LOCAL COHOMOLOGY MULTIPLICITIES IN POSITIVE CHARACTERISTIC

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Abstract. In this paper we give an interpretation of the invariants $\lambda_{a,i}(A)$ introduced by Lyubeznik in [Lyu93] in the case that A is of positive characteristic. The result computes these invariants for the newly introdruced class of *close to F–rational* varieties in terms of étale cohomology with $\mathbb{Z}/p\mathbb{Z}$–coefficients. This result presents the first application of Emerton and Kisin's Riemann–Hilbert type correspondence to local algebra. In fact our proof works in characteristic zero as well such that we obtain generalizations of results on these invariants which were previously obtained by analytic techniques.

1. Introduction

Let (R, m) be a regular local ring of dimension n and let $A = R/I$ be a homomophic image of R. In [Lyu93] Lyubeznik introduces new invariants $\lambda_{a,i}(A)$ (defined as the ath Bass number of $H^{n-i}_I(R)$) and shows that this is independent of the presentation of A as a quotient of a regular local ring. One can verify that

$$\begin{aligned}\lambda_{a,i}(A) &\stackrel{\text{def}}{=} \dim_k \operatorname{Ext}^a_R(k, (H^{n-i}_I(R))) \\ &= \dim_k \operatorname{Hom}_R(k, H^a_m(H^{n-i}_I(R))) \\ &= e(H^a_m(H^{n-i}_I(R))).\end{aligned}$$

The multiplicity $e(\_\_)$ (of the holonomic D_R–module if in characteristic zero and of the unit $R[F]$–module if in characteristic $p > 0$) can be described as follows which also shows the last equality (the second equality is mentioned in [Lyu93, Section 4]): The main results of [Lyu93, HS93] state that the module $H^a_m(H^{n-i}_I(R))$ is injective. As it is supported at the maximal ideal it is isomorphic to a finite direct sum of e copies of an injective hull $E_{R/m} \cong H^n_m(R)$ of the residue field of R. This integer e is the multiplicity.

Our main result is the following description of these invariants in the case that A has reasonable singularities.

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Theorem 1.1. *Let k be a field of characteristic $p > 0$ and let Y be a k-variety of dimension d which is close to F–rational outside the single point $x \in Y$. Let $A = \mathcal{O}_{Y,x}$. Then*

1. $\lambda_{0,i}(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^i_{\{x\}}(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$ *for* $1 \leq i \leq d-1$.
2. $\lambda_{a,d}(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^{d+1-a}_{\{x\}}(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$ *for* $2 \leq a \leq d-1$ *and* $\lambda_{d,d}(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^1_{\{x\}}(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z}) + 1$.
3. *All other* $\lambda_{a,i}(A)$ *vanish.*

The notion *close to F–rational* will be introduced and briefly discussed in Section 4. For now we just point out that (as one should hope) F–rational varieties are close to F–rational and thus so are smooth varieties. In particular the theorem applies in the case that Y has an isolated singularity at x.

This isolated singular case was motivated by the main result in [GLS98] where García López and Sabbah prove a topological description in the case that A is the local ring of an isolated complex singularity:[1]

Theorem 1.2. *Let Y be a complex variety of dimension d with an isolated singularity at the point $x \in Y$. Let $A = \mathcal{O}_{Y,x}$. Then*

1. $\lambda_{0,i}(A) = \dim_{\mathbb{C}} H^i_{\{x\}}(Y, \mathbb{C})$ *for* $1 \leq i \leq d-1$.
2. $\lambda_{a,d}(A) = \dim_{\mathbb{C}} H^{d+1-a}_{\{x\}}(Y, \mathbb{C})$ *for* $2 \leq a \leq d-1$ *and* $\lambda_{d,d}(A) = \dim_{\mathbb{C}} H^1_{\{x\}}(Y, \mathbb{C}) + 1$.
3. *All other* $\lambda_{a,i}(A)$ *vanish.*

Statement (1) was already pointed out in [Lyu93] to follow from a result of Ogus [Ogu73, Theorem 2.3]. Observing the proof in [GLS98] we first note that part (3) is independent of the characteristic whereas the other parts distinctively use characteristic zero.

In order to obtain the analog of Theorem 1.2 (a generalization thereof is our Theorem 1.1) in positive characteristic we have to work somewhat harder. The proof given in [GLS98] is our point of departure. They use the Riemann–Hilbert correspondence and duality for holonomic D–modules. Our idea is to replace the Riemann–Hilbert correspondence (*i.e.* de Rham theory) with the correspondence recently introduced by Emerton and Kisin [EK04] (*i.e.* Artin–Schreyer theory). The main obstacle is that the categories involved in the Emerton–Kisin correspondence do not have a duality which was an essential part in the proof of García López and Sabbah. Thus our first task is to give a new proof of Theorem 1.2 which as its main feature avoids the use of duality. In this proof we also show explicitly that part (1) and (2) are equivalent once part (3) is established.

In Section 3 we briefly recall the setup for the Emerton–Kisin correspondence and show that this allows us to carry our new characteristic zero proof

[1] To be precise, they state part (2) in its Poincaré dual form ($\lambda_{a,d} = \dim_{\mathbb{C}} H^{d+a}(Y, \mathbb{C})$ for $2 \leq a \leq d$), see Remark 2.2 why this is the case and also why we prefer our version.

over to positive characteristic. Thus we obtain Theorem 1.1 postponing the introduction and investigation of *close to F–rational* to the last section.

2. Duality free proof in characteristic zero.

2.1. **A spectral sequence computation.** We start with explaining that a vanishing condition (slightly weaker than the one in part (3) of Theorem 1.2) for the $\lambda_{a,i}(A)$ implies part (3) and also that part (1) and (2) are equivalent. This is done via a not so difficult spectral sequence argument. The condition we impose is that

$$\lambda_{a,i} = e(H^a_{[x]} H^{n-i}_{[Y]}(\mathcal{O}_X)) = 0 \tag{2.1}$$

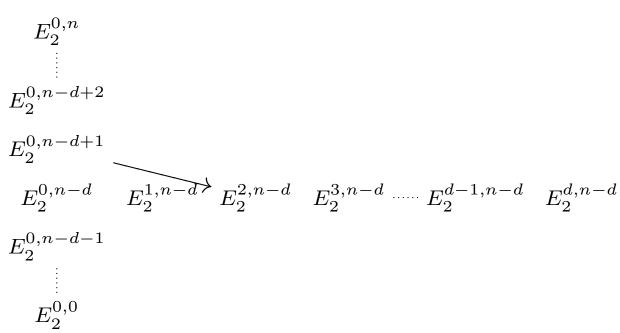
for all pairs (a, i) except possibly $(0, i)$ with $0 \leq i \leq n$ or $(a, n-d)$ with $0 \leq a \leq d$. We will then show that this implies that

$$\lambda_{0,1}(A) + 1 = \lambda_{d,d}(A) \quad \text{and}$$
$$\lambda_{0,d-r+1}(A) = \lambda_{r,d}(A) \text{ for } 2 \leq r \leq d-1$$

and that $\lambda_{1,d}(A) = 0$ and $\lambda_{0,i}(A) = 0$ for $i \geq d$ and $\lambda_{0,0}(A) = 0$. This clearly suffices to support all our claims. Now consider the spectral sequence

$$E_2^{a,j} = H^a_{[x]} H^j_{[Y]}(\mathcal{O}_X) \Rightarrow H^{a+j}_{[x]}(\mathcal{O}_X).$$

Since $\lambda_{a,i} = e(E_2^{a,n-i})$ the vanishing assumption (2.1) yields that the only possibly nonzero entries of the E_2 sheet of this spectral sequence are the ones illustrated in the picture:



Clearly, the only possibly nonzero arrow is the one indicated. We now assume that $d \geq 2$ and leave the easy cases $d = 1$ and $d = 0$ to the reader. Recall that by the Hartshorne–Lichtenbaum vanishing theorem one has $H^n_{[Y]}(\mathcal{O}_X) = 0$ and therefore $E_2^{0,n} = H^0_{[x]} H^n_{[Y]}(\mathcal{O}_X) = 0$ which just says that $\lambda_{0,0}(A) = 0$. Now we claim that for $r \geq 2$ the E_r sheet of the spectral sequence has only the following (possibly) nonzero entries

$$\begin{array}{cccccc}
0 & & & & & \\
E_2^{0,n-1} & & & & & \\
\vdots & & & & & \\
E_2^{0,n-d+r} & & & & & \\
E_2^{0,n-d+r-1} & & & & & \\
 & \searrow & & & & \\
0 & & E_2^{r,n-d} & E_2^{r+1,n-d} \cdots\cdots & E_2^{d-1,n-d} & E_2^{d,n-d}
\end{array}$$

where the only nonzero arrow is the one indicated which yields an isomorphism

$$H^0_{[x]} H^{n-d+r-1}_{[Y]}(\mathcal{O}_X) \cong H^r_{[x]} H^{n-d}_{[Y]}(\mathcal{O}_X) \tag{2.2}$$

for $r < d$. In the case $r = d$ this only nonzero arrow fits in a short exact sequence

$$0 \longrightarrow H^0_{[x]} H^{n-1}_{[Y]}(\mathcal{O}_X) \longrightarrow H^d_{[x]} H^{n-d}_{[Y]}(\mathcal{O}_X) \longrightarrow H^n_{[x]}(\mathcal{O}_X) \longrightarrow 0 \tag{2.3}$$

the right map being the edge map of the spectral sequence.[2] All these claims simply follow from the observation that in the target of the spectral sequence the only surviving term is $H^n_{[x]}(\mathcal{O}_X)$, and from the special shape of the sheets of having only one nonzero arrow. For $r < d$ the bottom left terms (the ones below the nonzero arrow) must vanish since they do not contribute to the limit (the only term that does is $E^{d,n-d}$) and since there are no nonzero arrows arriving at or departing from any of them in the rth or any higher sheet. Similarly the only nonzero arrow must be an isomorphism since otherwise there would be a surviving term in the next sheet which is impossible as we just argued. Rephrasing these observations in terms of the $\lambda_{a,i}(A)$ we obtain from Equation (2.2) that

$$\lambda_{0,d-r+1}(A) = \lambda_{r,d}(A)$$

for $2 \leq r \leq d-1$ and from Equation (2.3) that

$$\lambda_{0,1}(A) + 1 = \lambda_{d,d}(A)$$

where we used in the latter that $e(\_\_)$ is additive on short exact sequences and that $e(H^n_{[x]}(\mathcal{O}_X)) = 1$. Furthermore from the vanishing of $E_2^{1,n-d}$ and $E_2^{0,n-j}$ for $j \leq n-d$ (which already follows by considering sheet 2) we have that $\lambda_{1,d}(A) = \lambda_{0,i}(A) = 0$ for $i \geq d$.

[2]The injectivity on the left is clear since $E_2^{0,n-1}$ must die in the limit. The surjectivity on the right follows since $E_2^{0,0} = 0$, thus the term in the middle ($E_2^{d,d}$) is the only one that can contribute to the abutment term, thus has to surject onto it.

Remark 2.1. The vanishing condition (2.1) is satisfied precisely when $H^j_{[Y]}(\mathcal{O}_X)$ is supported at the point x for $j \geq n-d$. This in turn clearly holds whenever $Y - x$ is Cohen–Macaulay, so in particular whenever Y has an isolated singularity at x and is smooth otherwise.

Remark 2.2. In [GLS98] García López and Sabbah produce the Poincaré dual statement of part (2), namely that $\lambda_{a,d} = \dim_{\mathbb{C}} H^{d+a}(Y, \mathbb{C})$ for $2 \leq a \leq d$.[3] The reason for this lies in their computation of $\lambda_{a,d}(A)$ which uses duality for holonomic $\mathcal{D}$–modules which under Riemann–Hilbert corresponds to Poincaré duality, in that special case. Thus they obtain the equivalence of part (1) and part (2) as a consequence of Poincaré duality. Our observation though shows that Poincaré duality plays no role in this result as it follows completely formal from the structure of the invariants $\lambda_{a,i}$.

2.2. **Preparatory lemmata and proof of Theorem 1.2.** We start with some (probably well known) facts which will naturally lead to the proof of Theorem 1.2.

Lemma 2.3. *Let X be a smooth $\mathbb{C}$–variety of dimension n and let $k : x \hookrightarrow X$ be the inclusion of a point. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$–module, then*

$$\mathrm{Sol}(H^a_{[x]}(\mathcal{M})) = k_! k^{-1} H^{-a}(\mathrm{Sol}\, \mathcal{M}).$$

Proof. By definition of the symbols involved (${}^{\mathrm{p}}H$ denotes perverse cohomology, $\mathrm{Sol}(\_\_) \stackrel{\mathrm{def}}{=} \mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\_\_, \mathcal{O}_X)[n]$) we have

$$\mathrm{Sol}(H^a_{[x]}(\mathcal{M})) = \mathrm{Sol}\, H^a(\mathbf{R}\Gamma_{[x]}\mathcal{M}) = {}^{\mathrm{p}}H^{-a}(\mathrm{Sol}(\mathbf{R}\Gamma_{[x]}\mathcal{M})).$$

One has that $\mathrm{Sol} \circ \mathbf{R}\Gamma_{[x]} = k_! k^{-1} \circ \mathrm{Sol}$.[4] Thus the latter of the above is equal to

$${}^{\mathrm{p}}H^{-a}(k_! k^{-1}(\mathrm{Sol}(\mathcal{M}))) = k_! {}^{\mathrm{p}}H^{-a}(k^{-1}(\mathrm{Sol}(\mathcal{M})))$$

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[3] In order to recover our part (2) of Theorem 1.2 one proceeds as in [GLS98, Remark 1]) and uses Poincaré duality for the link $L_{(Y,x)}$ of the singularity (Y, x). The link is a real orientable compact manifold of dimension $2d-1$. We have, locally analytically around x that

$$\begin{aligned} H^{d+a-1}(Y - x, \mathbb{C}) &\cong H^{d+a-1}(L_{(Y,x)}, \mathbb{C}) \\ &\cong H^{d-a}(L_{(Y,x)}, \mathbb{C}) \qquad \text{(Poincaré duality)} \\ &\cong H^{d-a}(Y - x, \mathbb{C}). \end{aligned}$$

At the same time $H^{d+a}_{\{x\}}(Y, \mathbb{C}) \cong H^{d+a-1}(Y - x, \mathbb{C})$ and $H^{d-a+1}_{\{x\}}(Y, \mathbb{C}) \cong H^{d-a}(Y - x, \mathbb{C})$ for $a \neq d$. Furthermore ($a = d$) one has $\dim_{\mathbb{C}} H^1_{\{x\}}(Y, \mathbb{C}) = \dim_{\mathbb{C}} H^0(Y - x, \mathbb{C}) + 1$ and the claim follows.

[4] This follows from the fact that for any closed embedding $k : Z \longrightarrow X$ and complex of $\mathcal{D}_X$–modules $\mathcal{M}^\bullet$ one has the triangle

$$\mathbf{R}\Gamma_{[Z]}\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet \longrightarrow \mathbf{R}j_* j^* \mathcal{M}^\bullet \xrightarrow{+1}$$

where j denotes the open inclusion $X - Z \subseteq X$. Let us denote $\mathrm{Sol}(\mathcal{M}^\bullet)$ by $\mathcal{L}^\bullet$ and apply Sol to this triangle. Using the compatibility of Sol with the six operations, in particular $\mathrm{Sol} \circ j_* = j_! \circ \mathrm{Sol}$ and $\mathrm{Sol} \circ j^{-1} = j^! \circ \mathrm{Sol}$, we obtain the following triangle.

$$\mathbf{R}j_! j^! \mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathrm{Sol}(\mathbf{R}\Gamma_{[Z]}\mathcal{M}^\bullet) \xrightarrow{+1}$$

where the equality holds since $k_!$ is t–exact by [KW01, Lemma III.4.1]. After pullback along k we are on the point x on which perverse cohomology is the same as ordinary cohomology. Thus we may replace ${}^{\mathrm{p}}H^{-a}$ by H^{-a}. Using that k^{-1} is exact we get

$$k_!{}^{\mathrm{p}}H^{-a}(k^{-1}(\mathrm{Sol}(\mathcal{M}))) = k_!H^{-a}(k^{-1}(\mathrm{Sol}(\mathcal{M}))) = k_!k^{-1}H^{-a}(\mathrm{Sol}(\mathcal{M}))$$

as claimed. □

One of our tools is the intersection homology $\mathcal{D}_X$–module $\mathcal{L}(Y,X)$ of Brylinski and Kashiwara [BK81]. It is the middle extension

$$\mathcal{L}(Y,X) = \tilde{j}_{!*}H^{n-d}_{[Y-\mathrm{Sing}\,Y]}(\mathcal{O}_{(X-\mathrm{Sing}\,Y)})$$

where $\tilde{j}$ denotes the open inclusion $(X - \mathrm{Sing}\,Y) \subseteq X$. Its characterizing property is that it is the smallest $\mathcal{D}_X$–submodule of $H^{n-d}_{[Y]}(\mathcal{O}_X)$ which agrees with $H^{n-d}_{[Y]}(\mathcal{O}_X)$ away from the singular locus of Y. Thus in particular if Y is smooth then $\mathcal{L}(Y,X) = H^{n-d}_{[Y]}(\mathcal{O}_X)$.

Lemma 2.4. *Let X be a smooth k–variety of dimension n and let $i : Y \hookrightarrow X$ be a closed subvariety of dimension d and assume that for $x \in Y$ one has $\mathcal{L}(Y,X)|_{X-x} \cong H^{n-d}_Y(\mathcal{O}_X)|_{X-x}$. Then*

$$H^a_{[x]}(H^{n-d}_{[Y]}(\mathcal{O}_X)) \cong H^a_{[x]}(\mathcal{L}(Y,X))$$

for $a \geq 2$.

Proof. By assumption one has the short exact sequence

$$0 \longrightarrow \mathcal{L}(Y,X) \longrightarrow H^{n-d}_{[Y]}(\mathcal{O}_X) \longrightarrow \mathcal{C} \longrightarrow 0$$

whose cokernel $\mathcal{C}$ has support in the point x. Thus $H^a_{[x]}(\mathcal{C}) = 0$ for $a \geq 1$. By the long exact sequence of $H^\bullet_{[x]}(\_\_)$ applied to this short exact sequence the claim of the lemma follows. □

Lemma 2.5. *Let X be a smooth k–variety of dimension n and let $Y \subseteq X$ be a closed subvariety of dimension d. Assume that for $x \in Y$ one has $\mathcal{L}(Y,X)|_{X-x} \cong \mathbf{R}\Gamma^{n-d}_{[Y]}(\mathcal{O}_X)|_{X-x}[-(n-d)]$. Then*

$$\mathrm{Sol}(\mathcal{L}(Y,X)) = i_!j_{!*}\mathbb{C}_{(Y-x)}[d]$$

where j is the inclusion of $(Y-x) \hookrightarrow Y$.

Proof. Let us fix the notation $(X - \mathrm{Sing}\,Y) \xrightarrow{j''} (X-x) \xrightarrow{j'} X$. Then by definition of middle extension and our assumption one has

$$\begin{aligned}\mathcal{L}(Y,X) &= j'_{!*}j''_{!*}H^{n-d}_{[Y-\mathrm{Sing}\,Y]}(\mathcal{O}_{(X-\mathrm{Sing}\,Y)}) \\ &= j'_{!*}\mathcal{L}(Y-x, X-x) \\ &= j'_{!*}\mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{(X-x)})[n-d].\end{aligned}$$

Comparing with the [KS90, Triangle 2.6.33] standard triangle $\mathbf{R}j_!j^!\mathcal{L}^\bullet \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathbf{R}k_!k^{-1}\mathcal{L}^\bullet \xrightarrow{+1}$ one obtains the claim.

Denoting the inclusion $(Y-x) \hookrightarrow (X-x)$ by i' we recall the computation[4]

$$\begin{aligned}\mathrm{Sol}(\mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{(X-x)})[n-d]) &= i'_! i'^{-1}\,\mathrm{Sol}(\mathcal{O}_{(X-x)})[d-n] \\ &= i'_! i'^{-1}\mathbb{C}_{X-x}[n][d-n] = i'_!\mathbb{C}_{Y-x}[d]\end{aligned}$$

where we also used that $\mathrm{Sol}(\mathcal{O}_{(X-x)}) = \mathbb{C}_{X-x}[n]$. Now finish the proof with the following chain of equalities

$$\begin{aligned}\mathrm{Sol}(\mathcal{L}(Y,X)) &= j'_{!*}\,\mathrm{Sol}(\mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{(X-x)})[n-d]) \\ &= j'_{!*} i'_! \mathbb{C}_{Y-x}[d] \\ &= i_! j_{!*}\mathbb{C}_{Y-x}[d]\end{aligned}$$

the last of which follows from the fact that for a closed immersion $i_! = i_{!*}$ and thus the j and i can be exchanged as we have done. □

Remark 2.6. Granted, the assumption on $\mathcal{L}(Y,X)$ of the preceding two lemmata seems somewhat random. And indeed we do not have a reasonable interpretation of it in characteristic zero. The situation is different in positive characteristic where our notion of close to F–rational of the next section relates it to previous work on singularities, such as tight closure theory and the notion of F–depth as in [HS77].

Also note that if Y has an isolated singularity at x then the assumptions are (trivially) satisfied since in this case $\mathcal{L}(Y,X)|_{Y-x} = H^{n-d}_{Y-x}(\mathcal{O}_{X-x}) = \mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d]$. Thus we can apply these results in the following proof freely.

Proof of Theorem 1.2. Since x is assumed an isolated singularity of Y it follows from Remark 2.1 that part (3) holds and part (1) and (2) are equivalent. Thus it is enough to show, say, part (2):

As we pointed out in the introduction the $\mathcal{D}_X$–module $H^a_{[x]}H^i_{[Y]}(\mathcal{O}_X)$ is isomorphic to a finite direct sum of $\lambda_{a,i}(A)$ many copies of $H^n_{[x]}(\mathcal{O}_X)$, an injective hull of the residue field at x. By Lemma 2.3 together with $\mathrm{Sol}(\mathcal{O}_X) = \mathbb{C}_X[n]$ one has

$$\mathrm{Sol}(H^n_{[x]}(\mathcal{O}_X)) \cong k_! k^{-1} H^{-n}(\mathbb{C}_X[n]) \cong k_!\mathbb{C}_x$$

where we recall that k was just the inclusion of $x \hookrightarrow X$. Therefore $\lambda_{a,i}(A)$ is just the dimension of the fiber at x of $\mathrm{Sol}(H^a_{[x]}H^i_{[Y]}(\mathcal{O}_X))$. Thus, for $a \geq 2$ we can compute

$$\begin{aligned}\lambda_{a,d}(A) &= e(H^a_{[x]}(H^d_{[Y]}(\mathcal{O}_X))) \\ &= \dim_{\mathbb{C}}\left(\mathrm{Sol}(H^a_{[x]}(H^d_{[Y]}(\mathcal{O}_X)))\right)_x \\ &= \dim_{\mathbb{C}}\left(\mathrm{Sol}(H^a_{[x]}(\mathcal{L}(Y,X)))\right)_x \qquad \text{(by Lemma 2.4)} \\ &= \dim_{\mathbb{C}}\left(k_! k^{-1} H^{-a}(\mathrm{Sol}\,\mathcal{L}(Y,X))\right)_x \qquad \text{(by Lemma 2.3)} \\ &= \dim_{\mathbb{C}}\left(H^{-a}(i_! j_{!*}\mathbb{C}_{Y-x}[d])\right)_x \qquad \text{(by Lemma 2.5)} \\ &= \dim_{\mathbb{C}}\left(H^{-a}(j_{!*}\mathbb{C}_{Y-x}[d])\right)_x\end{aligned}$$

where i is the inclusion $Y \hookrightarrow X$ and j denotes the inclusion $(Y - x) \subseteq Y$. Since j is just the inclusion of the complement of a single point it follows that

$$j_{!*}\mathbb{C}_{Y-x}[d] \cong \tau_{\leq d-1}\mathbf{R}j_*\mathbb{C}_{Y-x}[d]$$

by [Bor84, V.2.2 (2)]. By definition of Deligne's truncation $\tau_{\leq d-1}$ one has for $a \geq 1$

$$\Big(H^{d-a}(j_{!*}\mathbb{C}_{Y-x})\Big)_x = \Big(\mathbf{R}^{d-a}j_*\mathbb{C}_{Y-x}\Big)_x.$$

Applying the following Lemma 2.7 we get for $2 \leq a \leq d-1$ that

$$\lambda_{a,d}(A) = \dim_{\mathbb{C}}\Big(\mathbf{R}^{d-a}j_*\mathbb{C}_{Y-x}\Big)_x = \dim_{\mathbb{C}} H^{d-a+1}_{\{x\}}(Y,\mathbb{C})$$

and (for $a = d$) that $\lambda_{d,d}(A) = H^1_{\{x\}}(Y,\mathbb{C}) + 1$. □

Lemma 2.7. *Let Y be a $\mathbb{C}$–variety and let $x \in Y$ be a closed point and C be a constant sheaf on Y. Then*

$$(\mathbf{R}^i j_* j^{-1}C)_x = H^{i+1}_{\{x\}}(Y, C)$$

for $i \geq 1$ and (for $i = 0$) one has the short exact sequence

$$0 \longrightarrow C_x \longrightarrow (\mathbf{R}^0 j_* j^{-1}C)_x \longrightarrow H^1_{\{x\}}(Y,C) \longrightarrow 0.$$

Proof. For the open inclusion $j : (Y - x) \hookrightarrow Y$ consider the triangle

$$\mathbf{R}\Gamma_{\{x\}}C \longrightarrow C \longrightarrow \mathbf{R}j_*j^{-1}C \xrightarrow{+1}$$

and take its fiber at the point x to obtain the following triangle:

$$(\mathbf{R}\Gamma_{\{x\}}C)_x \longrightarrow C_x \longrightarrow (\mathbf{R}j_*j^{-1}C)_x \xrightarrow{+1}$$

Since $H^i(C_x) = 0$ for $i > 0$ and $H^0_{\{x\}}(C) = 0$ (since C is a constant sheaf) the long exact sequence yields

$$0 \longrightarrow C_x \longrightarrow (\mathbf{R}^0 j_* j^{-1}C)_x \longrightarrow (H^1_{\{x\}}(Y,C))_x \longrightarrow 0$$

and for $i \geq 1$

$$(R^i j_* j^{-1}C)_x \cong (H^{i+1}_{\{x\}}(C))_x.$$

But clearly since $H^{i+1}_{\{x\}}(C)$ is supported on x we have $(H^{i+1}_{\{x\}}(C))_x = H^{i+1}_{\{x\}}(Y,C)$ which finishes the proof. □

3. The case of positive characteristic.

We very briefly recall the setup of the correspondence of Emerton and Kisin and point out the relevant facts which will make clear that the above proof in characteristic zero also works in positive characteristic.

3.1. **Emerton–Kisin correspondence.** Let k be a field of positive characteristic p and let X be a smooth k–variety. In [EK04] Emerton and Kisin establish an anti–equivalence (on the level of derived categories) between constructible $\mathbb{Z}/p\mathbb{Z}$–sheaves on $X_{\text{ét}}$ on one hand and locally finitely generated unit $\mathcal{O}_{F,X}$–modules on the other. Their construction closely models the Riemann–Hilbert correspondence and underlies the same formalism – except that there is no duality available on either side of the correspondence. This leads to the defect that their anti–equivalence is compatible with only three of Grothendieck's *six operations*, namely with analogs of $f^!$, f_* (denoted f_+ in [EK04]) and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}}$ on the $\mathcal{O}_{F,X}$–module side, which correspond to f^*, $f_!$ and $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p\mathbb{Z}}$ on the constructible étale $\mathbb{Z}/p\mathbb{Z}$ side.

We will recall the definition of $\mathcal{O}_{F,X}$–module and point out that the local cohomology modules are locally finitely generated as such. So that the formalism of Emerton–Kisin can be applied to our study of the numbers $\lambda_{a,i}$. For a nice introduction see [EK03]; or [EK04] for the most general theory.

Definition 3.1. A quasi–coherent $\mathcal{O}_X$–module $\mathcal{M}$ together with an $\mathcal{O}_X$–linear map

$$\vartheta : F^*\mathcal{M} \longrightarrow \mathcal{M}$$

is called an *$\mathcal{O}_{F,X}$–module.* Here F denotes the Frobenius morphism on X. If ϑ is an isomorphism, then $(\mathcal{M}, \vartheta)$ is called *unit.*

Locally, if $X = \operatorname{Spec} R$, an $\mathcal{O}_{X,F}$–module is nothing but a module M over the noncommutative ring

$$R[F] \overset{\text{def}}{=} \frac{R\langle F\rangle}{(r^pF - Fr \mid r \in R)}.$$

Such $R[F]$–module is called *finitely generated* if it is just that: finitely generated as an $R[F]$–module. Thus we have the notion of locally finitely generated for $\mathcal{O}_{F,X}$–modules.

One of the key results of the theory (which was essentially proved by Lyubeznik in [Lyu97]) is that the category of locally finitely generated unit $R[F]$–modules is abelian, and that every such $\mathcal{M}$ has finite length in that category [Lyu97, Theorem 3.2].

Example 3.2. Let $X = \operatorname{Spec} R$ be affine. Then, abusing the identification of $\mathcal{O}_X$–modules and R–modules one can write $F^*M = R^{(1)} \otimes_R M$ where $R^{(1)}$ is the R–R–bimodule with the usual left structure and the right structure via the Frobenius map. Thus one sees immediately that the natural map

$$F^*R = R^{(1)} \otimes_R R \longrightarrow R$$

sending $a \otimes r$ to ar^p is an isomorphism, showing that R is a fg (finitely generated) unit $R[F]$–module.

Let $g \in R$ be an element and consider the localization R_g. The natural map

$$F^* R_g = R^{(1)} \otimes_R R_g \longrightarrow R_g$$

has an inverse given by sending r/t to $rt^{p-1} \otimes 1/t$. Furthermore R_g is generated as an $R[F]$–module by $1/g$. Thus again R_g is a fg unit $R[F]$–module.

Since local cohomology modules $H^i_I(R)$ for I an ideal of R can be computed via Čech resolutions, whose entries are localizations of the type R_g, the aforementioned result that the category of fg unit $R[F]$–modules is abelian implies that local cohomology modules are fg unit.

These examples are a special instance of more general results showing that the *cohomology with supports* functors are defined in the category of locally finitely generated unit $\mathcal{O}_{F,X}$–modules [EK04, Proposition 5.11.5]. If $\mathcal{M}^\bullet$ is a bounded complex of such modules then so is $\mathbf{R}\Gamma_{[Y]}\mathcal{M}^\bullet$ for Y a closed subvariety of X and one has the usual triangle

$$\mathbf{R}\Gamma_{[Y]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_+ j^! \mathcal{M} \xrightarrow{+1}$$

where $j : X - Y \longrightarrow X$ denotes the open inclusion.

The correspondence of Emerton and Kisin is between the derived category of bounded complexes of $\mathcal{O}_{F,X}$ modules whose cohomology is locally finitely generated unit,

$$D^b_{lfgu}(\mathcal{O}_{F,X}) \underset{\sim}{\overset{\mathrm{Sol}}{\rightleftarrows}} D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

and the derived category of bounded complexes of $\mathbb{Z}/p\mathbb{Z}$ sheaves with constructible cohomology on $X_{\text{ét}}$. Furthermore, they define functors $f^!$, f_+ and $\_\_ \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \_\_$. They are not the same as (though closely related to) the functors of Grothendieck-Serre duality.

The canonical t–structure on the left induces via the anti–equivalence an exotic t–structure on $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, which in turn is just the t–structure for the middle perversity as described by Gabber [Gab00]. Thus one obtains a notion of *perverse sheaves* and thus of perverse cohomology.

3.2. **Intermediate extensions.** There is a theory of intermediate extensions. If $j : U \longrightarrow X$ is a locally closed immersion of smooth k–schemes and $\mathcal{M}$ a lfgu $\mathcal{O}_{F,U}$–module, then its intermediate extension $j_{!+}\mathcal{M}$ is defined as the smallest submodule $\mathcal{M}' \subseteq H^0(j_+\mathcal{M})$ such that $j^!\mathcal{M}' = \mathcal{M}$. Furthermore,

$$\mathrm{Sol}(j_{!+}(\mathcal{M})) = j_{!*}\,\mathrm{Sol}(\mathcal{M}) \overset{\text{def}}{=} \mathrm{Im}({}^{\mathrm{p}}H^0(j_!\,\mathrm{Sol}(\mathcal{M})) \longrightarrow {}^{\mathrm{p}}H^0(j_*\,\mathrm{Sol}(\mathcal{M})))$$

so that the intermediate extension is compatible with the correspondence [EK03, Section 4.3]. We will only apply this to $\mathcal{M} = H^{n-d}_{[Y-\operatorname{Sing} Y]}(\mathcal{O}_{(X-\operatorname{Sing} Y)})$

for Y a closed subset of X and j the open inclusion $(X - \operatorname{Sing} Y) \subseteq X$. In this case we get

$$\mathcal{L}(Y,X) \stackrel{\text{def}}{=} j_{!+} H^{n-d}_{[Y-\operatorname{Sing} Y]}(\mathcal{O}_{(X-\operatorname{Sing} Y)}) \subseteq H^{n-d}_{[Y]}(\mathcal{O}_X)$$

as its unique simple submodule. This important special case was already obtained in [Bli04]. The key point in obtaining these results is the aforementioned fact that lfgu $\mathcal{O}_{F,X}$–modules have finite length.

The following proposition lists the properties of the theory which are needed to be able to carry out the above proof in the new context.

Proposition 3.3.
(1) $\operatorname{Sol}(\mathcal{O}_X) = \mathbb{Z}/p\mathbb{Z}[n]$ *where n is the dimension of X.*
(2) *For a closed immersion of smooth k–schemes $k : Y \longrightarrow X$ one has* $\operatorname{Sol} \circ \mathbf{R}\Gamma_{[Y]} \cong k_! k^{-1} \circ \operatorname{Sol}$.
(3) *Let $k : Y \longrightarrow X$ be a closed immersion of smooth schemes. Then $k_!$ is t–exact.*

Proof. Part (1) is just Example 9.3.1 in [EK04].

For part (2) note that $\mathbf{R}\Gamma_{[Y]}\mathcal{M}$ is defined via the triangle

$$\mathbf{R}\Gamma_{[Y]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow j_+ j^! \mathcal{M} \xrightarrow{+1}$$

with $j : X - Y \longrightarrow X$ denoting the open inclusion. Applying Sol and using the fact that Sol interchanges j_+ with $j_!$ and $j^!$ with j^* by [EK04, Proposition 9.3, Proposition 9.5] we compare with the triangle

$$j_! j^* \operatorname{Sol}(\mathcal{M}) \longrightarrow \operatorname{Sol}(\mathcal{M}) \longrightarrow k_! k^{-1} \operatorname{Sol}(\mathcal{M}) \xrightarrow{+1}$$

in $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ to obtain the result.

Part (3) can be checked by hand (using Gabbers definition of the t–structure in [Gab00]), but also follows via the correspondence from the fact that k_+ is exact by [EK04, Remark 3.4.1]. □

Proof of Theorem 1.1. The assumption of close to F–rational implies by Proposition 4.2 that $\mathcal{L}(Y,X)|_{(X-x)} \cong \mathbf{R}\Gamma_Y(\mathcal{O}_X)|_{X-x}[n-d]$. This means in particular that $\mathcal{L}(Y,X)|_{(X-x)} = H^{n-d}_Y(\mathcal{O}_X)|_{X-x}$ and $H^i_Y(\mathcal{O}_X)$ is supported in x for $i \neq n-d$. Thus the vanishing condition (2.1) is satisfied and therefore (by Section 2.1) part (3) holds and part (1) and (2) are equivalent. Again we proof part (2) to finish the argument.

This is done by following the arguments in the preceding section step by step, working on the étale site and replacing $\mathbb{C}$ by $\mathbb{Z}/p\mathbb{Z}$ whenever appropriate. Here are some remarks on this task which finishes the proof.

(1) For Lemma 2.3 one uses that $\mathbf{R}\Gamma_{[x]}$ commutes with Sol in the way claimed. Furthermore we use that $k_!$ is t–exact. This is Proposition 3.3 part (2) and (3).
(2) As pointed out at the beginning of the proof the assumptions of Lemma 2.4 and Lemma 2.3 are satisfied. For Lemma 2.4 literally the same argument holds after the existence of the middle extension

$\mathcal{L}(Y,X)$ is established as discussed above. The same remark applies to Lemma 2.5.

(3) In the actual proof one should use that $\mathrm{Sol}(\mathcal{O}_X) = \mathbb{Z}/p\mathbb{Z}[n]$ and the discussed properties of middle extension, in particular its compatibility with Sol.

(4) Lemma 2.7 is even stated for general coefficients and the argument is valid for any k–variety with the étale topology. The only caveat is that we implicitly used excision in the last part; an étale version of which can be found in [Mil80, Proposition 1.27], for example.

□

4. Close to F–rational singularities.

We finish this note with a brief discussion of a new class of singularities, called close to F–rational.

Definition 4.1. Let (A, m) be a local noetherian ring of dimension d. Let $H^\bullet_m(A) = \oplus H^i_m(A)$ be the local cohomolgy with support in m. Then A is called *close to F–rational* if and only if

$$H^*_m(A)/0^F$$

is simple as an $A[F]$–module, where 0^F denotes all the elements of $H^*_m(A)$ which are annihilated by a power of the Frobenius.

If Y is a noetherian scheme then Y is called *close to F–rational* if for all closed points $y \in Y$ the local ring $\mathcal{O}_{Y,y}$ is close to F–rational.

Recall that F–rationality of A is equivalent to $H^*_m(A)$ being simple as an $A[F]$–module (at least if A is excellent). This implies that an F–rational ring is close to F–rational. The obstruction to F–rationality is the tight closure of zero 0^* in $H^*_m(A)$ (see [Hun96] for relevant notions from the theory of tight closure). Close to F-rational just means that this obstruction is, if not zero (F–rational) it is least F–nilpotent. Since one always has that $H^d_m(A) \neq 0^*$ [5] it follows that A is almost F–rational if and only if $H^d_m(A)/0^F_{H^d_m(A)}$ is $A[F]$–simple and $H^i_m(A)$ is F nilpotent for $i \neq d$.

The following characterization of close to F–rational singularities is the key point of our investigation.

Proposition 4.2. *Let Y be a subvariety of dimension d embedded in X which is a smooth k–variety of dimension n (*$\operatorname{char} k = p > 0$*). Then Y is close to F–rational if and only if*

$$\mathcal{L}(Y,X) = \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)[n-d]$$

where $\mathcal{L}(Y,X)$ denotes the unique simple unit $R[F]$ submodule of $H^c_{[Y]}(\mathcal{O}_X)$.

[5] For the versed in tight closure theory: existence of test elements is responsible for this, see for example [Hun96].

Proof. This is a slight extension of the main result in [Bli04]. Since by definition, close to F–rational is checked locally, we have to verify that for every point $y \in Y$ the local ring $(A, m) = \mathcal{O}_{Y,y}$ is close to F–rational if and only if $\mathcal{L}(A, R) = \mathbf{R}\Gamma_{[I]}(R)[n-d]$ where $R = \mathcal{O}_{X,x}$ such that $A = R/I$. For this reduction we used that $\mathcal{L}(Y, X)$ and $\mathbf{R}\Gamma_{[Y]}\mathcal{O}_X$ localize.

In this situation [Bli04, Theorem 4.9] states that $\mathcal{L}(A, R) = H^{n-d}_I(R)$ if and only if $0^* = 0^F$ holds in $H^d_m(A)$. This latter condition is equivalent to $H^d_m(A)/0^F$ being $A[F]$–simple since 0^* is the maximal proper $R[F]$–submodule of $H^d_m(A)$. It remains to point out that $H^{n-i}_I(R)$ is zero if and only if $H^i_m(A)$ is F–nilpotent. This is because, in the notation of [Lyu97, Example 4.8] we have

$$H^{n-i}_I(R) = \mathcal{H}_{R,A}(H^i_m(A)).$$

By [Lyu97,] one has $\mathcal{H}_{R,A}(\mathcal{M}) = 0$ if and only if the $A[F]$–module $\mathcal{M}$ is F–nilpotent. It follows that $H^{n-i}_I(R) = 0$ if and only $H^i_m(A)$ is F–nilpotent. □

Remark 4.3. Close to F–rational singularities are related to the notion of F–depth as introduced by Hartshorne and Speiser in [HS77, page 60]. One can verify that if A is close to F–rational then F–depth $A = \dim A$. This notion of F–depth is shown to be equal to the étale $\mathbb{Z}/p\mathbb{Z}$–depth.

Thus the notion of (close to) F–rational singularities yields a reasonable description of the class of varieties $Y \subseteq X$ for which $\mathcal{L}(Y, X) \cong \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)[n-d]$ and consequently Theorem 1.1 holds. Whether there is a useful interpretation of this property in zero characteristic remains unclear.

The setup in [HS77] combined with some results of [Lyu97] also allow for a more direct proof of a generalization of part (1) of Theorem 1.1.

Proposition 4.4. *Let k be a perfect field of characteristic $p > 0$. Let Y be a k–variety of pure dimension d. Let $A = \mathcal{O}_{Y,x}$ for some point $x \in X$. Then,*

$$\lambda_{0,i}(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^i_{\{x\}}(Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$$

for $0 \leq i \leq F$–depth$(\operatorname{Spec} A - x)$.

Proof. Let $A = R/I$ be represented as the quotient of the regular local ring (R, m) with residue field $k = R/m$. We may (and do) replace Y by $\operatorname{Spec} A$. By definition one has $\lambda_{0,i}(A) = \dim_k \operatorname{Hom}(k, H^{n-i}_I(R))$. By [HS77, Theorem 2.5] for $i < F$–depth$(Y - x) + 1$ the local cohomology module $H^{n-i}_I(R)$ is cofinite, supported in m. It follows by [Lyu93, Theorem 2.4] that $H^{n-i}_I(R)$ is injective and thus $H^0_m(H^{n-i}_I(R)) = H^{n-i}_I(R) \cong E^e_{R/m}$ for some integer $e = e(H^{n-i}_I(R)) = \lambda_{0,a}(A)$.

In the terminology of [Lyu97, Section 4] the local cohomology module $H^{n-i}_I(R)$ is obtained from the local cohomology module $H^i_m(A)$ via the functor $\mathcal{H}_{R,A}$ ([Lyu97, Example 4.8]):

$$H^{n-i}_I(R) \cong \mathcal{H}_{R,A}(H^i_m(A)).$$

Note that $\lambda_{0,i}(A) = e(H_I^{n-i}(R))$ is called the *corank* of $H_I^{n-i}(R)$ in the notation of [Lyu97]. Thus by [Lyu97, Proposition 4.10], we get

$$\lambda_{0,i}(A) = e(H_I^{n-i}(R)) = \dim_k(H_m^i(A))_s$$

where the subscript $(\_)_s$ denotes the Frobenius stable part (that is the largest k–subspace on which the Frobenius acts bijectively).

One finishes with an application of [HS77, Proposition 5.3][6] which states that for $i \leq F$–depth$(Y - x) + 1$ one has

$$H^i_{\{x\}}(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}/p\mathbb{Z}} k \cong H^i_m(A)_s.$$

Thus we conclude that $\lambda_{0,i}(A) = \dim_k H^i_m(A)_s = \dim_{\mathbb{Z}/p\mathbb{Z}} H^i_{\{x\}}(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ for $0 \leq i \leq F$–depth$(Y - x)$ as claimed. □

Remark 4.5. Similarly one can use the notion of de Rham depth of [Ogu73] to give a zero characteristic version of the last Proposition. Thus in positive (resp. zero characteristic) this gives a proof of part (1) of Theorem 1.1 (resp. Theorem 1.2). By our observation that part (1) and (2) are essentially equivalent we obtain an alternative proof of the full result.

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