron-Severi space of X . This generalizes the corresponding results for vol_X proved by the third author in [9, 2.2.C]. When X is a toric variety, these functions were studied in [7]. We refer to [5] for a survey of the circle of ideas surrounding asymptotic invariants.

Theorem A then implies:

Corollary B. *A class $\xi_0 \in N^1(X)_{\mathbb{R}}$ is ample if and only if*

$$\widehat{h}^i(\xi) = 0$$

for all $i > 0$ and all $\xi \in N^1(X)_{\mathbb{R}}$ in a small neighborhood of ξ_0 .

One can see this as an asymptotic analogue of Serre's criterion for amplitude. In the toric case, a somewhat stronger statement appears in [7].

The proof of the theorem combines some algebraic constructions from [8] with geometric facts about big line bundles that fail to be ample. Specifically, choose a collection of very general divisors $E_1, \dots, E_p \in |A|$, and assume for a contradiction that L is not nef, but that

$$\widehat{h}^i(X, L - tA) = 0$$

for $i > 0$ and small positive t . The first point is to show that this vanishing descends to the divisors E_{α} . By induction it follows each of the restrictions $L|_{E_{\alpha}}$ is ample, and we can assume furthermore that they are all very ample. Now consider the complex

$$(*) \quad H^0(\mathcal{O}_X(mL)) \xrightarrow{v} \bigoplus^p H^0(\mathcal{O}_{E_{\alpha}}(mL)) \xrightarrow{u} \bigoplus^{\binom{p}{2}} H^0(\mathcal{O}_{E_{\alpha} \cap E_{\beta}}(mL)),$$

the maps u and v being determined by restriction. The cohomology of this injects into $H^1(X, \mathcal{O}_X(mL - pA))$. On the other hand, using the assumption that L fails to be nef, we show that one can arrange things so that there is a non-trivial ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, vanishing on a set of dimension ≥ 1 , such that the base ideal of $|mL|$ grows like \mathfrak{a}^m , i.e.

$$\mathfrak{b}(|mL|) \subseteq \mathfrak{a}^m.$$

Therefore the image of v is contained in the subgroup

$$\bigoplus^p H^0(\mathcal{O}_{E_{\alpha}}(mL) \otimes \mathfrak{a}^m) \subseteq \bigoplus^p H^0(\mathcal{O}_{E_{\alpha}}(mL)),$$

and this leads to a lower bound on the dimension of $\ker(u)/\text{im}(v)$. In fact, a dimension count shows that if $p \sim m\delta$ for $0 < \delta \ll 1$, then $h^1(mL - pA)$ will grow like m^d , which produces the required cohomology.

This argument suggests that there is a relation between the amplitude of restrictions of L and its asymptotic higher cohomologies, and we explore this connection in Section 3. Fixing as above a very ample divisor A , define $a(L, A)$ to be the least integer k such that $L|_{E_1 \cap \dots \cap E_k}$ is ample for k very general divisors $E_1, \dots, E_k \in |A|$. So for example, $a(L, A) \leq \dim \mathbf{B}_+(L)$, where $\mathbf{B}_+(L)$ is the augmented stable base locus of L in the sense of [4], (cf. [9, Section 10.3]).¹ We prove

Proposition C. *If $i > a(L, A)$ then $\widehat{h}^i(\xi) = 0$ for all ξ in a neighborhood of $[L]$ in $N^1(X)_{\mathbb{R}}$.*

We also give examples to show that the converse can fail, i.e. that it can happen that $\widehat{h}^i(L - tA) = 0$ for small $t > 0$ when $i = a(L, A)$. It remains an interesting open question whether one can predict from geometric data the largest value of i for which $\widehat{h}^i(L - tA) \neq 0$ for small $t > 0$.

The paper is organized as follows. We start in Section 1 with a lemma concerning base ideals of big linear series. The main result appears in Section 2, and finally in Section 3 we study restrictions and asymptotic higher cohomology.

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0. CONVENTIONS AND BACKGROUND

0.1. We work throughout over the complex numbers. A *variety* is a reduced and irreducible scheme, and we always deal with closed points.

0.2. We follow the conventions of [9, Chapter 1] concerning divisors on a projective variety X . Thus a *divisor* on X means a Cartier divisor. A \mathbb{Q} - or \mathbb{R} -divisor indicates an element of

$$\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes \mathbb{Q} \quad \text{or} \quad \mathrm{Div}_{\mathbb{R}}(X) = \mathrm{Div}(X) \otimes \mathbb{R}.$$

$N^1(X)$ is the Néron-Severi group of numerical equivalence classes of divisors, while $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$ denote the corresponding groups for \mathbb{Q} - and \mathbb{R} -divisors.

0.3. Given a projective variety X of dimension d , and a divisor L on X , we set

$$(1) \quad \widehat{h}^i(X, L) = \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(mL))}{m^d/d!}.$$

It is established in [8, Proposition 5.15, Theorem 5.1] that this depends only on the numerical equivalence class of L , and that it satisfies the homogeneity

$$(2) \quad \widehat{h}^i(pL) = p^d \cdot \widehat{h}^i(L).$$

This allows one to define $\widehat{h}^i(L)$ for an arbitrary \mathbb{Q} -divisor by clearing denominators, giving rise to a function $\widehat{h}^i : N^1(X)_{\mathbb{Q}} \rightarrow \mathbb{R}$. The main result of [8] is that this extends uniquely to a continuous function

$$\widehat{h}^i : N^1(X)_{\mathbb{R}} \rightarrow \mathbb{R}$$

satisfying the same homogeneity property ([8, Theorem 5.1]). When L is an \mathbb{R} -divisor, we typically write $\widehat{h}^i(L)$ to denote the value of this function on the numerical equivalence class of

¹By definition, $\mathbf{B}_+(L)$ is the stable base-locus of the \mathbb{Q} -divisor $L - \varepsilon A$ for any small $\varepsilon > 0$ and ample A .

L . The \widehat{h}^i are called asymptotic cohomological functions in [8], although we occasionally use some slight variants of this terminology. Observe finally that the homogeneity (2) implies that if L is a Cartier divisor, then

$$(3) \quad \widehat{h}^i(X, L) = \frac{\widehat{h}^i(X, pL)}{p^d} = \limsup_{m \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(pmL))}{(pm)^d/d!}.$$

for every fixed integer $p > 0$. The analogous statement holds when L is a \mathbb{Q} -divisor, provided that p is sufficiently divisible to clear the denominators of L .

1. A LEMMA ON BASE-LOCI

This brief section is devoted to the proof of a useful fact concerning the base-ideals of big linear systems. The result in question asserts in effect that the base-ideals associated to multiples of a divisor which is not nef grow at least like powers of the ideal of a curve. While the result is known to experts, we include it here for lack of a convenient reference.

Proposition 1.1. *Let D be a divisor on a normal projective variety V , and denote by*

$$\mathfrak{b}(|\ell D|) \subseteq \mathcal{O}_V$$

the base-ideal of the indicated linear series. Assume that D is not nef. Then there exist positive integers q and c , and an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_V$ vanishing on a set of dimension ≥ 1 , such that

$$\mathfrak{b}(|mqD|) \subseteq \mathfrak{a}^{m-c}$$

for all $m > c$.

Proof. The assertion being trivial otherwise, we may suppose that D has non-negative Kodaira-Iitaka dimension $\kappa(V, D) \geq 0$. We assume first that V is smooth, in which case the statement is a small elaboration of the proof of [9, 11.2.18]. Specifically, consider for $q \geq 1$ the asymptotic multiplier ideals $\mathcal{J}(\|qD\|)$. We claim that for $q \gg 0$ this ideal must vanish on a curve. In fact, thanks to [9, 11.2.13], there is a fixed divisor A such that

$$\mathcal{O}_V(qD + A) \otimes \mathcal{J}(\|qD\|)$$

is globally generated for all $q > 0$. Therefore if $C \subseteq V$ is any curve not contained in the zeroes of $\mathcal{J}(\|qD\|)$, then $(qD + A) \cdot C \geq 0$. But if this happens for arbitrarily large values of q , then $(D \cdot C) \geq 0$, and hence if $\mathcal{J}(\|qD\|)$ vanished only on points for large q then D would be nef. On the other hand, it follows from the subadditivity theorem [9, 11.2.4] that

$$\mathfrak{b}(|mqD|) \subseteq \mathcal{J}(\|mqD\|) \subseteq \mathcal{J}(\|qD\|)^m,$$

which for suitably large q gives the desired statement (with $c = 0$).

Starting with a normal projective variety V , let $\mu : V' \rightarrow V$ be a resolution of singularities, and set $D' = \mu^*D$. Then $H^0(V, \mathcal{O}_V(\ell D)) = H^0(V', \mathcal{O}_{V'}(\ell D'))$ by normality, and in particular $\mathfrak{b}(|\ell D'|) = \mathfrak{b}(|\ell D|) \cdot \mathcal{O}_{V'}$. By the case already treated, we have

$$\mathfrak{b}(|mqD'|) \subseteq \mathfrak{q}^m$$

for suitable q and all $m > 0$, where $\mathfrak{q} \subseteq \mathcal{O}_{V'}$ is an ideal sheaf whose zeroes dominate a proper algebraic subset of V having dimension ≥ 1 . After further blowing up V' , we may suppose

that in fact $\mathfrak{q} = \mathcal{O}_{V'}(-F)$ for some effective Cartier divisor F on V' . Now set $\mathfrak{a} = \mu_*\mathcal{O}_{V'}(-F)$. Then

$$\mathfrak{b}(|mqD|) \subseteq \mu_*\mathcal{O}_{V'}(-mF) = \overline{\mathfrak{a}^m},$$

where as usual $\overline{\mathfrak{a}^m}$ denotes the integral closure of the ideal in question. But quite generally, given any ideal $\mathfrak{a} \subseteq \mathcal{O}_V$, there exists an integer $c > 0$ such that $\overline{\mathfrak{a}^{k+1}} = \mathfrak{a} \cdot \overline{\mathfrak{a}^k}$ for $k \geq c$ (cf. [9, (**)] on p. 218] and hence $\overline{\mathfrak{a}^m} \subseteq \mathfrak{a}^{m-c}$ for $m \geq c$. \square

Remark 1.2. We do not know whether the normality hypothesis is essential.

2. A CHARACTERIZATION OF AMPLE DIVISORS

This section is devoted to the statement and proof of our main result.

We start by fixing notation. In what follows, X will be a projective variety of dimension d over the complex numbers, and L will denote a Cartier divisor on X .

If the divisor L is ample, then the functions \widehat{h}^i vanish identically in a neighborhood of $[L]$ in $N^1(X)_{\mathbb{R}}$ for every $i \geq 1$; this follows easily from Serre vanishing, the continuity of the functions \widehat{h}^i , and the fact that the ample cone is open inside $N^1(X)_{\mathbb{R}}$. In particular these functions vanish at $L - tA$ for every very ample divisor A and every sufficiently small $t \geq 0$. The main result of this section is that this property characterizes amplitude.

Theorem 2.1. *Let X be a projective variety, and let L be a Cartier divisor on X . Assume that there exists a very ample divisor A on X and a number $\varepsilon > 0$ such that*

$$\widehat{h}^i(X, L - tA) = 0 \quad \text{for all } i > 0, 0 \leq t < \varepsilon.$$

Then L is ample.

Theorem A from the Introduction follows immediately. We will deduce Corollary B at the end of the section.

We now begin working towards the proof of Theorem 2.1. First of all, in order to eventually be able to apply Proposition 1.1, we reduce to the situation when the variety X is normal.

Lemma 2.2. *Assume that Theorem 2.1 holds for normal projective varieties. Then it holds in general.*

Proof. Let X be an arbitrary projective variety, and suppose that L and A are divisors on X satisfying the assumptions of Theorem 2.1, so that L is Cartier, A is ample, and there exists an $\varepsilon > 0$ such that $\widehat{h}^i(X, L - tA) = 0$ for all $i > 0$ and $0 \leq t < \varepsilon$. Consider the normalization $\nu: \widetilde{X} \rightarrow X$ of X . Since ν is a finite morphism, ν^*A is ample. Moreover

$$\widehat{h}^i(\widetilde{X}, \nu^*(L - tA)) = \widehat{h}^i(X, L - tA).$$

thanks to the birational invariance of higher cohomology functions ([8, Proposition 2.9]). Assuming the theorem for normal varieties we conclude that ν^*L is ample, and hence L is ample as well. \square

So we henceforth assume that X is normal. The plan of the proof is now to study the \widehat{h}^i via restrictions to divisors and use induction on dimension. Specifically, choose a sequence of very general divisors

$$E_1, E_2, \dots \in |A|.$$

Given $m, p > 0$ we take the first p of the E_α and form the complex $K_{m,p}^\bullet$:

$$(4) \quad \mathcal{O}_X(mL) \longrightarrow \bigoplus^p \mathcal{O}_{E_\alpha}(mL) \longrightarrow \bigoplus^{\binom{p}{2}} \mathcal{O}_{E_\alpha \cap E_\beta}(mL) \longrightarrow \dots,$$

obtained as a twist of the p -fold tensor product of the one-step complexes $\mathcal{O}_X \longrightarrow \mathcal{O}_{E_\alpha}$. Because it will be important to keep track of the number of summands, we denote by

$$\bigoplus^{\binom{p}{i}} \mathcal{O}_{E_{\alpha_1} \cap \dots \cap E_{\alpha_i}}$$

the direct sum of the sheaves $\mathcal{O}_{E_{\alpha_1} \cap \dots \cap E_{\alpha_i}}$ over all choices of i increasing indices. It is established in [8, Corollary 4.2] that $K_{m,p}^\bullet$ is acyclic, and hence resolves $\mathcal{O}_X(mL - pA)$. In particular,

$$(5) \quad H^r(X, \mathcal{O}_X(mL - pA)) = \mathbb{H}^r(K_{m,p}^\bullet).$$

The hypercohomology group on the right in (5) is in turn computed by a first-quadrant spectral sequence with

$$(6) \quad E_1^{i,j} = \begin{cases} H^j(\mathcal{O}_X(mL)) & i = 0 \\ \bigoplus^{\binom{p}{i}} H^j(\mathcal{O}_{E_{\alpha_1} \cap \dots \cap E_{\alpha_i}}(mL)) & i > 0. \end{cases}$$

As in [9, 2.2.37] or [8, Section 5] we may — and do — assume that the dimensions of all the groups appearing on the right in (6) are independent of the particular divisors E_α that occur. We will write these dimensions as

$$h^j(\mathcal{O}_{E_1}(mL)) \quad , \quad h^j(\mathcal{O}_{E_1 \cap E_2}(mL))$$

and so on.

The first point is to show that the vanishing hypothesis of the theorem descends to very general divisors in $|A|$.

Lemma 2.3. *Keeping notation as in Theorem 2.1, assume that there is a positive real number $\varepsilon > 0$ such that*

$$(7) \quad \widehat{h}^i(X, L - tA) = 0 \quad \text{for all } i > 0, 0 \leq t < \varepsilon.$$

Let $E \in |A|$ be a very general divisor. Then

$$(8) \quad \widehat{h}^i(E, (L - tA)|_E) = 0 \quad \text{for all } i > 0, 0 \leq t < \varepsilon.$$

Proof. Assuming (7), it's enough to prove

$$(9) \quad \widehat{h}^i(E, L_E) = 0 \quad \text{for all } i > 0.$$

For then the more general statement (8) follows (using the homogeneity and continuity of the higher cohomology functions on X and on E) upon replacing L by $L - \delta A$ for a rational number $0 < \delta < \varepsilon$.

Suppose then that (9) fails, and consider the complex $K_{m,p}^\bullet$. We compute a lower bound on the dimension of the group $E_\infty^{1,i}$ in the hypercohomology spectral sequence. Specifically, by looking at the possible maps coming into and going out from the $E_r^{1,i}$, one sees that

$$h^{i+1}(X, mL - pA) + h^i(X, mL) \geq p \cdot h^i(\mathcal{O}_{E_1}(mL)) - \binom{p}{2} \cdot h^i(\mathcal{O}_{E_1 \cap E_2}(mL)) - \binom{p}{3} \cdot h^{i-1}(\mathcal{O}_{E_1 \cap E_2 \cap E_3}(mL)) - \dots$$

Now we can find some fixed constant $C_1 > 0$ such that for all $m \gg 0$:

$$h^i(\mathcal{O}_{E_1 \cap E_2}(mL)) \leq C_1 \cdot m^{d-2}, \\ h^{i-1}(\mathcal{O}_{E_1 \cap E_2 \cap E_3}(mL)) \leq C_1 \cdot m^{d-3}, \text{ etc.}$$

Moreover, since we are assuming for a contradiction that $\widehat{h}^i(E, L_E) > 0$, we can find a constant $C_2 > 0$, together with a sequence of arbitrarily large integers m , such that

$$(10) \quad h^i(\mathcal{O}_{E_1}(mL)) \geq C_2 \cdot m^{d-1}.$$

Putting this together, we find that there are arbitrarily large integers m such that

$$(11) \quad h^{i+1}(X, mL - pA) + h^i(X, mL) \geq C_3 \cdot (pm^{d-1} - p^2m^{d-2} - p^3m^{d-3} - \dots)$$

for suitable $C_3 > 0$. Note that this constant C_3 is independent of p . At this point, we fix a very small rational number $0 < \delta \ll 1$. By the homogeneity of \widehat{h}^i on E_1 , we can assume that the sequence of arbitrarily large values of m for which (10) and (11) hold is taken among multiples of the denominator of δ (see 0.3). Then, restricting m to this sequence and taking $p = \delta m$, the first term on the RHS of (11) dominates provided that δ is sufficiently small. Hence

$$h^{i+1}(X, mL - pA) + h^i(X, mL) \geq C_4 \cdot \delta m^d$$

for a sequence of arbitrarily large m , and some $C_4 > 0$. But this implies that

$$\widehat{h}^{i+1}(X, L - \delta A) + \widehat{h}^i(X, L) > 0,$$

contradicting the hypothesis. \square

Proof of Theorem 2.1. By Lemma 2.2, we can suppose without loss of generality that X is normal. We assume that there exists $\varepsilon > 0$ such that

$$(12) \quad \widehat{h}^i(X, L - tA) = 0 \quad \text{for all } i > 0, 0 \leq t < \varepsilon,$$

but that L is not ample, and we'll aim to get a contradiction.

Note that the Theorem fails for L if and only if it fails for integral multiples of $L - \delta A$ when $0 < \delta \ll 1$. So we can suppose that we have a non-nef divisor L satisfying (12).

Let $E \in |A|$ be one of the very general divisors fixed at the outset. Thanks to Lemma 2.3, we can assume by induction on dimension that $\mathcal{O}_E(L)$ is ample. Replacing L again by a multiple, we can suppose in addition that $\mathcal{O}_E(L)$ is very ample with vanishing higher cohomology. This combination being an open condition in families, we can further assume that $\mathcal{O}_{E_\alpha}(L)$ is very ample for each of the E_α .

As above, form the complex $K_{m,p}^\bullet$, and consider in particular the beginning of the bottom row of the spectral sequence (6):

$$H^0(\mathcal{O}_X(mL)) \xrightarrow{v_{m,p}} \bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL)) \xrightarrow{u_{m,p}} \bigoplus^{\binom{p}{2}} H^0(\mathcal{O}_{E_\alpha \cap E_\beta}(mL)).$$

There is a natural injection

$$\frac{\ker(u_{m,p})}{\operatorname{im}(v_{m,p})} \subseteq H^1(X, mL - pA),$$

and the plan is to estimate from below the dimension of this subspace.

As in the proof of Lemma 2.3 there is a uniform bound having the shape

$$h^0(\mathcal{O}_{E_1 \cap E_2}(mL)) \leq C_1 \cdot m^{d-2}.$$

Therefore, considering $\ker(u_{p,m})$ as a subspace of $\bigoplus^p H^0(\mathcal{O}_{E_\alpha}(mL))$, one has

$$(13) \quad \operatorname{codim} \ker(u_{m,p}) \leq C_2 \cdot p^2 m^{d-2}$$

for some $C_2 > 0$ and all $m \gg 0$.

By Proposition 1.1, after possibly replacing L by a suitable multiple, we can find an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ vanishing on a set of dimension ≥ 1 , together with an integer $c \geq 0$ such that

$$\mathfrak{b}(|mL|) \subseteq \mathfrak{a}^{m-c} \quad \text{for all } m > c.$$

Then $v_{m,p}$ admits a factorization

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(mL) \otimes \mathfrak{a}^{m-c}) & \xrightarrow{v'_{m,p}} & \bigoplus^p H^0(E_\alpha, \mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^{m-c}) \\ \parallel & & \downarrow \\ H^0(X, \mathcal{O}_X(mL)) & \xrightarrow{v_{m,p}} & \bigoplus^p H^0(E_\alpha, \mathcal{O}_{E_\alpha}(mL)) \end{array}$$

We claim that there is a constant $C_3 > 0$ such that for all $m \gg 0$:

$$(14) \quad H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^{m-c}) \text{ has codimension } \geq C_3 \cdot m^{d-1} \text{ in } H^0(\mathcal{O}_{E_\alpha}(mL)).$$

Granting this, we have

$$(15) \quad \dim \frac{\ker(u_{m,p})}{\operatorname{im}(v_{m,p})} \geq C_4 \cdot (pm^{d-1} - p^2 m^{d-2})$$

for some constant $C_4 > 0$ and all $m \gg 0$. Once again fixing $0 < \delta \ll 1$, limiting m to multiples of the denominator of δ , and setting $p = \delta m$, one finds that

$$h^1(X, \mathcal{O}_X(m(L - \delta A))) \geq C_5 \cdot \delta m^d$$

for m large enough. This implies that $\widehat{h}^1(X, L - \delta A) > 0$, giving the required contradiction.

It remains to prove (14). Choose any point $x = x_\alpha \in \operatorname{Zeroes}(\mathfrak{a}) \cap E_\alpha$, and write $\mathfrak{m}_x \subseteq \mathcal{O}_{E_\alpha}$ for its maximal ideal: here we use that $\dim \operatorname{Zeroes}(\mathfrak{a}) \geq 1$ to know that such a point exists. Since

$$H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{a}^{m-c}) \subseteq H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{m}_x^{m-c}),$$

it is enough to bound the codimension of $H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{m}_x^{m-c})$ in $H^0(\mathcal{O}_{E_\alpha}(mL))$. But since $\mathcal{O}_{E_\alpha}(L)$ is very ample, it follows that $\mathcal{O}_{E_\alpha}(mL)$ separates $(m-c)$ -jets at the point x . The

dimension of the space of $(m - c)$ -jets at a point of a possibly singular variety is no smaller than the dimension of the space of $(m - c)$ -jets at a smooth point of a variety of the same dimension, and thus we have

$$\operatorname{codim} H^0(\mathcal{O}_{E_\alpha}(mL) \otimes \mathfrak{m}_x^{m-c}) \geq \binom{m - c + d}{d - 1},$$

as required. \square

Finally, we give the proof of Corollary B from the Introduction.

Proof of Corollary B. Consider the following three statements concerning a Cartier divisor L on X :

- (a) L is ample;
- (b) for all $i > 0$ the function \widehat{h}^i vanishes in a neighborhood of $[L]$;
- (c) $\widehat{h}^i(L - tA) = 0$ for some ample divisor A and all $i > 0$ and $0 \leq t \ll 1$.

We have (a) \Rightarrow (b) by Serre's vanishing and the continuity of \widehat{h}^i , the implication (b) \Rightarrow (c) is obvious, and Theorem 2.1 yields (c) \Rightarrow (a). Therefore (a) \Leftrightarrow (b), which is the content of the Corollary. \square

3. AMPLITUDE OF RESTRICTIONS AND COHOMOLOGY

In the course of this section we will study how the vanishing of higher asymptotic cohomology relates to the amplitude of the restrictions of a line bundle to certain very general complete intersections.

We consider as before an arbitrary complex projective variety X , and a very ample divisor A on X . For an arbitrary \mathbb{Q} -Cartier divisor L on X , we introduce the following invariants:

$$\begin{aligned} a(L, A) &\stackrel{\text{def}}{=} \min \{k \mid L|_{E_1 \cap \dots \cap E_k} \text{ is ample for very general } E_i \in |A|\}, \\ b(L) &\stackrel{\text{def}}{=} \dim \mathbf{B}_+(L), \\ c(L) &\stackrel{\text{def}}{=} \max \{i \mid \widehat{h}^i \text{ is not identically zero in any neighborhood of } [L] \text{ in } N^1(X)_\mathbb{R}\}. \end{aligned}$$

These quantities express to some degree how far L is from being ample, with smaller values corresponding to 'more positive' divisors. Note that they each depend only on the numerical equivalence class of L .

The invariants $a(L, A), b(L), c(L)$ satisfy the following relation.

Proposition 3.1. *For every L and A as above, we have*

$$c(L) \leq a(L, A) \leq b(L).$$

Proof of Proposition 3.1. The inequality $a(L, A) \leq b(L)$ follows from the observation that the restriction of a Cartier divisor to a general hyperplane section strictly reduces the dimension of the augmented base locus, and the fact that a divisor with empty augmented base locus is ample. Hence it remains to show the inequality $c(L) \leq a(L, A)$.

At this point, it will be convenient to fix Cartier divisors L_1, \dots, L_ρ whose numerical classes form a basis for $N^1(X)_{\mathbb{R}}$, and to work in the computations that follow with \mathbb{Q} -divisors on X are taken to be linear combinations of the L_k . The point of this is that given very general divisors

$$E_1, E_2, \dots \in |A|,$$

we can assume as above that for every $\vec{m} = (m_k) \in \mathbb{Z}^\rho$, the dimensions of the cohomologies of the restriction of $\sum m_k L_k$ to any intersection of j distinct E_α are independent of the particular E_α chosen. We will implicitly make use of this assumption in the following paragraphs.

We now begin the proof that $c(L) \leq a(L, A)$. By replacing L by an integral multiple of an arbitrarily close rational class in $N^1(X)_{\mathbb{R}}$, it suffices to show that $\widehat{h}^i(X, L) = 0$ for every $i > a(X, L)$. The plan is to proceed by induction on $a(L, A)$. If $a(L, A) = 0$, then the claim follows by Serre's vanishing, so we can assume that $a(L, A) \geq 1$ and suppose that the property holds for $a(L, A) - 1$.

We claim that for every $i > a(L, A)$ the function

$$g_i(t) \stackrel{\text{def}}{=} \widehat{h}^i(L + tA)$$

is differentiable with zero derivative on $(0, \infty)$. Granting this for the moment, we will finish the proof. Indeed we have $g_i(t) = 0$ for $t \gg 0$ by Serre's vanishing, and thus the claim implies that g_i vanishes on $(0, \infty)$, hence at 0 by continuity.

Turning to the proof of the claim, consider to begin with an arbitrary divisor D on X , and arbitrary integers $m, p \geq 0$. By looking again at the spectral sequence (6), one sees that

$$(16) \quad \begin{aligned} \left| h^i(mD - pA) - h^i(mD) \right| &\leq p \cdot (h^i(mD|_{E_1}) + h^{i-1}(mD|_{E_1})) \\ &+ \binom{p}{2} \cdot (h^{i-1}(mD|_{E_1 \cap E_2}) + h^{i-2}(mD|_{E_1 \cap E_2})) + \dots \end{aligned}$$

Now fix rational numbers $t, \delta > 0$, with $\delta \ll 1$, and set $p = \delta m$ where m clears the denominators of t and δ . Applying (16) with $D = L + tA$, and taking m to range over a sequence of suitably divisible integers computing the largest of $\widehat{h}^i(D)$ and $\widehat{h}^i(D - \delta A)$, one finds that

$$(17) \quad \begin{aligned} \frac{|\widehat{h}^i(D - \delta A) - \widehat{h}^i(D)|}{\delta} &\leq C_1 \cdot (\widehat{h}^i(D|_{E_1}) + \widehat{h}^{i-1}(D|_{E_1})) \\ &+ C_2 \cdot \delta \cdot (\widehat{h}^{i-1}(D|_{E_1 \cap E_2}) + \widehat{h}^{i-2}(D|_{E_1 \cap E_2})) + \dots, \end{aligned}$$

where the C_j are positive numerical constants depending only on the dimension d . A similar argument using $D + \delta A$ in place of D gives

$$(18) \quad \begin{aligned} \frac{|\widehat{h}^i(D + \delta A) - \widehat{h}^i(D)|}{\delta} &\leq C_1 \cdot (\widehat{h}^i((D + \delta A)|_{E_1}) + \widehat{h}^{i-1}((D + \delta A)|_{E_1})) \\ &+ C_2 \cdot \delta \cdot (\widehat{h}^{i-1}((D + \delta A)|_{E_1 \cap E_2}) + \widehat{h}^{i-2}((D + \delta A)|_{E_1 \cap E_2})) + \dots \end{aligned}$$

Suppose now that $i > a(L, A)$, and as above let $D = L + tA$. Since

$$a(D|_{E_1 \cap \dots \cap E_k}) = a(D, A) - k \leq a(L, A) - k < i - k$$

for all $1 \leq k \leq a(D, A)$, the same holding with $D + \delta A$ in place of D , all the asymptotic cohomological dimensions other than \widehat{h}^0 appearing in the RHS of (17) and (18) are zero by

induction, and hence each of the dimensions \widehat{h}^0 compute the self-intersection of the given divisor. Thus we can find a positive constant C , independent of δ , such that

$$(19) \quad \left| \frac{\widehat{h}^i(L \pm \delta A) - \widehat{h}^i(L)}{\delta} \right| \leq C \cdot \delta^{i-1} \cdot \left((L + \delta A)^{d-i} \cdot A^i + \delta \cdot (L + \delta A)^{d-i-1} \cdot A^{i+1} \right).$$

So far we have assumed that $\delta \in \mathbb{Q}$, but then by the continuity of both sides, the inequality must actually hold for every real $0 < \delta \ll 1$. Therefore $g_i(t)$ is differentiable with zero derivative at every rational $t > 0$. To see that this property extends over the whole interval $(0, \infty)$, we fix an arbitrary $t > 0$, and let δ be any real number with $0 < |\delta| \ll 1$ and such that $t + \delta \in \mathbb{Q}$. Then (19) yields

$$\begin{aligned} \left| \frac{\widehat{h}^i((L + tA) + \delta A) - \widehat{h}^i(L + tA)}{|\delta|} \right| &= \left| \frac{\widehat{h}^i((L + (t + \delta)A) - \delta A) - \widehat{h}^i(L + (t + \delta)A)}{|\delta|} \right| \\ &\leq C \cdot |\delta|^{i-1} \cdot \left((L + (t + \delta + |\delta|)A)^{d-i} \cdot A^i + |\delta| \cdot (L + (t + \delta + |\delta|)A)^{d-i-1} \cdot A^{i+1} \right). \end{aligned}$$

Since the RHS goes to zero as $\delta \rightarrow 0$, this implies that $g_i(t)$ is differentiable at $t > 0$ and, moreover, we have $g'_i(t) = 0$. This proves the claim, hence completes the proof of the theorem. \square

One might ask whether in fact equality holds in $c(L) \leq a(L, A)$ for all very ample divisors A . Although the idea is tempting, it turns out that in general this is not the case.

Example 3.2. Let $X = \mathbb{F}_1 \times \mathbb{P}^1$, where $\mathbb{F}_1 \cong \text{Bl}_p \mathbb{P}^2$, and denote by $p : X \rightarrow \mathbb{F}_1$ and $q : X \rightarrow \mathbb{P}^1$ the two projections. Let $E \subset \mathbb{F}_1$ be the (-1) -curve, let $F \subset \mathbb{F}_1$ be a fiber of the ruling, and let $H \subset \mathbb{P}^1$ be a point. We consider the divisors

$$L = p^*(\lambda E + F) + q^*H \quad \text{and} \quad A = p^*(E + \mu F) + q^*H \quad \text{for some} \quad \lambda, \mu \in \mathbb{Z}_{\geq 2}.$$

Notice that A is very ample and L is a big divisor. Moreover, the base locus of L coincides with its augmented base locus and is equal to $B \stackrel{\text{def}}{=} p^{-1}(E)$. In particular $b(L) = 2$.

On the other hand, Künneth's formula for asymptotic cohomological functions (see [8, Remark 2.14]), and the fact that L is not ample imply that $c(L) = 1$. Fix a general element $Y \in |A|$ cutting out a smooth divisor D on B . Note that $\mathcal{O}_B(D) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu - 1, 1)$ via the isomorphism $B = E \times \mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Therefore, since D is smooth, it must be irreducible; moreover, p induces an isomorphism $D \cong \mathbb{P}^1$. We observe that the base locus of $L|_Y$ is contained in the restriction of the base locus of L , hence in D , and that $\mathcal{O}_D(L|_Y) \cong \mathcal{O}_{\mathbb{P}^1}(\mu - \lambda)$. We conclude that

$$a(L, A) = \begin{cases} 2 = b(L) & \text{if } \lambda \geq \mu, \\ 1 = c(L) & \text{if } \lambda < \mu. \end{cases}$$

Remark 3.3. Outside the big cone, the equation $a(L, A) = c(L)$ fails badly on every projective variety. Indeed, if L is a Cartier divisor on a d -dimensional projective variety X such that neither L nor $-L$ is pseudo-effective, then we have $c(L) < d$ and $c(-L) < d$, and either $a(L) = d$ or $a(-L) = d$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400 EAST, SALT LAKE CITY, UT 48112-0090

Current address: Institute for Advanced Study, School of Mathematics, 1 Einstein Drive, Princeton, NJ 08540

E-mail address: defernex@math.utah.edu

MATHEMATICAL INSTITUTE, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, P.O. Box 91, BUDAPEST, H-1152 HUNGARY

UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, FB 6 MATHEMATIK, D-45117 ESSEN, GERMANY

E-mail address: alex.kuronya@math.bme.hu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, USA

E-mail address: rlaz@umich.edu