LYUBEZNIK'S NUMBERS FOR COHOMOLOGICALLY ISOLATED SINGULARITIES.

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Abstract. In this note I give a description of Lyubeznik's local cohomology invariants for a certain natural class of local rings, namely the ones which have the same local cohomology vanishing as one expects from an isolated singularity. This strengthens our results of [BB04] while at the same time somewhat simplifying the proofs. Through examples I further point out the bad behavior of these invariants under reduction mod p.

1. Introduction

Let $A = R/I$ for I an ideal in a regular (local) ring (R, m) of dimension n and containing a field k. The main results of [Lyu93, HS93] state that the local cohomology module $H^a_m(H^{n-i}_I(R))$ is injective and supported at m. Therefore it is a finite direct sum of $e = e(H^a_m(H^{n-i}_I(R)))$ many copies of the injective hull $E_{R/m}$ of the residue field of R. Lyubeznik shows in [Lyu93] that this number

$$\lambda_{a,i}(A) \stackrel{\text{def}}{=} e(H^a_m(H^{n-i}_I(R)))$$

does not depend on the auxiliary choice of R and I. If A is a complete intersection, these invariants are essentially trivial (all are zero except $\lambda_{d,d} = 1$ where $d = \dim A$). The goal of this paper is to describe these invariants for a large class of rings, including those which are complete intesections away from the closed point. Alternatively this class of rings can be viewed as consisting of the rings which behave cohomologically like an isolated singularity.

Theorem 1.1. *Let $A = \mathcal{O}_{Y,x}$ for Y a closed k–subvariety of a smooth variety X. If for $i \neq d$ the modules $H^{n-i}_{[Y]}(\mathcal{O}_X)$ are supported in the point x then*

(1) *For $2 \leq a \leq d$ one has*

$$\lambda_{a,d}(A) - \delta_{a,d} = \lambda_{0,d-a+1}(A)$$

and all other $\lambda_{a,i}(A)$ vanish.

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(2)
$$\lambda_{a,d}(A) - \delta_{a,d} = \begin{cases} \dim_{\mathbb{F}_p} H^{d-a+1}_{\{x\}}(Y_{\acute{e}t}, \mathbb{F}_p) & \textit{if } \operatorname{char} k = p \\ \dim_{\mathbb{C}} H^{d-a+1}_{\{x\}}(Y_{\mathrm{an}}, \mathbb{C}) & \textit{if } k = \mathbb{C} \end{cases}$$
where $\delta_{a,d}$ is the Kroneker delta function.

In the case that A has only an isolated singularity and $k = \mathbb{C}$, this was shown by Garcia Lopez and Sabbah in [GLS98]. In [BB04] Bondu and myself proved part (1) in all characteristics but part (2) only under an additional assumption. The point of this note is to show that this additional assumption was unnecessary, an observation which also makes the proof much clearer.

2. Proof of the result

Let us first fix our notation. Let $Y \subseteq X$ be a closed subscheme of X. Let X be smooth of dimension n and Y is of dimension d. Let $x \in Y$ be a point. We take the freedom to shrink X (and Y) since everything is local at x. We denote the inclusions as follows:

$$\begin{array}{ccc} Y & \xrightarrow{\quad i' \quad} & X \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j'} \\ Y - x & \xrightarrow{\quad i \quad} & X - x \end{array}$$

To ease notation we carry out the argument in the case when $\operatorname{char} k$ is positive. The proof of the characteristic zero result is exactly the same, replacing the Emerton–Kisin correspondence by the Riemann–Hilbert correspondence.

Building upon [BB04] it remains to show that
$$\lambda_{a,d}(A) - \delta_{a,d} = \dim H^{d-a+1}_{\{x\}}(Y_{\text{ét}}, \mathbb{F}_p)$$
for $2 \leq i \leq d$. By definition of the invariants
$$\lambda_{a,i}(A) = e(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X)))$$
this comes down to computing $\dim \operatorname{Sol}(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X))$. The only trick in the proof is to replace $H^{n-i}_{[Y]}(\mathcal{O}_X)$ by something more accessible – that is by something whose solutions Sol can readily be computed. The rest is mere computation.

Quite generally one has a short exact sequence
$$0 \longrightarrow K \longrightarrow j'_{!*} H^{n-i}_{[Y]}(\mathcal{O}_X)|_{X-x} \longrightarrow H^{n-i}_{[Y]}(\mathcal{O}_X) \longrightarrow C \longrightarrow 0$$
where, by construction, the kernel K and quotient C are both supported on x.[1,2] Splitting this four term sequence into two short exact sequences and

[1] The functors we call j_* and $j_{!*}$ are denoted by j_+ and $j_!+$ in [EK04]. In order to adjust to the notation used in the Riemann–Hilbert correpondence I changed this notation here.

[2] The intermediate extension $j_{!*}M$ of a module ($\mathcal{D}_{(X-x)}$– or $\mathcal{O}_{F,(X-x)}$–module if in characteristic 0 or p respectively) is defined as the unique smallest submodule M' of

using the long exact sequence for $\Gamma_{[x]}(\_\_)$ we get that for $a \geq 2$

$$H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X)) \cong H^a_{[x]}(j'_{!*}(H^{n-i}_{[Y]}(\mathcal{O}_X)|_{X-x})). \tag{2.1}$$

This first substitution, combined with [BB04, Lemma 2.3], we record as a Lemma:

Lemma 2.1. *With notation as above, and without any assumptions on the singularities of* $A = \operatorname{Spec} \mathcal{O}_{Y,x}$

$$\lambda_{a,d}(A) = \dim(H^{-a} j'_{!*} \operatorname{Sol} H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}))_x$$

for $2 \leq a \leq d$.

Proof. By definition we have

$$\begin{aligned}
\lambda_{a,d}(A) &= e(H^a_{[x]} H^{n-d}_{[Y]}(\mathcal{O}_X)) \\
&= \dim(\operatorname{Sol} H^a_{[x]} H^{n-d}_{[Y]}(\mathcal{O}_X)))_x \\
&= \dim(k_! k^{-1} H^{-a} \operatorname{Sol} H^{n-d}_{[Y]}(\mathcal{O}_X)))_x \qquad \text{[BB04, Lemma 2.3]} \\
&= \dim(H^{-a} \operatorname{Sol} j'_{!*}(H^{n-d}_{[Y]}(\mathcal{O}_X)|_{X-x}))_x
\end{aligned}$$

where $k : x \longrightarrow X$ is the inclusion of the point. The commutation of Sol with $j'_{!*}$ now finishes the argument. ∎

The assumption that $H^{n-i}_{[Y]}(\mathcal{O}_X)$ is supported at x for $i \neq d$ we rephrase by saying that one has a quasi-isomorphism of complexes

$$H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}) \cong \mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d].$$

and the solutions of the latter can be computed easily[3] (and is preverse!) as done in [BB04]:

$$\operatorname{Sol}(\mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d]) = i'_!(\mathbb{F}_p)_{Y-x}[d]$$

Since j is just the inclusion of complement of a point we have that

$$j_{!*}(\_\_) \cong \tau_{\leq d-1} \mathbf{R} j'_*(\_\_)$$

$H^0 j_*(M)$ such that $j^! M' \cong M$, where $j : X - x \longrightarrow X$ is the open inclusion of the complement of x into X. The intermediate extension appears here as a substitute for $j_!$ which does not exist in the characteristic p context. It was this realization (replace $j_!$ with $j_{!*}$) which made it possible to make the argument work in all characteristics.

[3] In the case $k = \mathbb{C}$ this computation yields $i_! \mathbb{C}_{Y-x}[d]$ instead.

by [Bor84, V.2.2 (2)]. Continuing the computation of Lemma 2.1 using these observations we get for $a \geq 1$ that

$$\begin{aligned}
\lambda_{a,i}(A) &= \dim(H^{-a} j'_{!*} \operatorname{Sol} H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}))_x \\
&= \dim(H^{-a} j'_{!*} \operatorname{Sol}(\mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d]))_x \\
&= \dim(H^{-a} j'_{!*} i'_{!}(\mathbb{F}_p)_{Y-x}[d])_x \\
&= \dim(H^{-a} i_{!} j_{!*}(\mathbb{F}_p)_{Y-x}[d])_x \\
&= \dim(H^{d-a} \tau_{\leq d-1} \mathbf{R} j_*(\mathbb{F}_p)_{Y-x})_x \\
&= \dim(H^{d-a} \mathbf{R} j_*(\mathbb{F}_p)_{Y-x})_x
\end{aligned}$$

The latter was computed in [BB04, Lemma 2.7] to be equal to $H^{d-a+1}_{\{x\}}(Y_{\text{ét}}, \mathbb{F}_p) + \delta_{a,d}$ as required.

Remark 2.2. A slight refinement of the same techniques yield also to a more general statement. Namely, if one requires vanishing of $H^a_m H^{n-i}_I(R)|_{\operatorname{Spec} R - \text{pt}}$ below the diagonal $a = i + m$ then the result remains true in the range $d - m + 2 \leq a \leq d$.

3. Examples

This section is to provide some examples of the bad behavior of the invariants $\lambda_{a,i}$ under reduction to positive characteristic. The uniformity of the Theorem for all characteristics seems, on the first sight, to suggest that that one can expect a good behavior of the invariants under reduction mod p. This impression is however quickly shattered, essentially for reasons that local cohomology is well known to behave poorly under reduction. Alternatively one also can observe that the cohomology theory corresponding to $H^i_x(Y_{an}, \mathbb{C})$ under reduction is not $H^i_x(Y_{\text{ét}}, \mathbb{F}_p)$ but rather p-adic rigid cohomology or crystalline cohomology, of which $H^i_x(Y_{\text{ét}}, \mathbb{F}_p)$ is only a small part, namely the slope zero part.

The examples that follow are standard examples for the bad behaviour of local cohomology under reduction mod p. I learned them from a talk by Anurag Singh at Oberwofach in March 2005. Our general setup is as follows: Let $A = R/I$ where R is a polynomial ring over $\mathbb{Z}$ and I is a homogeneous ideal. We denote by $A_0 = A \otimes_{\mathbb{Z}} \mathbb{C}$ the generic characteristic zero model and by $A_p = A \otimes \mathbb{F}_p$ for all p prime the positive characteristic models.

Example 3.1. Let $R = \mathbb{Z}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$ and $I = (\delta_1, \delta_2, \delta_3)$ be the ideal of 2×2 minors of the displayed matrix of variables. Then $A = R/I$ has a free resolution (as an R–module) given by

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix}} R^3 \xrightarrow{(\delta_1\ \delta_2\ \delta_3)} R \longrightarrow 0$$

This shows that $\mathrm{Ext}^3(R/I,R)=0$ and therefore, reducing mod p, that $\mathrm{Ext}^3(R_p/I^{[p^e]},R_p)=0$ for all e by the flatness of the Frobenius. This implies that $H^3_I(R_p)=0$ as well since in positive characteristic $H^i_I(R_p)=\lim \mathrm{Ext}^i(R_p/I^{[p^e]},R_p)$. On the other hand, it is well known that in zero characteristic, $H^3_I(R_{\mathbb{Q}})$ is not zero. Hence this provides an example where for the characteristic zero model we have $\lambda_{0,3}=\lambda_{2,4}\neq 0$ whereas in all positive characteristics $\lambda_{0,3}=\lambda_{2,4}=0$.

The next example even shows that the vanishing of $\lambda_{a,i}$ can vary in an arithmetic progression:

Example 3.2. Let A be the homogeneous coordinate ring of $\mathbb{P}^1\times E$ where E is the elliptic curve $\mathrm{Proj}\,\frac{\mathbb{Z}[x,y,z]}{x^3+y^3+z^3}$. Then A is given as the quotient of R (as above) by the ideal

$$I=(\delta_1,\delta_2,\delta_3,x^3+y^3+z^3,ux^2+vy^2+zw^2,u^2x+v^2y+w^2z,u^3+v^3+w^3).$$

The resolution of $A=R/I$ can be computed to be equal to

$$0\longrightarrow R\longrightarrow R^6\longrightarrow R^{11}\longrightarrow R^7\longrightarrow R\longrightarrow 0$$

and one verifies that $H^4_I(R_p)=0$ if and only if $\mathrm{char}\,k\equiv 2 \mod 3$ (this essentially follows from the fact that depending on the modulus of p mod 3 the elliptic curve is supersingular or not). Hence we have that $\lambda_{0,2}=\lambda_{2,3}=0$ if and only if $\mathrm{char}\,k\equiv 2 \mod 3$.

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