THE INTERSECTION HOMOLOGY D–MODULE IN FINITE CHARACTERISTIC.

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Abstract. For Y a closed normal subvariety of codimension c of a smooth $\mathbb{C}$–variety X, Brylinski and Kashiwara showed in [BK81] that the local cohomology module $\mathcal{H}^c_Y(X,\mathcal{O}_X)$ contains a unique simple $\mathcal{D}_X$–submodule, denoted by $\mathcal{L}(Y,X)$. In this paper the analogous result is shown for X and Y defined over a perfect field of finite characteristic. Moreover, a local construction of $\mathcal{L}(Y,X)$ is given, relating it to the theory of tight closure. From the construction one obtains a criterion for the $\mathcal{D}_X$–simplicity of $\mathcal{H}^c_Y(X)$.

1. Introduction

Let Y be a closed codimension c subvariety of the smooth $\mathbb{C}$ variety X and let Z be the singular locus of Y. Denote by $\mathcal{D}_X$ the sheaf or differential operators on X. In [BK81, Proposition 8.5], Brylinski and Kashiwara show the existence (and usefulness) of a unique holonomic $\mathcal{D}_X$ module $\mathcal{L} = \mathcal{L}(Y,X)$ satisfying the properties

$$\mathcal{L}|_{X-Z} \cong \mathcal{H}^c_{Y-Z}(X-Z,\mathcal{O}_{X-Z})$$
$$\mathcal{H}^0_Z(\mathcal{L}) = \mathcal{H}^0_Z(\mathcal{L}^*) = 0,$$

where the star stands for duality of holonomic $\mathcal{D}$–modules and $\mathcal{H}^i_Y$ denotes the higher derived sections with support in Y. The proof of this result is rather formal and uses duality theory for holonomic $\mathcal{D}_X$–modules. Furthermore, they show that $\mathcal{L}(Y,X)$ is the unique simple, selfdual holonomic $\mathcal{D}_X$–module agreeing with $\mathcal{H}^c_Y(X,\mathcal{O}_X)$ on X'. This result is obtained by showing that $\mathcal{L}(Y,X)$ corresponds, via the Riemann–Hilbert correspondence, to the intersection homology complex π_Y of middle perversity, which, by construction, is simple and selfdual. All these constructions, such as holonomicity, duality and the Riemann–Hilbert correspondence completely rely on characteristic zero – on analytic techniques even, if one is strict.

The question answered in this paper is: What is the situation if X is defined over a field of positive characteristic? Somewhat surprisingly, the existence of a unique simple $\mathcal{D}_X$–submodule $\mathcal{L}(Y,X)$ can be proved, up to the use of metaresults, independent of the characteristic. The only key

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2000 *Mathematics Subject Classification.* 14B15,13N10.

Key words and phrases. D-modules, unit F-modules, Intersection Homology, Tight Closure.

fact needed is that $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ has finite length as a $\mathcal{D}_X$–module. This is guaranteed by holonomicity in characteristic 0 and by [Lyu97, Theorem 5.7] in positive characteristic, respectively.

We state the result and sketch the simple argument – for a complete proof refer to Theorem 4.1.

Theorem. *Let X be a smooth k–variety and let Y be a closed irreducible subvariety of codimension c. Then $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ has a unique simple $\mathcal{D}_X$–submodule $\mathcal{L}(Y, X)$. Furthermore, $\mathcal{L}(Y, X)$ agrees with $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ on $X - \operatorname{Sing} Y$.*

Proof. (Sketch) Since $H^c_Y(X, \mathcal{O}_X)$ has finite length as a $\mathcal{D}_X$–module it has some simple $\mathcal{D}_X$–submodule $\mathcal{L}$. Denote the inclusion $X' \overset{\text{def}}{=} X - \mathrm{SingY} \subseteq X$ by i and write Y' for $Y - \operatorname{Sing} Y$. One sees easily that the restriction of $\mathcal{L}$ to X' is nonzero. As the restriction of $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ is equal to $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$, and by smoothness of Y' the latter is $\mathcal{D}_X$–simple it follows that $\mathcal{L}|_{X'} = \mathcal{H}^c_Y(X, \mathcal{O}_X)|_{X'}$. Since this holds for any simple submodule of $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ the same argument shows that any two such have nonzero intersection, thus they are equal. This shows the uniqueness. □

This existence proof gives very little information about the concrete structure of $\mathcal{L}(Y, X)$. Even in characteristic zero, to explicitly determine $\mathcal{L}(Y, X)$ is difficult. The best results in this case are due to Vilonen [Vil85] for Y a complete intersection with an isolated singularity. He uses analytic techniques to characterize the sections of $\mathcal{H}^c_Y(X, \mathcal{O}_Y)$ belonging to $\mathcal{L}(Y, X)$. They are precisely the ones vanishing under a certain residue map. Furthermore he gives a canonical generator, the canonical class associated to $Y \subseteq X$, for $\mathcal{L}(Y, X)$ in this case.

To explicitly determine $\mathcal{L}(Y, X)$ in positive characteristic is the main purpose of this paper. The strategy is to use the Frobenius instead of the differential structure. This substitution is justified by the close relationship of so called unit $\mathcal{O}_X[F^e]$–structures and $\mathcal{D}_X$–structures, described in [Lyu97, Bli02, EK00]. Our construction is local in nature. If we denote by R and $A = R/I$ the local rings of X and Y at a point $x \in Y$, we roughly show the following, for precise statements see Section 4.1.

Theorem. *Let R be regular, local and F–finite. Let $A = R/I$ be a normal domain. Then the unique simple D_R–submodule, $\mathcal{L}(A, R)$, of $H^c_I(R)$ is dual to the unique simple $A[F^e]$–module quotient of $H^d_m(A)$.*

The duality we are referring to is an extension of Matlis duality incorporating Frobenius actions. Furthermore, the construction is explicit enough to identify (non canonical) generators for $\mathcal{L}(A, R)$. What we have gained is that the unique simple $A[F^e]$–module quotient of $H^d_m(A)$ is well studied and fairly well understood; it is the quotient of $H^d_m(A)$ by the tight closure of zero, $0^*_{H^d_m(A)}$. The vanishing of $0^*_{H^d_m(A)}$ is governed by F–rationality of A, which is a positive characteristic analog of rational singularities. As a consequence of this connection we obtain the following D_R–simplicity criterion for $H^c_I(R)$:

Theorem. *Let R be regular, local and F–finite. Let $A = R/I$ be a Cohen–Macaulay domain of codimension c. Then, if A is F–rational then $H^c_I(R)$ is D_R–simple.*

More precise simplicity criteria for $H^c_I(R)$ are given in Section 4.2.

The paper is structured as follows. In Sections 2 and 3 we recall and further develop some necessary machinery from the theory of $R[F^e]$–modules and tight closure. As the techniques used later are local in nature, the notation reflects this and we are using the commutative algebra language. These sections do not concretely deal with the applications to constructing $\mathcal{L}(Y, X)$ but derive general results which constitute the technical underpinning of what follows. As a notable byproduct we answer a question posed by Lyubeznik showing that minimal roots exist for finitely generated unit $R[F^e]$–modules for any regular, local ring R. In [Lyu97] this was only shown in the complete case.

Section 4 contains the main results discussed above and generalizations thereof. Furthermore, as an application to tight closures theory we show that the parameter test module commutes with localization. We finish this section with a complete characterization of D_R–simplicity in the case of curves, providing a finite characteristic analog of results of Yekutueili and S. P. Smith.

Finally we remark that the substitution of the $\mathcal{D}_X$–module structure by a unit $\mathcal{O}_X[F^e]$–structure in the study of $\mathcal{L}(A, R)$ enables one to place $\mathcal{L}(Y, X)$, in finite characteristic, in the context of a Riemann–Hilbert type correspondence. That such a correspondence exists is recent work of Emerton and Kisin [EK99, EK00], where an equivalence (on the level of derived categories) of the category of finitely generate unit $\mathcal{O}_X[F^e]$–modules and the category of constructible $\mathbb{F}_{p^e}$–sheaves is developed. Within this correspondence, the simple unit $\mathcal{O}_X[F^e]$–module $\mathcal{L}(Y, X)$ constructed here does indeed correspond to certain middle extensions on the constructible $\mathbb{F}_p$–site. These connections will not be discussed here but should appear in the final version of [EK00].

It is a pleasure to thank my advisor Karen Smith for her expertise and encouragement during my dissertation [Bli01], which contains most of the results presented here. Furthermore, I thank Matt Emerton, Brian Conrad and Gennady Lyubeznik for valuable conversations on various parts of this paper.

2. Background on $R[F^e]$–modules

Throughout this paper, R denotes a noetherian ring of dimension n containing a field k of positive characteristic p, unless stated otherwise. For an ideal I of height c we denote the quotient R/I by A. This is a ring of dimension $d = n - c$. In general we assume that R is regular and F–finite, *i.e.* R is a finite module over its subring of pth powers.

The *(absolute) Frobenius map* on R, *i.e.* the ring map sending each element to its pth power, is denoted by $F = F_R$. The associated map on $X = \operatorname{Spec} R$ we also denote by the same letter $F = F_X$.

If $\mathcal{M}$ is an R–module, then $\mathcal{M}^e$ denotes the R–R–bimodule, which, as a left module is just $\mathcal{M}$, but with right structure twisted by the eth iterate of the Frobenius, *i.e.* for $r \in R$ and $m \in \mathcal{M}$ one has $m \cdot r = r^{p^e} m$. With this notation Psekine and Szpiro's *Frobenius functor* is defined as $F^*(\mathcal{M}) = R^1 \otimes \mathcal{M}$. Thinking of F as a map on $\operatorname{Spec} R$ this is just the pullback functor for the Frobenius map. Since F^* is just tensoring with an R–module, it follows that F^* commutes with direct limits and direct sums. If R is regular, F^* is flat; therefore it commutes with finite intersections. The flatness of F^* in the regular case is where the theory draws its power from. Even though the the following constructions make sense for almost any ring, the theory is most useful if R is regular. Therefore we mostly restrict to this case in what follows.

The same construction can be done for higher powers of F, clearly we have $(F^e)^* = (F^*)^e$ which we denote by F^{e*}.

We review the definition and basic properties of modules with Frobenius action. Since we are being extremely brief with this here, we advise the reader with no prior exposure to modules with Frobenius action might want to first consult Section 2 of [Bli02]. For a thorough introduction see [Bli01], Chapter 2, or [EK99, Lyu97].

Definition 2.1. An $R[F^e]$*–module* is an R–module $\mathcal{M}$ together with an R–linear map

$$\vartheta^e : F^{e*}\mathcal{M} = R^e \otimes \mathcal{M} \longrightarrow \mathcal{M}.$$

If ϑ^e is an isomorphism, then $(\mathcal{M}, \vartheta^e)$ is called a *unit* $R[F^e]$*–module.*

By adjointness of extension and restriction of scalars for the Frobenius map on $\operatorname{Spec} R$, these maps $\vartheta^e \in \operatorname{Hom}(F^{e*}\mathcal{M}, \mathcal{M})$ are in one-to-one correspondence with maps $F^e_{\mathcal{M}} \in \operatorname{Hom}(\mathcal{M}, F^e_*\mathcal{M})$ where $F^e_{\mathcal{M}}(m) = \vartheta^e_{\mathcal{M}}(1 \otimes m)$. Thus, alternatively, the $R[F^e]$–module is determined by a map $F^e_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ satisfying the p^e–linearity condition $F^e_{\mathcal{M}}(rm) = r^{p^e} F^e_{\mathcal{M}}(m)$ for all $r \in R$ and $m \in \mathcal{M}$. Therefore, an $R[F^e]$–module is nothing but a module over the ring $R[F^e]$, where $R[F^e]$ denotes the (non commutative) ring which is obtained from R by adjoining the non commutative variable F^e to R and forcing the relations $r^{p^e} F^e = F^e r$ of p^e–linearity.

In this sense one defines the category of $R[F^e]$–modules, $R[F^e]$–mod, as the module category over this ring $R[F^e]$. As the module category over an associative ring, $R[F^e]$–mod is an abelian category. The category of unit $R[F^e]$–modules, $uR[F^e]$–mod, is the full subcategory whose objects are those $R[F^e]$–modules which are unit. Using that for a regular ring the Frobenius functor F^{e*} is exact, it follows that $uR[F^e]$–mod is also an abelian category (see [Bli01, Chapter 2] for details).

Notational convention concerning Frobenius. If $\mathcal{N}$ is an R–submodule of the $R[F^e]$–module $(\mathcal{M}, \vartheta^e, F^e)$, then $F^{e*}\mathcal{N}$ is a submodule of $F^{e*}\mathcal{M}$. Its image in $\mathcal{M}$ under ϑ^e we denote by $RF^e(\mathcal{N}) = \vartheta^e(F^{e*}\mathcal{N})$; or briefly just by $F^e(\mathcal{N})$. It is the R–submodule of $\mathcal{M}$ generated by the elements $F^e(n)$ for $n \in \mathcal{N}$. If $\mathcal{M}$ is unit, then $F^{e*}\mathcal{N}$ is, via ϑ^e, isomorphic to $RF^e(\mathcal{N})$. More specifically, on the category of R–submodules of $\mathcal{M}$, the two functors F^{e*} and $RF^e(\_\_)$ are isomorphic since $RF^e(\_\_) = \vartheta^e \circ F^{e*}(\_\_)$. It turns out to be convenient to work with $F^e(\_\_) = RF^e(\_\_)$ instead of F^{e*} as one never leaves the ambient module $\mathcal{M}$ which makes many arguments more transparent.

Definition 2.2. An $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ is called finitely generated if it is a finitely generated module over the ring $R[F^e]$.

Let $\varphi : M \longrightarrow F^{e*}M$ be an R–linear map. Consider the directed system one obtains by taking higher Frobenius powers of this map. The limit one obtains

$$\mathcal{M} = \varinjlim(M \xrightarrow{\varphi} F^{e*}M \xrightarrow{F^*\varphi} F^{2e*}M \longrightarrow \cdots \quad)$$

carries a natural unit $R[F^e]$–module structure since $F^{e*}M$ is computed by the same directed system truncated at the first map.

If a unit $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ arises in such a fashion one calls φ a *generator* of $(\mathcal{M}, \vartheta^e)$. If M is finitely generated it is called a finite generator, and if, in addition, φ is injective, then $\mathcal{M}$ is called a *root* of $\mathcal{M}$. In this case one identifies M with its isomorphic image in $\mathcal{M} = \varinjlim F^{er*}M$. Thus a root of a unit $R[F]$–module $\mathcal{M}$ is a finitely generated R–submodule M, such that $M \subseteq RF^e(M)$ and $\mathcal{M} = \bigcup_r RF^{er}(M) = R[F^e]M$. A key observation is the following proposition , see [Bli02, Proposition 2.5] or [EK99] for proof:

Proposition 2.3. *Let R be regular. A unit $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ is finitely generated if and only if $\mathcal{M}$ has a root.*

With this at hand one can easily show that the category of finitely generated unit $R[F^e]$–modules is an abelian subcategory of the category of $R[F^e]$–modules which is closed under extensions. Significantly more work (for the second part) is involved in showing the next important theorem, found as Proposition 2.7 and Theorem 3.2 in [Lyu97].

Theorem 2.4. *Let R be regular and let $\mathcal{M}$ be a finitely generated unit $R[F^e]$–module. Then $\mathcal{M}$ has ACC in the category of unit $R[F^e]$–modules*

If R is also a finitely generated algebra over a regular local ring, then $\mathcal{M}$ has DCC, i.e. $\mathcal{M}$ has finite length as a unit $R[F^e]$–module.

Examples 2.5. $F^{e*}R$ is canonically isomorphic to R via the map $r' \otimes r \mapsto r'r^{p^e}$, thus R is a unit $R[F^e]$–module. An ideal $I \subseteq R$ is an $R[F^e]$–submodule of R since $F(I) = I^{[p^e]} \subseteq I$. Note that in general I is however not a unit $R[F^e]$–submodule since the inclusion $I^{[p^e]} \subseteq I$ strict by Nakayama's lemma. Thus R is a *simple* unit $R[F^e]$–module.

Let $S \subseteq R$ be a multiplicatively closed subset of R. Then the localization $S^{-1}R$ is naturally a unit $R[F^e]$–module. Its structural map $\vartheta^e_{S^{-1}R} : R^e \otimes$

$S^{-1}R \longrightarrow S^{-1}R$ is given by sending $r' \otimes \frac{r}{s}$ to $r'(\frac{r}{s})^{p^e}$. Its inverse is the map $rs^{-1} \mapsto s^{p^e-1}r \otimes s^{-1}$. Furthermore, the natural localization map $R \longrightarrow S^{-1}R$ is a map of unit $R[F^e]$–modules.

The local cohomology modules, $H^i_I(R)$, of R with support in I can be calculated as the cohomology modules of the Čech complex,

$$\check{\mathrm{C}}(R; x_1, \dots, x_n) = R \longrightarrow \oplus R_{x_i} \longrightarrow \oplus R_{x_i x_j} \longrightarrow \dots \longrightarrow R_x$$

where $x_1, \dots, x_n$ are a set of generators of I, and x denotes the product of the x_i's. The modules of $\check{\mathrm{C}}(R; x_1, \dots, x_n)$ are localizations of R and therefore unit $R[F^e]$–modules. The maps of the Čech complex are just signed sums of localization maps and therefore maps of $R[F^e]$–modules. Thus the local cohomology modules $H^i_I(R) = \check{\mathrm{C}}(R; x_1, \dots, x_n)$ are $R[F^e]$–modules as the category of $R[F^e]$–modules is abelian. If R is regular then the local cohomology modules $H^i_I(R)$ are unit $R[F^e]$–modules as the category of unit $R[F^e]$–modules is abelian.

Also note that if (A, m) is local of dimension d, then the top local cohomology module $H^n_m(A)$ is a unit $A[F^e]$–module even if A is not regular. This follows since $H^n_m(A)$ is the cokernel of the last map of an appropriate Čech complex arising from a system of parameters of A. As F^{e*}_A is always right exact this is enough to conclude that the cokernel is in fact unit. Consequently, the injective hull $E_{A/m}$ of A/m carries a unit $A[F^e]$–structure whenever it is isomorphic to $H^d_m(A)$, *i.e.* whenever A is Gorenstein.

2.1. **Roots and base change.**

Let $\pi : R \longrightarrow S$ be a map of rings. We observe that the base change functor $\pi^* = S \otimes_R \_\_$ extends to a functor from (unit) $R[F^e]$–modules to (unit) $S[F^e]$–modules. This follows since on $R[F^e]$–modules one has

$$S \otimes_R \_\_ \cong S \otimes_R R[F^e] \otimes_{R[F^e]} \otimes \_\_ \cong S[F^e] \otimes_{R[F^e]} \_\_.$$

It is also a consequence of the fact that base change commutes with Frobenius as the following isomorphism of S–R–bimodules shows.

$$S^e \otimes_S S \cong S^e \cong S \otimes_R R^e \tag{2.1}$$

Therefore, given a (unit) $R[F]$–module $(\mathcal{M}, \vartheta^e)$, the (unit) $S[F]$–module structure on $S \otimes_R \mathcal{M}$ is given by

$$S^e \otimes_S S \otimes_R \mathcal{M} \cong S \otimes_R R^e \otimes_R \mathcal{M} \xrightarrow{\mathrm{id}_S \otimes \vartheta^e} S \otimes_R \mathcal{M}.$$

Clearly, this is an isomorphism if and only if ϑ^e is an isomorphism. In terms of the corresponding Frobenius actions we have $F^{e*}_{S \otimes \mathcal{M}} = F^{e*}_S \otimes F^{e*}_{\mathcal{M}}$. The next proposition summarizes some properties of base change for $R[F^e]$–modules.

Proposition 2.6. *Let $R \longrightarrow S$ be a map of rings. Let $\mathcal{M}$ be a finitely generated unit $R[F^e]$–module with generator M. Then $S \otimes M$ is a generator of the finitely generated unit $S[F^e]$–module $S \otimes \mathcal{M}$.*

If R and S are regular, then the image of $S \otimes M$ in $S \otimes \mathcal{M}$ is a root of $S \otimes \mathcal{M}$.

If $R \longrightarrow S$ is also flat and M is a root of $\mathcal{M}$, then $S \otimes M$ itself is a root of $S \otimes \mathcal{M}$. If $R \longrightarrow S$ is faithfully flat, then a submodule M of $\mathcal{M}$ is a root of $\mathcal{M}$ if and only if $S \otimes M$ is a root of $S \otimes \mathcal{M}$.

Proof. The first statement follows from the fact that tensor commutes with Frobenius and direct limits: By definition of generator, $\mathcal{M} = \varinjlim F^{er*}(M)$. Applying $S \otimes \_\_$ we get

$$S \otimes \mathcal{M} = S \otimes \varinjlim F_R^{er*}(M) = \varinjlim S \otimes F_R^{er*}(M) = \varinjlim F_S^{er*}(S \otimes M).$$

Thus, $S \otimes \mathcal{M}$ has the S–finitely generated generator $S \otimes \mathcal{M}$.

All members of this direct limit come with a natural map to $S \otimes \mathcal{M}$, obtained from the map $F^{er}(M) \longrightarrow \mathcal{M}$. Thus, taking images in $S \otimes \mathcal{M}$, and denoting the image of $S \otimes M$ in $S \otimes \mathcal{M}$ by $\overline{M}$, the direct limit becomes a union over all $F^{er}(\overline{M})$ (we used that, by flatness of Frobenius, images are preserved by F^{er}). In particular, $\overline{M} \subseteq F^e(\overline{M})$, thus $\overline{M}$ is a root of $S \otimes \mathcal{M}$.

The flatness of S ensures that $S \otimes M$ is already a submodule of $S \otimes \mathcal{M}$. Thus $S \otimes M$ is a root. If $S \otimes M$ is a root, then $S \otimes M \subseteq S \otimes F^e(M)$. Assuming faithfully flatness of $R \longrightarrow S$ this ensures that $M \subseteq F^e(M)$ and $\mathcal{M} = \bigcup F^{er}(M)$. Therefore M is a root of $\mathcal{M}$. □

Still fixing the data of a map of rings $\pi : R \longrightarrow S$, any $S[F^e]$–module $(\mathcal{N}, \vartheta^e)$ naturally carries an $R[F^e]$–module structure given by

$$R^e \otimes_R \mathcal{N} \xrightarrow{\pi \otimes \mathrm{id}} S^e \otimes_S \mathcal{N} \xrightarrow{\vartheta^e} \mathcal{N}.$$

This, of course, is because π induces a ring homomorphism $R[F^e] \longrightarrow S[F^e]$ and therefore an $S[F^e]$–module is, via restriction, also an $R[F^e]$–module. Note that in general, a unit $S[F^e]$–module, viewed as an $R[F^e]$–module, is *not* unit. It is unit, if and only if the natural map $R^e \otimes_R S \longrightarrow S^e$ sending $r \otimes s$ to $\pi(r)s^{p^e}$ is an isomorphism of R–S–bimodules. The following proposition summarizes a few cases where this happens:

Proposition 2.7. *Let $\pi : R \longrightarrow S$ be a map of rings. In the following cases is the natural map $R^e \otimes_R S \longrightarrow S^e$ sending $r \otimes s$ to $\pi(r)s^{p^e}$ an isomorphism.*

1. *S is the localization of R at some multiplicative set $T \subseteq R$, and π is the localization map.*
2. *R is a k–algebra, k is a field and $S = R \otimes_k K$ where K is a separable extension field of k, such that the empty set is a p–basis of K over k (e.g. K is algebraic over k, or K and k are perfect).*
3. *$\pi : R \longrightarrow S$ is an étale map with R and S reduced.*
4. *R is regular local and F–finite, and S is the I–adic completion of R with respect to some ideal I of R.*

In all these cases it follows that a unit $S[F]$–module is also unit as a $R[F]$–module.

Proof. For (1) note that if $S = T^{-1}R$, the inverse to the natural map $R^e \otimes_R T^{-1}R \longrightarrow (T^{-1}R)^e$ is given by sending rt^{-1} to $rt^{p^e-1} \otimes t^{-1}$.

In (2) we immediately reduce to showing that $k^e \otimes_k K$ and K^e are naturally isomorphic via the map $s \otimes t \mapsto st^{p^e}$. This implies (2) since then

$$\begin{aligned} R^e \otimes_R S &= R^e \otimes_R R \otimes_k K = R^e \otimes_k K = R \otimes_k k^e \otimes_k K \\ &\cong R \otimes_k K^e = S^e. \end{aligned}$$

The assumptions on $k \subseteq K$ ensure, by [Eis95], Theorem A1.3 and Theorem A1.4, that $k^{1/q} \otimes_k K = K^{1/q}$. This is just a different way of writing that $k^e \otimes_k K \cong K^e$ via the natural map described above.

For (3) observe that locally, an étale map can be factored as $R \longrightarrow R' \longrightarrow S$ where $R \longrightarrow R'$ is the spectrum of an open subset of $\operatorname{Spec} R$, and $R' \longrightarrow S$ is finite and étale. As the localization was already treated in (1) we can assume that $R \longrightarrow S$ is finite étale. Then, the argument in [HH90], Discussion 6.3 reduces to the case that R is a field and S is a separable algebraic extension. This case is treated in (2).

Let $R \longrightarrow \widehat{R}$ be the I-adic completion as in (4). By assumption, R^e is a finitely generated right R–module. Therefore $R^e \otimes_R \widehat{R} = \varprojlim R^e/R^e I^t$ by [Eis95], Theorem 7.2. Using that the sequence $R^e I^t = I^{t[p^e]}$ is cofinal within the powers I^t of I we conclude

$$R^e \otimes_R \widehat{R} \cong \varprojlim \frac{R^e}{R^e I^t} = \varprojlim \frac{R^e}{I^{t[p^e]} R^e} \cong \widehat{R}^e.$$

$\square$

As a special case of (1) of the above proposition we assume that T consists of the powers of a single element of R, *i.e.* $T^{-1}R = R_x$ for some $x \in R$. We claim that if $\mathcal{M}$ is a finitely generated unit $R_x[F^e]$–module then $\mathcal{M}$ is finitely generated as a unit $R[F^e]$–module. Observe that $R[F^e]\frac{1}{x} = R_x[F^e]$ since any fraction $\frac{r}{x^{p^{er}}}$ can be written as $rF^{er}\frac{1}{x}$. Now let $m_1, \dots, m_n$ be $R_x[F^e]$–module generators of $\mathcal{M}$. Then

$$\mathcal{M} = R_x[F^e]\langle m_1, \dots, m_n \rangle = R[F^e]\frac{1}{x}\langle m_1, \dots, m_n \rangle = R[F^e]\langle \frac{m_1}{x}, \dots, \frac{m_n}{x} \rangle.$$

Therefore $\frac{m_1}{x}, \dots, \frac{m_n}{x}$ are a set of $R[F^e]$–module generators of $\mathcal{M}$. Thus $\mathcal{M}$ is finitely generated even as an $R[F^e]$–module.

Similarly, if S is module finite and étale over R, and $\mathcal{N}$ a finitely generated S–module such that $S[F^e]\mathcal{N} = \mathcal{N}$ for an $S[F^e]$–module $\mathcal{N}$, then $R[F^e]\mathcal{N} = \mathcal{N}$ since by (3) above $RF^{er}\mathcal{N} = SF^{er}\mathcal{N}$. Since, by module finiteness of S over R, $\mathcal{N}$ is a finitely generated R–module, thus, $\mathcal{N}$ is finitely generated as a $R[F^e]$–module. We summarize these findings in a proposition.

Proposition 2.8. *Let $\mathcal{M}$ be a finitely generated (unit) $R[F^e]$–module and x an element of R. Then $\mathcal{M}_x$ is a finitely generated (unit) $R[F^e]$–module.*

If S is finite étale over R, then $S \otimes_R \mathcal{M}$ is a finitely generated (unit) $R[F^e]$–module.

We note the following global version of this result. For a thorough account on a global theory of $R[F^e]$–modules, called $\mathcal{O}_X[F^e]$–modules or $\mathcal{O}_{X,F^e}$–modules, we refer to [EK99]. On the fairly basic level we are using the global theory, the adaption from the affine case is straightforward.

Proposition 2.9. *Let $U \subseteq X = \operatorname{Spec} R$ be an open inclusion, where R is regular. Let $\mathcal{M}$ be a locally finitely generated unit $\mathcal{O}_U[F^e]$–module. Then $\mathcal{M}$ (or more precisely $i_*\mathcal{M}$ if i denotes the inclusion) is a finitely generated unit $R[F^e]$–module. The same holds for $i : U \longrightarrow X$ étale.*

Proof. Cover U by finitely many basic open sets $U_i = \operatorname{Spec} R_{f_i}$ for $f_i \in R$. Then the first map of the Čech complex of $\mathcal{M}$ associated to this cover is

$$\oplus_i \mathcal{M}|_{U_i} \longrightarrow \oplus_{i<j} \mathcal{M}|_{U_{ij}}.$$

By definition, objects appearing are finitely generated unit $R_{f_i}[F^e]$–modules ($R_{f_i f_j}[F^e]$–modules respectively), and therefore, by the last Proposition, they are finitely generated unit $R[F^e]$–modules. As this category is abelian, one concludes that the kernel of this map, $\mathcal{M}$, is also a finitely generated unit $R[F^e]$–module. With this at hand one can reduce the étale case to the finite étale case by factoring the étale map into a finite étale map followed by an open inclusion. Thus we reduced to Proposition 2.8. □

Finally, let us investigate the behavior of roots in the context of Proposition 2.7.

Corollary 2.10. *Let R be regular and $R \longrightarrow S$ such that $F_R^{e*} \cong F_S^{e*}$, e.g. one of the cases in Proposition 2.7. Let $\mathcal{M}$ be a finitely generated unit $R[F^e]$–module with root $M \subseteq \mathcal{M}$. Let $\mathcal{N}$ be a finitely generated unit $S[F^e]$–submodule of $S \otimes_R \mathcal{M}$. Then $\mathcal{N} \cap M$ is a root of the finitely generated unit $R[F^e]$–module $\mathcal{N} \cap \mathcal{M}$.*

Proof. By assumption, the finitely generated unit $S[F^e]$–modules $S \otimes_R \mathcal{M}$ and $\mathcal{N}$ are unit $R[F^e]$–modules (though quite likely *not* finitely generated as $R[F^e]$–modules). The intersection of the two unit $R[F^e]$–submodules $\mathcal{M}$ and $\mathcal{N}$ is a unit $R[F^e]$–module. As it is a submodule of the finitely generated $\mathcal{M}$, it follows that $\mathcal{M} \cap \mathcal{N}$ is a finitely generated unit $R[F^e]$–module since the category of finitely generated unit $R[F^e]$–modules is abelian.

To check that the finitely generated $N \overset{\text{def}}{=} \mathcal{N} \cap M$ is a root of $\mathcal{N}$ means that $\bigcup F_S^{er}(N) = \mathcal{N}$ and $N \subseteq F_S^e(N)$. Thus

$$F_R^e(\mathcal{N} \cap M) = F_R^e(\mathcal{N}) \cap F_R^e(M) = F_S^e(\mathcal{N}) \cap M \supseteq \mathcal{N} \cap M$$

and

$$\bigcup F_R^{er}(\mathcal{N} \cap M) = \bigcup (F_S^{er}(\mathcal{N}) \cap M) = \mathcal{N} \cap M.$$

The key point was that for the S–submodule $\mathcal{N}$ of $S \otimes \mathcal{M}$ one has

$$F_R^e(\mathcal{N}) = F_S^e(\mathcal{N})$$

by assumption. □

It is important to keep in mind that we did not exclude that case that $\mathcal{M} \cap \mathcal{N}$ is zero in the last corollary. In particular it follows that $\mathcal{N} \cap \mathcal{M} = 0$ if and only if $\mathcal{N} \cap M = 0$. Also note that $N \overset{\text{def}}{=} S \otimes M \cap \mathcal{N}$ is a root of the $S[F^e]$–module $\mathcal{N}$ and naturally $N \cap M = \mathcal{N} \cap M$ is the root of the $R[F^e]$–module $\mathcal{N} \cap \mathcal{M}$.

2.2. **The minimal root.**

Building on the last proposition and corollary we prove a result on the existence of minimal roots for regular, F–finite local rings. This was previously only known in the complete case [Lyu97, Theorem 3.5]. First we recall:

Proposition 2.11. *Let R be regular. The intersection of finitely many roots of a finitely generated unit $R[F^e]$–module $\mathcal{M}$ is a root of $\mathcal{M}$. If $\mathcal{N}$ is a unit $R[F^e]$–submodule of $\mathcal{M}$, and M a root of $\mathcal{M}$, then $M \cap \mathcal{N}$ is a root of $\mathcal{N}$.*

Proof. It is enough to show that the intersection of the two roots M_1 and M_2 is again a root. Clearly, $F^e(M_1 \cap M_2) = F^e(M_1) \cap F^e(M_2) \supseteq M_1 \cap M_2$. Since M_2 is finitely generated, it is contained in $F^{er'}(M_1)$ for some $r' > 0$, since $\mathcal{M} = \bigcup F^{er}(\mathcal{M}_1)$. Thus $F^{er}(M_2) \subseteq F^{e(r+r')}(M_1 \cap M_2)$ for all r. Therefore

$$\mathcal{M} = \bigcup F^{er}(M_2) \subseteq \bigcup F^{er}(M_1 \cap M_2),$$

which finishes the argument that $M_1 \cap M_2$ is a root of $\mathcal{M}$. For the second part, we observe that $F^e(M \cap \mathcal{N}) = F^e(M) \cap \mathcal{N} \supseteq M \cap \mathcal{N}$, and $\mathcal{N} = \mathcal{N} \cap \mathcal{M} = \mathcal{N} \cap \bigcup F^{er}(M) = \bigcup F^{er}(\mathcal{N} \cap M)$. The key point in all this was that the Frobenius commutes with finite intersections. $\square$

Theorem 2.12. *Let R be regular local and F–finite and let $\mathcal{M}$ be a finitely generated unit $R[F^e]$–module. Then $\mathcal{M}$ has a unique minimal root.*

Proof. If $\widehat{R}$ denotes the m–adic completion we are in case (4) of Proposition 2.7. By [Lyu97], Theorem 3.5, or [Bli01], Proposition 2.20, the finitely generated unit $\widehat{R}[F^e]$–module $\widehat{R} \otimes \mathcal{M}$ has a unique minimal root N. By Corollary 2.10 $M \overset{\text{def}}{=} N \cap \mathcal{M}$ is a root of the unit $R[F^e]$–module $\mathcal{M} \cap (\widehat{R} \otimes \mathcal{M}) = \mathcal{M}$. Clearly, $\widehat{R} \otimes M \subseteq N$. Since $\widehat{R} \otimes M$ is a root of $\widehat{R} \otimes \mathcal{M}$, it contains N by minimality of N. Therefore $N = \widehat{R} \otimes M$. Now it follows easily that M is indeed the unique minimal root of $\mathcal{M}$. If $M' \subseteq M$ is another root, we have, by minimality of N the inclusion of roots $N \subseteq \widehat{R} \otimes M' \subseteq \widehat{R} \otimes M$ of $\widehat{R} \otimes \mathcal{M}$. Since the first and last are equal we have that M' and M are equal upon completion. Thus $M = M'$ by faithfully flat descent. $\square$

A consequence of the above proof is the following corollary.

Corollary 2.13. *Let (R, m) be regular local and F–finite. Let $\mathcal{M}$ be a finitely generated unit $R[F]$–module with minimal root M. Then $\widehat{R} \otimes M$ is the minimal root of $\widehat{R} \otimes \mathcal{M}$.*

The question of whether minimal roots exist for not necessarily local rings R remains open.

2.3. **Duality for $R[F^e]$–modules.** A key tool in local algebra is Matlis Duality. If (R,m) is local then the Matlis dual functor is defined as $D(\_\_) \overset{\text{def}}{=} \operatorname{Hom}(\_\_, E_{R/m})$. We seek to extend $D(= D_R)$ to a Functor from $R[F^e]$–modules to $R[F^e]$–modules. How this can be done is described in [Bli01], Chapter 4, in complete detail, as a consequence of a general investigation on how to extend contravariant functors to incorporate Frobenius action. Here we only give the bare minimum to establish the extension of D, most of the material can already be found in [Lyu97], Section 4.

Proposition 2.14. *Let $f : R \longrightarrow S$ be a flat map of noetherian rings. Then the natural map*

$$\psi : S \otimes \operatorname{Hom}_R(\mathcal{M}, \mathcal{N}) \longrightarrow \operatorname{Hom}_S(S \otimes \mathcal{M}, S \otimes \mathcal{N})$$

is an isomorphism if either S is module finite over R or if $\mathcal{M}$ is finitely presented.

Proof. The second part of the proposition is well known, see, for example, [Eis95], Proposition 2.10. For lack of a reference for the first part we give the following argument. Using geometric notation to denote restriction and extension of scalars by f_* and f^*, respectively, we have to show that $f^* \operatorname{Hom}(\mathcal{M}, \mathcal{N}) \cong \operatorname{Hom}(f^*\mathcal{M}, f^*\mathcal{N})$ via the map $\psi(s \otimes \varphi) = s \cdot f^*(\varphi)$. Showing that this is an isomorphism can be done locally and thus we can assume that R is local, and therefore $S = \oplus_{i=1}^n e_i R$ is a finitely generated, free R–module. We define the map ψ' by following the diagram below

$$\begin{tikzcd}
f_* f^* \operatorname{Hom}_R(\mathcal{M}, \mathcal{N}) \arrow[r, "\psi'"] \arrow[d, "\cong"] & f_* \operatorname{Hom}_S(f^*\mathcal{M}, f^*\mathcal{N}) \\
\oplus e_i \operatorname{Hom}_R(\mathcal{M}, \mathcal{N}) \arrow[d, "\cong"] & \\
\operatorname{Hom}_R(\mathcal{M}, \oplus e_i N) \arrow[r, "\cong"] & \operatorname{Hom}_R(\mathcal{M}, f_* f^* \mathcal{N}) \arrow[uu, "\cong", "\text{adj}"']
\end{tikzcd}$$

Besides the adjointness of f^* and f_*, which is responsible for the right map, all other isomorphisms either come from the direct sum decomposition of S as an R module or from Hom commuting with finite direct sums in the second argument. Chasing through this definition, an R–module generator $e_i \otimes \varphi$ of the left hand side gets mapped as follows:

$$e_i \otimes \varphi \;\mapsto\; e_i \varphi \;\mapsto\; (e_i \cdot \_\_) \circ \varphi \;\mapsto\; (e_i \otimes \_\_) \circ \varphi \;\mapsto\; e_i \cdot f^*(\varphi)$$

Thus, by R–linearity of ψ', we have $\psi'(s \otimes \varphi) = s \cdot f^*(\varphi) = \psi(s \otimes \varphi)$ for all $s \in S$ and $\varphi \in \operatorname{Hom}_R(\mathcal{M}, \mathcal{N})$. Thus $\psi = \psi'$ which is an isomorphism by construction. □

If R is regular and F–finite, the conditions of the last proposition are satisfied for the Frobenius map $F^e : R \longrightarrow R$ and we get as a corollary:

Corollary 2.15. *Let R be regular. Then the natural map*

$$\psi : F^{e*}\operatorname{Hom}(\mathcal{M},\mathcal{N}) \longrightarrow \operatorname{Hom}(F^{e*}\mathcal{M}, F^{e*}\mathcal{N})$$

sending $r \otimes \varphi$ to $rF^{e*}(\varphi)$ is an isomorphism if R is F–finite or $\mathcal{M}$ is finitely presented.

Since our interest is in the Matlis dual functor D, we strengthen the last corollary in this special case.

Lemma 2.16. *Let R be regular complete and local. The natural map*

$$\psi_{\mathcal{M}} : F^{e*}\operatorname{Hom}(\mathcal{M}, E_R) \longrightarrow \operatorname{Hom}(F^{e*}\mathcal{M}, F^{e*}E_R)$$

is an isomorphism if R is F–finite or if $\mathcal{M}$ is finitely presented or cofinite.

In these cases we have an isomorphism of functors $\Psi : D \circ F^{e*} \cong F^{e*} \circ D$

Proof. For the first part it remains to treat the case that $\mathcal{M}$ is a cofinite R–module. We take the beginning of an injective resolution $0 \longrightarrow \mathcal{M} \longrightarrow E_1 \longrightarrow E_2$ where E_i is a finite direct sum of copies of E_R, the injective hull of R/m. By applying F^{e*} and Hom in either order to this resolution we reduce the task of showing that $\psi_{\mathcal{M}}$ is an isomorphism to the case $\mathcal{M} = E_i$. Since Hom and F^{e*} both commute with finite direct sums we further reduce to the case $\mathcal{M} = E_R$. Thus, it remains to show that $\psi : F^{e*}\operatorname{Hom}(E_R, E_R) \longrightarrow \operatorname{Hom}(F^{e*}E_R, F^{e*}E_R)$ is an isomorphism. Since R is regular E_R itself carries a natural unit $R[F^e]$–structure $F^{e*}E_R \cong E_R$ (see Examples 2.5). Via this isomorphism we identify $\operatorname{Hom}(F^{e*}E_R, F^{e*}E_R) \cong \operatorname{Hom}(E_R, E_R) \cong R$. Then the above map $\psi : R^e \otimes R \longrightarrow R$ becomes the natural unit $R[F^e]$–structure on R, in particular ψ is an isomorphism.

Combining this isomorphism with the natural unit $R[F^e]$–structure on E_R we get a natural isomorphism

$$D(F^{e*}\mathcal{M}) \cong \operatorname{Hom}(F^{e*}\mathcal{M}, F^{e*}E_R) \cong F^{e*}\operatorname{Hom}(\mathcal{M}, E_R) = F^{e*}D(\mathcal{M})$$

as desired. □

Assume that R is complete. Let $F^{e*}\mathcal{M} \xrightarrow{\vartheta^e} \mathcal{M}$ be an $R[F^e]$–module which is finitely generated or cofinite as an R–module or that R is F–finite.

Applying the Matlis dual functor $D(\_\_) = \operatorname{Hom}(\_\_, E_R)$ to the structural morphism of $\mathcal{M}$ and composing with the isomorphism of Lemma 2.16 one obtains a map

$$\beta^e : D(\mathcal{M}) \xrightarrow{D(\vartheta^e)} D(F^{e*}\mathcal{M}) \xrightarrow{\Psi_{\mathcal{M}}} F^{e*}(D(\mathcal{M}))$$

whose second part is just the isomorphism Ψ form the last lemma.

Definition 2.17. Let R be complete and $(\mathcal{M}, \vartheta^e)$ an $R[F^e]$–module (finitely generated or cofinite as an R–module, if R is not F–finite). If $\beta^e \overset{\text{def}}{=} \Psi_{\mathcal{M}} \circ D(\vartheta^e)$, then

$$\mathcal{D}(\mathcal{M}) \overset{\text{def}}{=} \varinjlim(D(\mathcal{M}) \xrightarrow{\beta^e} F^{e*}D(\mathcal{M}) \xrightarrow{F^{e*}\beta^e} F^{2e*}D(\mathcal{M}) \longrightarrow \dots \quad)$$

is the unit $R[F^e]$–module generated by β^e. On $R[F^e]$–modules which are cofinite as R–modules this defines an exact functor.

The exactness claim is clear since Matlis duality and direct limits are exact functors, and $\mathcal{D}$, as a composition of these, is therefore also exact. This extension of Matlis duality $\mathcal{D}$ was defined in [Lyu97], Chapter 4, in a slightly less general setting and denoted differently as $\mathcal{H}_{A,R}$. What follows is a summary of the properties of $\mathcal{D}$, referring to [Lyu97] or [Bli01] for most of the proofs.

If $\mathcal{M}$ is a unit $R[F^e]$–module then $\mathcal{D}(\mathcal{M}) = D(\mathcal{M})$, since β^e is an isomorphism in this case. If $\mathcal{M}$ is cofinite as an R–module then $D(\mathcal{M})$ is a finitely generated R–module. Therefore $\mathcal{D}(\mathcal{M})$ is a finitely generated unit $R[F^e]$–module, since $D(\mathcal{M})$, its generator, is a finitely generated R–module. If in addition ϑ^e is surjective, then β^e is injective and therefore $D(\mathcal{M})$ is a root of $\mathcal{D}(\mathcal{M})$.

Notation 2.18. We introduce some notation from [HS77]. An element $m \in \mathcal{M}$ of the $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ is called F–nilpotent if $F^{re}(m) = 0$ for some r. Then $\mathcal{M}$ is called *F–nilpotent* if $F^{er}(\mathcal{M}) = 0$ for some $r \geq 0$. It is possible that every element of $\mathcal{M}$ is F–nilpotent but $\mathcal{M}$ itself is not, since F–nilpotency for $\mathcal{M}$ requires that all $m \in \mathcal{M}$ are killed by the *same* power of F^e. In particular the sub $F[R^e]$–module consisting of all F–nilpotent elements $\mathcal{M}_{\text{nil}}$ need not be nilpotent in general. If ϑ^e is surjective, then $\mathcal{M}$ is called *F–full.* Note that F–fullness does not mean F^e is surjective but merely that the submodule $F^e(\mathcal{M}) = \vartheta^e(F^{e*}\mathcal{M})$ is all of $\mathcal{M}$. Finally we say that $\mathcal{M}$ is *F–reduced* if F^e acts injectively.

The above notions are the same if we view $\mathcal{M}$ as an $R[F^{er}]$–module for some $r \geq 0$. Therefore they are valid without reference to a specific e.

We are lead to some functorial constructions for $R[F^e]$–modules. The $R[F^e]$–submodule consisting of all F–nilpotent elements of $\mathcal{M}$ we denote by $\mathcal{M}_{\text{nil}} = \{\, m \in \mathcal{M} \,|\, F^{er}(m) = 0 \text{ for some } r \,\}$. The quotient $\mathcal{M}/\mathcal{M}_{\text{nil}}$ is the biggest F–reduced quotient, we denote it by $\mathcal{M}_{\text{red}}$. The $R[F^e]$–submodule $F^\infty\mathcal{M} = \bigcap F^{er}(\mathcal{M})$ is the largest F–full submodule. If $\mathcal{M}$ is a cofinite R–module, then the decreasing chain of $R[F^e]$–submodules $F^{er}(\mathcal{M})$ stabilizes and we have $F^\infty\mathcal{M} = F^{er}(\mathcal{M})$ for some $r > 0$. We note some properties in the following lemma.

Lemma 2.19. (1) *The operation of taking F–nilpotent parts is left exact. If $\mathcal{N} \subseteq \mathcal{M}$, then $\mathcal{N}_{\text{red}} \subseteq \mathcal{M}_{\text{red}}$.*

(2) *Submodules of F–reduced $R[F^e]$–modules are also F–reduced. Quotients of F–full $R[F^e]$–modules are F–full. The property of F–nilpotency passes to quotients and submodules.*

(3) *$\mathcal{M}/F^{er}(\mathcal{M})$ is F–nilpotent for all r.*

(4) *The operations $F^\infty(\_\_)$ and $(\_\_)_{\text{red}}$ mutually commute which makes the F–full and F–reduced subquotient $\mathcal{M}_{\text{fred}} = (F^\infty\mathcal{M})_{\text{red}} = F^\infty(\mathcal{M}_{\text{red}})$ of an $R[F^e]$–module $\mathcal{M}$ well defined.*

Proof. For (1), note that if $\mathcal{N} \subseteq \mathcal{M}$ then $\mathcal{N}_{\text{nil}} = \mathcal{M}_{\text{nil}} \cap \mathcal{N}$. This, together with the fact that $\mathcal{M}_{\text{nil}}$ is a submodule of $\mathcal{M}$ implies that $(\_\_)_{\text{nil}}$ is left exact. The same formula $\mathcal{M}_{\text{nil}} = \mathcal{M}_{\text{nil}} \cap \mathcal{N}$ also implies that $\mathcal{N}_{\text{red}} = \mathcal{N}/\mathcal{N}_{\text{nil}}$ is a submodule of $\mathcal{M}_{\text{red}}$. Note that $(\_\_)_{\text{red}}$ is not left exact, in general.

(2) and (3) are clear, but we point out that a quotient of an F–reduced module might not be F–reduced, and similarly, submodules of F–full $R[F^e]$–modules might not be F–full.

To show part (4) we observe that $F^\infty\mathcal{M} \twoheadrightarrow F^\infty(\mathcal{M}_{\text{red}})$ is a surjection with kernel $F^\infty\mathcal{M} \cap \mathcal{M}_{\text{nil}}$. This follows since an element $m \in F^\infty\mathcal{M}$ is mapped to zero in $F^\infty(\mathcal{M}_{\text{red}})$ if and only if the image of m in $\mathcal{M}_{\text{red}}$ is zero (the full parts $F^\infty\mathcal{M}$ are submodules of $\mathcal{M}$!). This is the case if and only if $m \in \mathcal{M}_{\text{nil}}$. On the other hand $F^\infty\mathcal{M} \cap \mathcal{M}_{\text{nil}} = (F^\infty\mathcal{M})_{\text{nil}}$ and by definition $(F^\infty\mathcal{M})_{\text{red}}$ is the cokernel of the inclusion $(F^\infty\mathcal{M})_{\text{nil}} \hookrightarrow F^\infty\mathcal{M}$. Thus we conclude that, in fact, $F^\infty(\mathcal{M}_{\text{red}}) \cong (F^\infty\mathcal{M})_{\text{red}}$. □

The relevance of these notions for the study of the extension $\mathcal{D}$ of Matlis duality is demonstrated by the following two propositions.

Proposition 2.20. *Let R be a complete, regular local ring. Let $\mathcal{M}$ be an $R[F^e]$–module which is a cofinite R–module. Then $\mathcal{M}$ is F–nilpotent if and only if $\mathcal{D}(\mathcal{M}) = 0$.*

Proof. Note that $(\mathcal{M}, \vartheta^e)$ is F–nilpotent if and only if for some $r \geq 0$ we have $F^{er}(\mathcal{M}) = 0$. Since nothing depends on the specific e here we replace e by er and assume that $F^e(\mathcal{M}) = 0$, which is equivalent to ϑ^e being the zero map. This implies that the generator $\beta^e = \Psi_{\mathcal{M}} \circ D(\vartheta^e)$ is also the zero map, and thus the module $\mathcal{D}(\mathcal{M})$ generated by β^e is zero.

Conversely, $\mathcal{D}(\mathcal{M})$ is the unit $R[F^e]$–module generated by $\beta^e : D(\mathcal{M}) \longrightarrow F^{e*}(D(\mathcal{M}))$, thus, $\mathcal{D}(\mathcal{M})$ is zero if and only if the image of $D(\mathcal{M})$ in $\mathcal{D}(\mathcal{M})$ is zero. This, in turn, is equivalent to $D(\mathcal{M}) = \bigcup \ker \beta^{er}$ where β^{er} is defined inductively by $\beta^{e(r+1)} = \beta^{er} \circ F^{er*}(\beta^e)$. Since $D(\mathcal{M})$ is finitely generated this increasing union stabilizes. Therefore, $\beta^{er} = 0$ for some $r > 0$. Up to the natural transformation $\Psi : F^{er*} \circ D \cong D \circ F^{er*}$ the map β^{er} is just $D(\vartheta^{er})$ and we conclude that also $\vartheta^{er} = 0$. This implies that $F^{er}(\mathcal{M}) = 0$, *i.e.* $\mathcal{M}$ is F–nilpotent. □

Hartshorne and Speiser show in [HS77, Proposition 1.11] that if $\mathcal{M}$ is an $R[F^e]$–module which is cofinite as an R–module, and $\mathcal{M} = \mathcal{M}_{\text{nil}}$, then, in fact, $\mathcal{M}$ is F–nilpotent, *i.e.* a uniform power of F kills all elements of $\mathcal{M}$. Thus, *a posteriori*, in the above proposition one could replace the condition that $\mathcal{M}$ be F–nilpotent with the weaker condition that $\mathcal{M} = \mathcal{M}_{\text{nil}}$. It would be interesting to know if this modified version is true for not necessarily cofinite $R[F^e]$–modules.

Following are the most important properties of $\mathcal{D}$; we only sketch the proofs since they can also be found in Section 4 of [Lyu97].

Proposition 2.21. *Let (R, m) be a regular, complete k–algebra and let $\mathcal{M}$ be a $R[F^e]$–module that is cofinite as an R–module. Then*

(1) *$\mathcal{D}(\mathcal{M}) = 0$ if and only if $\mathcal{M}$ is F–nilpotent. $\mathcal{D}(\mathcal{M}) \cong \mathcal{D}(\mathcal{N})$ if and only if $\mathcal{M}_{\mathrm{fred}} = \mathcal{N}_{\mathrm{fred}}$.*
(2) *If $\mathcal{M}$ is F–full (see Notation 2.18), then $D(\mathcal{M})$ is a root of $\mathcal{D}(\mathcal{M})$. If $\mathcal{M}$ is also F–reduced, then $D(\mathcal{M})$ is the unique minimal root.*
(3) *Every unit $R[F^e]$–submodule $\mathcal{M}'$ of $\mathcal{D}(\mathcal{M})$ arises as $\mathcal{D}(N)$ for some $R[F^e]$–submodule of $\mathcal{M}$.*
(4) *$\mathcal{D}$ is an isomorphism between the lattice of graded $R[F^e]$–modules quotients of $\mathcal{M}$ (up to $(\_\_)_{\mathrm{fred}}$) and the lattice of unit $R[F^e]$–submodules of $\mathcal{D}(\mathcal{M})$.*

Proof. (Sketch) The first part of (1) we proved as Proposition 2.20 above. The remaining part is a nontrivial consequence, using the above mentioned fact that for cofinite $\mathcal{M}$ we have that $\mathcal{M}_{\mathrm{nil}}$ is F–nilpotent and also the existence of minimal roots of Theorem 2.12, see [Bli01], Proposition 4.23. For part (2) we already observed that if ϑ^e is surjective, then $D(\mathcal{M})$ is a root. The minimality statement is a consequence of the proof of part (1), again see [Bli01], Proposition 4.23. We sketch the argument of part (3): Assuming that $\mathcal{M}$ is F–full and F–reduced, part (2) yields that $D(\mathcal{M}) \longrightarrow F^{e*}D(\mathcal{M})$ is the minimal root of $\mathcal{D}(\mathcal{M})$. Then $N' \overset{\mathrm{def}}{=} D(\mathcal{M}) \cap \mathcal{N}$ is a root of $\mathcal{M}'$. All that is left to show is that $N \overset{\mathrm{def}}{=} D(N')$ is a $R[F^e]$–module quotient of $\mathcal{M}$. $N' = D(N)$ being a root is equivalent to β^e restricting to an injection $D(N) \longrightarrow F^{e*}D(N)$. This, in turn, is equivalent to ϑ^e restricting to a surjection $F^{e*}N \longrightarrow N$, which says nothing but that N is a $R[F]$–module quotient of $\mathcal{M}$. Part (4) now follows formally from the other parts. □

As a final remark we point out that if M is a simple F–full $R[F^e]$–module, then $\mathcal{D}(M)$ is nonzero and therefore a simple unit $R[F^e]$–module. This follows since a simple $R[F^e]$–module is F–full if and only if it is F–reduced and therefore by the last Proposition $\mathcal{D}(M)$ is simple (and automatically nonzero). If F^e had a kernel it would be a nontrivial $R[F^e]$–submodule and thus if M is simple the kernel of F^e must be all of M. Thus $F^e(M) = 0$ which contradicts the F–fullness since this exactly means that $F^e(M) = M$.

2.4. **The main example: $H^d_m(A)$.** Let (R, m) be complete regular local of dimension n, let I be an ideal of height $c = n - d$. The quotient R/I we denote by A.

The top local cohomology module $H^d_m(A)$ is an $A[F^e]$–module (*cf.* Examples 2.5) and, by restriction, also an $R[F^e]$–module. As discussed in Examples 2.5 $H^d_m(A)$ is a unit $A[F^e]$–module. As an $R[F^e]$–module it is generally not unit, but at least the structural map

$$R^e \otimes_R H^d_m(R/I) \longrightarrow H^d_m(R/I)$$

is surjective. It is equivalent to the map induced by the projection $R/I^{[p^e]} \longrightarrow R/I$ under the identification of $R^e \otimes_R H^d_m(R/I)$ with $H^d_m(R/I^{[p^e]})$. Thus, by definition, $\mathcal{D}(H^d_m(R/I))$ is the limit of

$$D(H^d_m(R/I)) \longrightarrow D(H^d_m(R/I^{[p^e]})) \longrightarrow D(H^d_m(R/I^{[p^{2e}]})) \longrightarrow \dots \tag{2.2}$$

Using local duality [BH98, Theorem 3.5.8] for the complete, regular and local ring R this directed sequence is isomorphic to the following

$$\operatorname{Ext}^c_R(R/I,R) \longrightarrow \operatorname{Ext}^c_R(R/I^{[p^e]},R) \longrightarrow \operatorname{Ext}^c_R(R/I^{[p^{2e}]},R) \longrightarrow \dots \tag{2.3}$$

where, again, the maps are the ones induced from the natural projections. Since the Frobenius powers of an ideal are cofinite within the normal powers, we get that the limit of this sequence is just $H^{n-i}_I(R)$. This is because an alternative definition of $H^{n-i}_I(R)$ is as the right derived functor of the functor $\Gamma_I(M) = \varinjlim \operatorname{Hom}(R/I^t,R)$ of sections with support in $\operatorname{Spec} R/I$.[1] The not very serious issue on whether the unit $R[F^e]$–structure on $H^c_I(R)$ coming from the computation via Ext's is the same as the one coming from the Čech complex is dealt with in [Lyu97], Propositions 1.8 and 1.11. Summarizing we get:

Proposition 2.22. *Let (R,m) be regular, local, complete and F–finite. Let $A = R/I$ for some ideal I of R of height $c = n-d$. Then*

$$\mathcal{D}(H^d_m(R/I)) \cong H^c_I(R)$$

as unit $R[F^e]$–modules.

By definition of $\mathcal{D}(\_\_)$, a root for $\mathcal{D}(H^d_m(A))$ is given by

$$\beta^e : \operatorname{Ext}^c_R(R/I,R) \longrightarrow \operatorname{Ext}^c_R(R/I^{[p^e]},R) \xrightarrow{\cong} R^e \otimes \operatorname{Ext}^c_R(R/I,R) \tag{2.4}$$

where the first part is induced from the surjection $R/I^{[p^e]} \longrightarrow R/I$, and the second is the isomorphism coming from the natural transformation $\Psi : R^e \otimes \operatorname{Hom}(\_\_,R) \cong \operatorname{Hom}(R^e \otimes \_\_,R)$, see Corollary 2.15. It is straightforward that this natural transformation for Hom induces a natural transformation on its right derived functors, the Ext's.

If we drop the assumption of completeness in the preceding discussion, and just assume that (R,m) is local we still have that $H^c_I(R)$ arises as the direct limit of

$$\operatorname{Ext}^c_R(R/I,R) \longrightarrow \operatorname{Ext}^c_R(R/I^{[p^e]},R) \longrightarrow \operatorname{Ext}^c_R(R/I^{[p^{2e}]},R) \longrightarrow \cdots$$

where the maps are induced from the natural projections $R/I^{[p^{er}]} \longrightarrow R/I^{[p^{e(r-1)}]}$. Together with the natural transformation identifying $\operatorname{Ext}^c_R(R/I^{[p^{er}]},R)$ with $R^{er} \otimes \operatorname{Ext}^c_R(R/I,R)$ this shows that $\operatorname{Ext}^c_R(R/I,R)$ is a generator for $H^c_I(R)$. Upon completion we get the generator

$$\operatorname{Ext}^c_{\widehat{R}}(\widehat{R}/I\widehat{R},\widehat{R}) \xrightarrow{\beta^e} \widehat{R}^e \otimes \operatorname{Ext}^c_{\widehat{R}}(\widehat{R}/I\widehat{R},\widehat{R})$$

of $H^c_{I\widehat{R}}(\widehat{R}) \cong \widehat{R} \otimes H^c_I(R)$. We also used Proposition 2.14 applied to the completion map to see that $\widehat{R} \otimes \operatorname{Ext}^c_R(R/I,R) \cong \operatorname{Ext}^c_{\widehat{R}}(\widehat{R}/I\widehat{R},\widehat{R})$.

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[1] See [BH98, Theorem 3.5.6] for the equivalence with our definition of local cohomology via Čech complexes.

3. Brief tight closure review

Tight closure is a powerful tool in commutative algebra introduced by Mel Hochster and Craig Huneke about fifteen years ago [HH88]. There is a strong connection between the singularities arising in the minimal model program and singularities obtained from tight closure theory [Smi97b]. One of the most significant is the equivalence of the notions of rational singularity and F–rational type which was established by Smith [Smi97a] and Hara [Har98], and independently by Mehta and Shrinivas [MS97]. The notion of F–rationality arises naturally from tight closure: the local ring (A, m) is called F–rational if all ideals I generated by a full system of parameters are tightly closed, *i.e.* $I = I^*$. In this section we briefly review the tight closure theory needed for our local construction of $\mathcal{L}(Y, X)$ given below. For a more detailed introduction to this beautiful subject we recommend [Smi01, Hun96] and later the more technical original papers [HH90, HH89].

Let A be a noetherian ring. We denote by A° the subset of elements of r that are not contained in any minimal prime of A. Let $N \subseteq M$ be a submodule of M. We denote by $N^{[p^e]}$ the image of $F^{e*}N$ in $F^{e*}M$. The tight closure N^*_M of N inside of M is defined as follows:

Definition 3.1. Let A be noetherian and $N \subseteq M$. The tight closure N^*_M (or just N^* if M is clear from the context) consists of all elements $m \in M$, such that there exists a $c \in A^\circ$, such that for all $e \gg 0$

$$c \otimes m \in N^{[p^e]}.$$

Here $N^{[p^e]}$ denotes the image of $F^{e*}N$ in $F^{e*}M$ and $c \otimes m$ is an element of $F^{e*}M$.

If $N = I$ is just an ideal of A, the definition is much more transparent. In this case $r \in A$ is in I^* if and only if there is $c \in A^\circ$ such that $cr^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. A module is tightly closed if $N^* = N$. We have that $N \subseteq N^*$ as one expects from a decent closure operation. If N is noetherian, then $N^* = (N^*)^*$. There are two related closure operations which are important for us.

Definition 3.2. Let $N \subseteq M$ be A–modules. The *finitistic tight closure* of N inside of M consists of all elements $m \in (N \cap M_0)^*_{M_0}$ for some finitely generated $M_0 \subseteq M$. It is denoted by N^{*fg}_M.

The *Frobenius closure* N^F_M consists of all elements $m \in M$ such that $1 \otimes m \in N^{[p^e]}$ for some $e \geq 0$.

We immediately see that $N^{*fg} \subseteq N^*$ and that equality holds if M is finitely generated. Clearly, $N^F \subseteq N^*$. For the zero submodule of the top local cohomology module of an excellent, local, equidimensional ring A, the finitistic tight closure is equal to the tight closure, *i.e.* $0^{*fg}_{H^d_m(A)} = 0^*_{H^d_m(A)}$ (see [Smi93, Proposition 3.1.1]). In general, it is a hard question to decide if the tight closure equals the finitistic tight closure, and it is related to aspects of the localization problem in tight closure theory (*cf.* [LS01]).

As our focus lies on modules with Frobenius actions we study the above closure operations in this case more closely. The following is an important proposition which is proved in [LS01], Proposition 4.2.

Proposition 3.3. *Let A be noetherian and let (M, ϑ^e) be an $A[F^e]$–module. If N is a $A[F^e]$–submodule, then so are N^*_M, N^{*fg}_M and N^F_M.*

This is checked by observing that $(N^*)^{[p^e]} \subseteq (N^{[p^e]})^*$. Then apply ϑ^e and use the easily verifiable fact that $\vartheta^e(\_^*) \subseteq \vartheta^e(\_)^*$ to see that

$$F^e(N^*) = \vartheta^e((N^*)^{[p^e]}) \subseteq \vartheta^e((N^{[p^e]})^*) \subseteq (F^e(N))^* \subseteq N^*$$

which finishes the argument. From this we get as an immediate corollary that the tight closure of the zero $A[F^e]$–submodule is a Frobenius stable submodule of any $A[F^e]$–module.

Corollary 3.4. *Let A be a ring and let (M, F^e) be an $A[F^e]$–module. Then 0^{*fg}_M, 0^*_M and $0^F_M = M_{\mathrm{nil}}$ are $A[F^e]$–submodules of M.*

3.1. **Test ideals and test modules.** The elements "c" occurring in the definition of tight closure play a special role. Those amongst them, that work for all tight closure tests for all submodules of all finitely generated A–modules are called the test elements of A.

Definition 3.5. An element $c \in A^\circ$ is called a *test element* if for all submodules $N \subseteq M$, of every finitely generated A–module M, we have $cN^*_M \subseteq N$. A test element is called *completely stable test element* if its image in the completion of every local ring of A is a test element.

It is shown in [HH90, Proposition 8.33], that it is enough to range over all ideals of A in this definition, *i.e.* c is a test element if and only if for all ideals I and all $x \in I^*$ we have $cx^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. Thus, the test elements are those elements c occurring in the definition of tight closure which work for all tight closure memberships of all submodules of all finitely generated A–modules. A nontrivial key result is that in most cases, test elements (and even completely stable test elements) exist:

Proposition 3.6. *Let A be reduced and of finite type over an excellent local ring. Then A has completely stable test elements. Specifically, any element $c \in A^\circ$ such that A_c regular has a power which is a completely stable test element.*

The proof of this is quite technical and can be found in Chapter 6 of [HH89]. Results in lesser generality (for example, when A is F–finite) are obtained fairly easily: for a good account see [Smi01, Hun96].

The ideal τ_A generated by all test elements is called the test ideal. As remarked, $\tau_A = \bigcap (I :_A I^*)$ where the intersection ranges over all ideals I of A. This naturally leads one to consider variants of the test ideal by restricting the class of ideals this intersection ranges over. The *parameter test ideal* of a local ring (A,m) is the ideal $\widetilde{\tau}_A = \bigcap (I :_A I^*)$ where the intersection ranges over all ideals generated by a full system of parameters. If A is Cohen–Macaulay, it follows from the definition of $H^d_m(A)$ as $\varinjlim A/(x_1,\dots,x_d)^{[p^e]}$ that $\widetilde{\tau}_A = \operatorname{Ann}_A(0^*_{H^d_m(A)})$ [Smi93, Proposition 4.1.4] where $x_1,\dots,x_d$ is a system of parameters for the local ring (A,m). If A is only an excellent domain, then $\widetilde{\tau}_A \subseteq \operatorname{Ann}_A(0^*_{H^d_m(A)})$. Further generalizing, the *parameter test module* is defined as $\tau_{\omega_A} = \operatorname{Ann}_{\omega_A} 0^*_{H^d_m(A)} = \omega_A \cap \operatorname{Ann}_{\omega_{\widehat{A}}} 0^*_{H^d_m(\widehat{A})}$ where the action of ω_A on $H^d_m(A)$ is the one coming from the Matlis duality pairing $H^d_m(A) \times \omega_{\widehat{A}} \longrightarrow E_A$. Of course we require here that A has a canonical module.

Lemma 3.7. *Let A be reduced, excellent, local and equidimensional with canonical module ω_A. If c is a parameter test element, then $c\omega_A \subseteq \tau_{\omega_A}$. In particular, τ_{ω_A} is nonzero.*

Proof. Let c be a parameter test element. In particular, c annihilates the finitistic tight closure of zero in $H^d_m(A)$. Therefore, for every $\varphi \in \omega_A$ and $\eta \in 0^*_{H^d_m(A)} = 0^{*fg}_{H^d_m(A)}$ we have $c\varphi\cdot\eta = \varphi\cdot(c\eta) = \varphi\cdot 0 = 0$ where "$\cdot$" represents the Matlis duality pairing. This shows that $c\omega_A \subseteq \tau_{\omega_A}$. The hypotheses on A ensure by [HH94, Remark 2.2(e)] that the canonical module is faithful, *i.e.* $c\omega_A \neq 0$. Therefore the last part of the lemma follows from the existence of test elements (Proposition 3.6), since a test element is also a parameter test element. $\square$

3.2. ***F*-rationality and local cohomology.** The tight closure of zero in the top local cohomology module $H^d_m(A)$ of a local ring (A,m) plays a role as the obstruction to F–rationality of A. Its distinguishing property is that it is the maximal proper $A[F^e]$–submodule of $H^d_m(A)$. Precisely the following is the case:

Theorem 3.8. *Let (A,m) be reduced, excellent and analytically irreducible. Then, the tight closure of zero, $0^*_{H^d_m(A)}$, in $H^d_m(A)$ is the unique maximal proper $A[F^e]$–submodule of $H^d_m(A)$.*

The quotient $H^d_m(A)/0^*_{H^d_m(A)}$ is a nonzero simple F–reduced and F–full $A[F^e]$–module.

Proof. The case $e = 1$ of the first part was shown by Smith in [Smi93], Theorem 3.1.4. The case $e \geq 1$ can be obtained similarly, see [Bli01], Theorem 5.9. Because $0^*_{H^d_m(A)}$ is the maximal proper $A[F^e]$–submodule, $H^d_m(A)/0^*_{H^d_m(A)}$ is a simple $A[F^e]$–module quotient. It remains to show that it is F–reduced (a simple $A[F^e]$–module is F–full if and only if it is F–reduced). For this note that the kernel of F is a $A[F^e]$–submodule and, by simplicity, it must either be zero (F–reduced) or all of $H^d_m(A)/0^*_{H^d_m(A)}$. In the second case, this implies that $F(H^d_m(A)) \subseteq 0^*_{H^d_m(A)}$. Since $H^d_m(A)$ is a unit $A[F^e]$–module (enough that the structural map ϑ is surjective) we have that $F(H^d_m(A)) = H^d_m(A)$. This contradicts the fact that $0^*_{H^d_m(A)}$ is a proper submodule. Thus the quotient is F–reduced and F–full. $\square$

To avoid the assumption of analytically irreducible we give a version of the above for the case that A is an excellent equidimensional ring. As the statement is about $H^d_m(A)$ which does not discriminate between A and its completion, we state the result for a complete A; in general one has to take the minimal primes of the completion of A in the statement below.

Corollary 3.9. *Let A be a complete, local, reduced and equidimensional ring of dimension d. Let $P_1, \dots, P_k$ be the minimal primes of the A. Then the maximal proper $A[F^e]$–submodules are precisely*

$$M_i \overset{\text{def}}{=} \ker(H^d_m(A) \longrightarrow H^d_m(A/P_i)/0^*_{H^d_m(A/P_i)})$$

where $i = 1 \dots k$. Furthermore, the tight closure of zero, $0^*_{H^d_m(A)}$, in $H^d_m(A)$ is the intersection of all maximal proper $A[F^e]$–submodules. Even though $H^d_m(A)/0^*_{H^d_m(A)}$ night not be simple as an $A[F^e]$–module, it is still F–full and F–reduced.

Proof. Since tight closure can be checked modulo minimal primes, the last statement is immediate.[2] By the last Theorem $0^*_{H^d_m(A/P_i)}$ is the maximal proper $A[F^e]$–submodule. Thus $H^d_m(A)/M_i \cong H^d_m(A/P_i)/0^*_{H^d_m(A/P_i)}$ is simple. Thus M_i is a maximal proper $A[F^e]$–submodule. To check that the M_i's are all the maximal proper $A[F^e]$–submodule let M be a $A[F^e]$–submodule of $H^d_m(A)$ not contained in any of the M_i. This implies that for all i the image of M in $H^d_m(A/P_i) = H^d_m(A)/P_iH^d_m(A)$ is all of $H^d_m(A/P_i)$ (it is an $A[F^e]$–module not contained in $0^*_{H^d_m(A/P_i)}$, thus must be all of $H^d_m(A/P_i)$ by last Theorem). But this implies, by the following lemma, that $M = H^d_m(A)$ and we are done.

It remains to remark that a possible kernel of F^e on $H^d_m(A)/0^*_{H^d_m(A)}$ would reduce to all of $H^c_m(A/P_i)/0^*_{H^c_m(A/P_i)}$ for some i and imply that $F(H^c_m(A/P_i)) \subseteq 0^*_{H^c_m(A/P_i)}$ which is a contradiction to F–fullness of $H^c_m(A/P_i)$. $\square$

Lemma 3.10. *Let A be a noetherian ring, and let $M \subseteq H$ be an A–modules such that for every minimal prime P of A one has $M + PH = H$. Then $M = H$.*

Proof. One immediately reduces to the case $M = 0$. Successive application of the assumption $H = PH$ implies that

$$H = (P_1 \cdot \dots \cdot P_k)^n H$$

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[2] This is generally proved for the tight closure of ideals in the literature (see [HH90], Proposition 6.25), but the same proof can be adapted for submodules.

where the P_i's are the minimal primes. But for large enough n, a power of the product of all minimal primes is zero, thus $H = 0$. □

If A is Cohen–Macaulay, the vanishing of the tight closure of zero in $H^d_m(A)$ characterizes F–rationality of A by [Smi97a], Theorem 2.6. By definition, A is called F–rational if and only if every ideal that is generated by a system of parameters is tightly closed.

4. The intersection homology module

First we give a detailed proof of the main existence theorem as sketched in the introduction.

Theorem 4.1. *Let X be an irreducible smooth k–scheme, essentially of finite type over k, and let Y be a closed irreducible subscheme of codimension c. Then $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ has a unique simple $\mathcal{D}_X$–submodule $\mathcal{L}(Y, X)$. This submodule is also the unique simple $\mathcal{O}_X[F^e]$–module and agrees with $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ on the complement of any closed set containing the singular locus of Y.*

Proof. Write $Z = \operatorname{Sing} Y$ and $Y' = Y - Z$ and $X' = X - Z$ and denote the open inclusion $X' \subseteq X$ by i. First we assume that the characteristic of k is positive; at the end of the proof we indicate how the proof is adapted to characteristic zero.

We first show that $H^c_{Y'}(X', \mathcal{O}_{X'})$ is simple as a unit $\mathcal{O}_X[F^e]$–module: Quite generally we note that, $\mathcal{O}_{Y'}$ is a simple unit $\mathcal{O}_{Y'}[F^e]$–module by observing that a nontrivial ideal $\mathcal{I} \subsetneq \mathcal{O}_{Y'}$ is never a unit submodule as the containment $\mathcal{I}^{[p^e]} \subsetneq \mathcal{I}$, *cf.* Examples 2.5. Using that Y' is smooth and irreducible, it follows that $\mathcal{O}_{Y'}$ is also simple as a $\mathcal{D}_{Y'}$–module. This can be reduced, by étale invariance of Dd'_Y, to the case $Y' = \operatorname{Spec}(k[x_1, \ldots, x_d])$ where one can check it by hand. Under Kashiwara's equivalence for $\mathcal{D}_X$–modules ([Haa88]), and for unit $\mathcal{O}_X[F^e]$–modules ([EK99], Theorem 5.10.1 or [Lyu97], Proposition 3.1), the module $\mathcal{O}_{Y'}$ corresponds to $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$ (*cf.* [EK99], Example 5.11.6). Therefore, $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$, is a simple $\mathcal{D}_{X'}$–module (simple unit $\mathcal{O}_{X'}[F]$–module, respectively).

Since $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ is a locally finitely generated unit $\mathcal{O}_X[F^e]$–module and therefore, by [Lyu97], Theorem 5.6, it has finite length as a $\mathcal{D}_X$–module. This assures the existence of simple $\mathcal{D}_X$–submodules of $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two such. Observe the exact sequence (see [Har67], Chapter 1)

$$0 = \mathcal{H}^c_Z(X, \mathcal{O}_X) \longrightarrow \mathcal{H}^c_Y(X, \mathcal{O}_X) \longrightarrow \mathcal{H}^c_{Y'}(X, \mathcal{O}_X) \cong i_*\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$$

where the last isomorphism is by excision and the vanishing of the first module is because the codimension of Z in X is strictly bigger than c. From this it follows that $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ and therefore $\mathcal{L}_i$ are submodules of $i_*\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$. By adjointness of restriction and extension we have

$$0 \neq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_i, i_*\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})) \cong \operatorname{Hom}_{\mathcal{O}_{X'}}(\mathcal{L}_i|_{X'}, \mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'}))$$

which shows that $\mathcal{L}_i|_{X'}$ are nonzero submodules of $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$. By simplicity of the latter all three have to be equal. In particular, the intersection of $\mathcal{L}_1$ with $\mathcal{L}_2$ is nonzero. As both are simple, this implies that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}(Y, X)$ as claimed. Furthermore, since $F^e(\mathcal{L}(Y, X))$ is also simple, it follows from the uniqueness that $F^e(\mathcal{L}(Y, X)) = \mathcal{L}(Y, X)$ and therefore $\mathcal{L}(Y, X)$ is also the unique simple $\mathcal{O}_X[F^e]$–submodule for all e.

Essentially the same proof works in characteristic zero. The key fact then is that $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ is a holonomic $\mathcal{D}_X$–module and that holonomic modules have finite length. Also observe that for the smooth Y' the structure sheaf $\mathcal{O}_{Y'}$ is $\mathcal{D}_{Y'}$–simple which is well known and easy to check by hand for the case $Y' = \operatorname{Spec}(k[x_1, \dots, x_d])$. Then Kashiwara's equivalence implies that the corresponding $\mathcal{H}^c_{Y'}(X', \mathcal{O}'_X)$ is a simple $\mathcal{D}_{X'}$–module. For all of the above statements, see [BGK+87]. □

This proof is pretty much identical for zero and positive characteristic. The metaresults which are used though are proved by very different techniques in each case.

If Y is not irreducible, the above result breaks down since then $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$ is no longer simple. In the case that Y is equidimensional one can give a complete description of the simple submodules of $\mathcal{H}^c_Y(X, \mathcal{O}_X)$. They correspond to the irreducible components $Y_1, \dots, Y_k$ of Y. For each component we have an inclusion

$$\mathcal{L}(Y_i, X) \subseteq H^c_{Y_i}(X, \mathcal{O}_X) \subseteq H^c_Y(X, \mathcal{O}_X)$$

which establishes $\mathcal{L}(Y_i, X)$ as simple submodules of $H^c_Y(X, \mathcal{O}_X)$. That the right map is an inclusion uses equidimensionality and follows from [Har67], Proposition 1.9 and Chapter 3. To see that these are all the simple submodules of $\mathcal{H}^c(X, \mathcal{O}_X)$ we show that any submodule $\mathcal{N}$ of $H^c_Y(X, \mathcal{O}_X)$ does contain one of the $\mathcal{L}(Y_i, X)$. At least for one i, the restriction of $\mathcal{N}$ to a open subset of X containing Y_i but none of the other components is a nonzero submodule of $H^c_{Y_i}(X, \mathcal{O}_X)$ (using excision). But then $\mathcal{N}$ clearly contains $\mathcal{L}(Y_i, X)$ by its uniqueness and simplicity. We get as a corollary:

Corollary 4.2. *Let Y be an equidimensional reduced subscheme of the smooth k–variety X of codimension c. Let $Y = Y_1 \cup \ldots \cup Y_k$ be its decomposition into irreducible components. Then the simple $\mathcal{D}_X$–submodules of $H^c_Y(X, \mathcal{O}_X)$ are precisely the $\mathcal{L}(Y_i, X)$ for $i = 0, \dots, k$. In this case we denote by $\mathcal{L}(Y, X)$ the (direct) sum of all the $\mathcal{L}(Y_i, X)$. Furthermore, away from the singular locus of Y we have that $\mathcal{L}(Y, X)$ agrees with $\mathcal{H}^c_Y(X, \mathcal{O}_X)$.*

The similarity of Theorem 3.8 and Corollary 3.9 with Theorem 4.1 and the last corollary was the original motivation which lead to the construction of $\mathcal{L}(Y, X)$ which is given in the next section.

Note that by the uniqueness of $\mathcal{L}(Y, X)$ it is clear that it localizes, *i.e.* if U is a open subset of X then $\mathcal{L}(Y, X)|_U = \mathcal{L}(Y \cap U, U)$.

In the case of positive characteristic, we state the following slightly stronger local version of the last theorem.

Theorem 4.3. *Let R be a regular local ring of positive characteristic. Let $A = R/I$ be equidimensional of codimension c in R. Then the sum of all simple unit submodules of $H^c_I(R)$ is*

$$\mathcal{L}(A, R) = \oplus \mathcal{L}(R/P_i, R)$$

where the P_i range over the primes minimal over I and $\mathcal{L}(R/P_i, R)$ is the unique simple unit $\mathcal{O}_X[F]$–submodule of $H^c_{P_i}(R)$. Moreover, if $f \in R$ is such that A_f is regular, then $\mathcal{L}(A, R)_f \cong H^c_I(R)_f$. If R is F–finite the same holds for the D_R–module structure.

Proof. The local results of [Lyu97] are not restricted to finitely generated algebras over a field. With the assumption above (sufficient to ensure that a finitely generated unit $R[F^e]$–module has finite length as such) the previous proof goes through verbatim. $\square$

4.1. **Local construction of $\mathcal{L}(A, R)$ in positive characteristic.** From now on we assume that k is of positive characteristic and R is F–finite. With the last theorem we can dispose of the D_R–structure and entirely work with the $R[F^e]$–structure in our investigation $\mathcal{L}(A, R)$. As the construction of $\mathcal{L}(Y, X)$ which we are about to present is local in nature, the language is adjusted accordingly. Moreover, the local construction of $\mathcal{L}(A, R)$ can be reduced to the complete case with help of the results on minimal roots of Section 2.2. Thus we assume for now that (R, m) is a complete, regular local and F–finite, and that $A = R/I$ equidimensional of codimension c. The philosophy behind the description of the simple unit $R[F^e]$–module $\mathcal{L}(A, R)$ is to identify its minimal root. As it turns out, the minimal root of $\mathcal{L}(A, R)$ is the parameter test module, which, under Matlis duality, corresponds by definition to the tight closure of zero $0^*_{H^d_m(A)}$ in $H^d_m(A)$.

Theorem 4.4. *Let (R, m) be a complete regular local and F–finite and let $A = R/I$ be equidimensional and of codimension c. Then, we have that*

$$\mathcal{L}(A, R) = \mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)})$$

where $0^*_{H^d_m(A)}$ is the tight closure of zero in $H^d_m(A)$.

Proof. First assume that A is a domain. By Theorem 3.8 the unique simple $R[F^e]$–module quotient of $H^d_m(A)$ is $H^d_m(A)/0^*_{H^d_m(A)}$. Therefore

$$\mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)}) \subseteq \mathcal{D}(H^d_m(A)) \cong H^c_I(R)$$

is a nonzero simple unit $R[F^e]$–submodule of $H^d_m(A)$. As $\mathcal{L}(A, R)$ is the unique such they are equal.

If A is only equidimensional, let $P_1, \dots, P_k$ be its minimal primes. In Corollary 3.9 we show that

$$0^*_{H^d_m(A)} = \ker(H^d_m(A) \longrightarrow \oplus_{i=1}^k H^d_m(A/P_i)/0^*_{H^d_m(A/P_i)})$$

Applying the functor $\mathcal{D}$ and using the domain case for A/P_i as just proved one checks that

$$\mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)}) = \oplus_{i=1}^k \mathcal{L}(A/P_i, R) = \mathcal{L}(A, R)$$

where the last equality is by definition. □

A more careful investigation of the construction of $\mathcal{L}(A, R)$ via the duality functor $\mathcal{D}$ shows its connection with the parameter test module. By definition, $\mathcal{D}(H^d_m(A)/0^*) = \mathcal{L}(A, R)$ is the unit $R[F^e]$–module generated by the Matlis dual of the $R[F^e]$–module structure on $H^d_m(A)/0^*_{H^d_m(A)}$. The Matlis dual of $H^d_m(A)$ is the canonical module $\omega_A = \operatorname{Ext}^c_R(A, R)$ of A. The dual of $H^d_m(A)/0^*_{H^d_m(A)}$ is found as the annihilator of $0^*_{H^d_m(A)}$ under the Matlis duality pairing $\omega_A \times H^d_m(A) \longrightarrow E_{R/m}$. By definition, this annihilator $\operatorname{Ann}_{\omega_A} 0^*_{H^d_m(R)}$ is the parameter test module τ_{ω_A}. Thus the inclusion of unit $R[F^e]$–modules $\mathcal{L}(A, R) \subseteq H^c_I(R)$ arises as the limit of the following map between their generators:

$$\begin{tikzcd}
\omega_A \arrow[r] & R^e \otimes \omega_A \arrow[r] & \cdots & = H^c_I(R) \\
\tau_{\omega_A} \arrow[u, hook] \arrow[r, hook] & R^e \otimes \tau_{\omega_A} \arrow[u, hook] \arrow[r, hook] & \cdots & = L(A, R) \arrow[u, hook]
\end{tikzcd} \tag{4.1}$$

Since, $H^d_m(A)/0^*_{H^d_m(A)}$ is F–reduced and F–full, the bottom map $\tau_{\omega_A} \hookrightarrow R^e \otimes \tau_{\omega_A}$ is, by Proposition 2.21(2), the unique minimal root of $\mathcal{L}(A, R)$.

Proposition 4.5. *Let (R, m) be complete regular local and F–finite. Let $A = R/I$ be equidimensional of codimension c in R. Then the parameter test module τ_{ω_A} is the unique minimal root of $\mathcal{L}(A, R)$.*

Now we drop the assumption that R be complete and only assume it to be regular, local and F–finite.

Theorem 4.6. *Let (R, m) be regular local and F–finite. Let $A = R/I$ be a domain of codimension c. Then $\mathcal{L}(A, R)$, the unique simple unit $R[F^e]$–submodule of $H^c_I(R)$, has the parameter test module τ_{ω_A} as its minimal root.*

Furthermore, upon completion, $\widehat{R} \otimes \mathcal{L}(A, R) = \mathcal{L}(\widehat{A}, \widehat{R})$.

Proof. By Corollary 2.10, $\mathcal{N} \overset{\text{def}}{=} L(\widehat{A}, \widehat{R}) \cap H^c_I(R)$ is a unit $R[F^e]$-submodule of $H^c_I(R)$. A root of $\mathcal{N}$ is found by intersecting $\mathcal{L}(\widehat{A}, \widehat{R})$ with the root ω_A of $H^c_I(R)$. Again by Corollary 2.10 this intersection is equal to $\tau_{\omega_A} = \omega_A \cap \tau_{\omega_{\widehat{A}}}$ which is the nonzero parameter test module by Lemma 3.7. By simplicity of $\mathcal{L}(A, R)$ it is therefore contained in $L(\widehat{A}, \widehat{R}) \cap H^c_I(R)$, or put differently,

$$\widehat{R} \otimes \mathcal{L}(A, R) \subseteq \mathcal{L}(\widehat{A}, \widehat{R}). \tag{4.2}$$

If R were a domain we would be done. Unfortunately, the completion of a domain might not be a domain, but at least it is equidimensional. If $P_1, \dots, P_k$ are the primes minimal over $I\widehat{R}$, then Theorem 4.3 states that

$\mathcal{L}(\widehat{A},\widehat{R}) = \oplus\mathcal{L}(\widehat{R}/P_i,\widehat{R})$. It remains to show that $\mathcal{L}(\widehat{R}/P_i,\widehat{R}) \subseteq \widehat{R}\otimes\mathcal{L}(A,R)$. This is, by $\mathcal{L}(\widehat{R}/P_i,\widehat{R})$ being the unique simple submodule of $\mathcal{H}^c_{P_i}(\widehat{R})$, equivalent to $\widehat{R}\otimes\mathcal{L}(A,R)\cap\mathcal{H}^c_{P_i}(\widehat{R}) \neq 0$. To see this let $f \in R - P_i$ such that A_f is regular, then, by the last part of Theorem 4.3,

$$\widehat{R}\otimes\mathcal{L}(A,R)_f = \widehat{R}\otimes H^c_I(R)_f = H^c_{I\widehat{R}}(\widehat{R})_f \supseteq H^c_{P_i}(\widehat{R})_f \neq 0$$

which shows the reverse inclusion of (4.2). The statement about the parameter test module now follows from the beginning of the proof as we just showed that $\mathcal{L}(A,R) = \mathcal{N}$. □

As done before this proof can be adjusted to work for equidimensional A from the start. For simplicity we treated the domain case and will do so from now on.

Remark 4.7. This construction of $\mathcal{L}(A,R)$ from its minimal root τ_{ω_A} enables one to explicitly construct D_R–module generators for $\mathcal{L}(A,R)$: The image of any element of τ_{ω_A} in $H^c_I(R)$ is a generator of $\mathcal{L}(A,R)$, in particular, if $c \in R$ is a test element such that $c^n\cdot\eta \neq 0$ for $\eta\in\omega_A$, then $c\cdot\eta$ generates $\mathcal{L}(A,R)$ as a D_R–module.

As another consequence of Corollary 2.10 one sees that the minimal root of $\mathcal{L}(A,R)$ is $\tau_{\omega_{\hat{A}}}$, the minimal root of $\mathcal{L}(\widehat{A},\widehat{R})$, intersected with ω_A, the root of $H^c_I(R)$. By definition, this is the parameter test module τ_{ω_A} of A. Using Theorem 2.12 it follows immediately that the parameter test module commutes with completion, which was to the best of our knowledge, unknown until now. We state this as a Proposition.

Proposition 4.8. *Let A be a domain which is a quotient of a regular local and F–finite ring. Then the parameter test module commutes with completion, i.e. $\tau_{\omega_{\hat{A}}} = \widehat{A}\otimes\tau_{\omega_A}$.*

4.2. **Simplicity criteria for $H^c_I(R)$.** With the connection between the tight closure of zero in $H^d_m(A)$ and the simple unit $R[F^e]$–module $\mathcal{L}(A,R)$ just derived, the characterization of Smith showing that $0^*_{H^d_m(A)}$ governs the F–rationality of A, easily implies a simplicity criteria for $H^c_I(R)$.

Theorem 4.9. *Let R be regular local and F–finite. Let I be an ideal such that $A = R/I$ is a domain. Then $H^c_I(R)$ is D_R–simple if and only if the tight closure of zero in $H^d_m(A)$ is F–nilpotent.*

Proof. $H^c_I(R)$ is D_R–simple if and only if it is equal to $\mathcal{L}(A,R)$. Then $\widehat{R}\otimes\mathcal{L}(A,R) = \mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)})$ is all of $\widehat{R}\otimes H^c_I(R)$ if and only if $\mathcal{D}(0^*_{H^d_m(A)}) = 0$, by exactness of $\mathcal{D}$. This is the case if and only if $0^*_{H^d_m(A)}$ is F–nilpotent by Proposition 2.20. □

Corollary 4.10. *Let R be regular, local and F–finite. Let $A = R/I$ be a domain of codimension c. If A is F–rational, then $H^c_I(R)$ is D_R–simple. If*

A is F–injective (i.e. F acts injectively on $H^d_m(A)$), then A is F–rational if and only if $H^c_I(R)$ is D_R–simple.

Proof. By [Smi97a], Theorem 2.6, F–rationality of A is equivalent to $0^*_{H^d_m(A)} = 0$. Therefore, by the last theorem, $\mathcal{L}(A,R) = H^c_I(R)$ if A is F–rational. Conversely, if $\mathcal{L}(A,R) = H^c_I(R)$ then $0^*_{H^d_m(A)}$ is F–nilpotent. Under the assumption the $H^d_m(A)$ is F–reduced this implies that $0^*_{H^d_m(A)} = 0$, therefore A is F–rational. $\square$

This should be compared to the following characterization of F–regularity in terms of D_A–simplicity due to Smith:

Proposition 4.11 ([Smi95, 2.2(4)]). *Let A be an F–finite domain which is F–split. Then A is strongly F–regular if and only if A is simple as a D_A–module.*

Note that this proposition is a statement about the D_A–module structure of A, *i.e.* a statement about the differential operators on A itself. This is different from our approach as we work with the differential operators D_R of the regular R. Nevertheless, the similarity of the result is striking and should be understood from the point of view Kashiwara's equivalence, *i.e.* the the D_A–module A should be studied via the corresponding D_R–module $H^c_I(R)$.

We reformulate the simplicity criterion of $H^c_I(R)$ such that it is a criterion solely on A, not referring to $H^d_m(A)$.

Theorem 4.12. *Let R be regular, local and F–finite. Let I be an ideal such that $A = R/I$ is a domain. If for all parameter ideals of A we have $J^F = J^*$, then $H^c_I(R)$ is D_R–simple.*

If A is Cohen–Macaulay, then $H^c_I(R)$ is D_R–simple if and only if $J^* = J^F$ for all parameter ideals J.

Proof. We show that if $J^* = J^F$ for all parameter ideals, then $0^*_{H^d_m(A)}$ is F–nilpotent, *i.e.* $0^*_{H^d_m(A)} = 0^F_{H^d_m(A)}$. Let $\eta \in H^d_m(A)$ represented by $z + (x_1, \dots, x_d)$ for some parameter ideal $J = (x_1, \dots, x_d)$, thinking of $H^d_m(A)$ as the limit $\varinjlim A/J^{[p^e]}$. Then the colon capturing property of tight closure shows that $z \in 0^*_{H^d_m(A)}$ if and only if $z \in J^*$ (*cf.* [Smi93, Proposition 3.1.1]). By our assumption $J^* = J^F$, this implies that $z^{p^e} \in J^{[p^e]}$ for some $e > 0$. Consequently, $F^e(\eta) = z^{p^e} + J^{[p^e]}$ is zero and thus every element of $0^*_{H^d_m(A)}$ is F–nilpotent.

Under the assumption that A is Cohen–Macaulay the same argument can be reversed using that the limit system defining $H^c_m(A)$ is injective. $\square$

As we were dealing with the domain case above we remark that in most cases when A is not analytically irreducible the equality $\mathcal{L}(A,R) = H^c_I(R)$ cannot hold, in particular $H^c_I(R)$ cannot be simple.

Proposition 4.13. *Let $A = R/I$ be equidimensional, local and satisfy Serre's S_2 condition. Suppose that A is* not *analytically irreducible, then $\mathcal{L}(A, R) \neq H^c_Y(A)$.*

Proof. Equidimensionality and S_2–ness implies by [HH94], Corollary 3.7, that $H^d_m(A)$ is indecomposable. The properties of the duality functor $\mathcal{D}$ show now that $\mathcal{D}(H^d_m(A)) = H^c_{I\widehat{R}}(\widehat{R})$ is indecomposable as a unit $R[F^e]$–module. If $\widehat{A}$ is not a domain it follows, essentially by definition, that $\mathcal{L}(\widehat{A}, \widehat{R})$ is decomposable, thus it cannot be equal to $H^c_{I\widehat{R}}(\widehat{R})$. As $\mathcal{L}$ behaves well under completion, it follows that $\mathcal{L}(A, R) \neq H^c_I(R)$. □

As an application of Theorem 4.12 we extend the last proposition to a characterization of the simplicity of $H^c_I(R)$ for the class of all domains A which have only an isolated singularity and whose normalization is F–rational. In particular this yields a characterization of the simplicity of $H^c_I(R)$ for $A = R/I$ a one dimensional domain.

Proposition 4.14. *Let R be regular local and F–finite. Let $A = R/I$ be a local S_2 domain with isolated singularity such that the normalization $\overline{A}$ is F–rational. Then $H^c_I(A)$ is D_R–simple if and only if $H^c_I(R)$ is analytically irreducible.*

Proof. If $H^c_I(R)$ is not analytically irreducible then, by Proposition 4.13, $H^c_I(R)$ is not D_R–simple (equiv. unit $R[F^e]$–simple).

Since R is excellent, $H^c_I(A)$ is analytically irreducible if and only if $\overline{A}$ is local by [Gro65], (7.8.31 (vii)). Let $z \in J^*$ for a parameter ideal $J = (y_1, \ldots, y_d)$ of A. Then, since $\overline{A}$ is F–rational and the expansion $\overline{J}$ of J to $\overline{A}$ is also a parameter ideal, one concludes that $z \in \overline{J}^* = \overline{J}$. Let, for some $a_i \in \overline{A}$,

$$z = a_1 y_1 + \ldots + a_d y_d$$

be an equation witnessing this ideal membership. As we observe in Lemma 4.15 below, for some big enough e, all a^{p^e} are in A. Therefore $z^{p^e} = a_1^{p^e} y_1^{p^e} + \ldots + a_d^{p^e} y_d^{p^e}$, which shows that $z^{p^e} \in J^{[p^e]}$ since all $a^{p^e} \in A$. Thus $J^* = J^F$ and Theorem 4.12 implies that $H^c_I(R)$ is D_R–simple. □

Lemma 4.15. *Let (A, m) be a local domain with at worst isolated singularities. Then the normalization $\overline{A}$ of A is local if and only if for all $x \in \overline{A}$ some power of x lies in A.*

Proof. If there were M_1 and M_2, maximal ideals of $\overline{A}$ lying over m, then, by assumption, for some $t \gg 0$ we have $(M_1)^t \subseteq m \subseteq M_2$. Since M_2 is prime it follows that already $M_1 \subseteq M_2$. The reverse inclusion follows by symmetry and thus $M_1 = M_2$, therefore $\overline{A}$ is local.

Conversely, if $\overline{A}$ is local with maximal ideal M it follows that $\sqrt{m\overline{A}} = M$. Therefore, if $x \in M$ it follows that $x^{n_0} \in m\overline{A}$ for sufficiently big n_0. We want to conclude that $x^n \in m$ and thus is in A for big enough n. For this note that, by the assumption of isolated singularity, the conductor ideal

$C = (A :_{\overline{A}} \overline{A})$ is M primary, *i.e.* sufficiently high powers of x lie in C. Now, if $x^{n_0} \in m\overline{A}$ and $x^n \in C$, then $x^{n_0+n} = x^{n_0}x^n$ is in m itself. Finally, a unit u of $\overline{A}$ can be written as $u = \frac{ux}{x}$ for x a nonunit of $\overline{A}$. Since both, x and ux are not units, sufficiently big powers are in A. Thus also sufficiently big powers of u will be in A. ☐

In the case that R is one–dimensional this yields a finite characteristic analog of results of S.P. Smith [Smi88] and Yekutieli [Yek98].

Corollary 4.16. *Let $A = R/I$ be a one–dimensional local domain with R regular and F–finite. Then $H^c_m(A)$ is D_R–simple (equiv. unit $R[F^e]$–simple) if and only if A is unibranch.*

Proof. As remarked in the proof of Proposition 4.14, A is unibranch if and only if A is analytically irreducible. As one dimensional domains have at worst isolated singular points and since the normalization is regular (and thus F–rational), Proposition 4.14 applies. ☐

This last result that for curves $\mathcal{L}(A, R)$ is described in the same way in positive characteristic as it is in characteristic zero is somewhat misleading. In higher dimensions one expects that $\mathcal{L}(A, R)$ behaves significantly different depending on the characteristic. For example, consider the ideal $I = (xy - zw) \subseteq R = k[x, y, z, w]$. Then $A = R/I$ is the coordinate ring of the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with only singular point being the vertex. The localization of A at the vertex is F–rational. Therefore, our results above shows that $H^1_I(R)$ is simple as a D_R–module in finite characteristic. Nevertheless, in characteristic zero the module $H^1_I(R)$ is not D_R–simple since the Bernstein-Sato polynomial of $xy - zw$ is $(s - 1)(s - 2)$ and therefore has an integral zero of less than -1. This shows that the D_R–submodule generated by $(xy - zw)^{-1} \in H^1_I(R)$ does not contain $(xy - zw)^{-2}$. Therefore $H^c_I(R)$ has a proper D_R–submodule and is therefore not D_R–simple.

This is in accordance with the Riemann–Hilbert type correspondences in either characteristic. For zero characteristic, the classical Riemann–Hilbert correspondence relates holonomic $\mathcal{D}_X$–modules to constructible $\mathbb{C}$–vectorspaces by means of a vast generalization of de Rham theory, *i.e.* to an ultimatively topological theory. In positive characteristic, on the other hand, the Emerton-Kisin correspondence relates finitely generated unit $\mathcal{O}_X[F^e]$–modules to constructible $\mathbb{F}^{p^e}$–sheaves on X, generalizing Artin-Schreyer theory, which ultimately is a coherent theory. This is one reasons why there is no surprise for the failure of a complete analogy of the description of the intersection homology module $\mathcal{L}(Y, X)$ in positive and zero characteristic.

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