

KOSZUL ALGEBRAS AND THE QUANTUM MACMAHON MASTER THEOREM

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ABSTRACT. We give a new proof of the quantum version of MacMahon’s Master Theorem due to Garoufalidis, Le and Zeilberger by deriving it from known facts about Koszul algebras.

1. INTRODUCTION

In [3], Garoufalidis, Le and Zeilberger prove a quantum version of MacMahon’s celebrated “Master Theorem” [6, pp. 97–98]. As stated in [3], the generalization was motivated in part by considerations in quantum topology and knot theory, and it also answers a long-standing open question by G. Andrews [1, Problem 5]. An abundance of different proofs of the original Master Theorem can be found in the literature; the quantum-generalization in [3] is proved by an adaptation of the “operator-elimination” proof given in [10]. Our goal here is to derive the quantum MacMahon Master Theorem of Garoufalidis, Le, and Zeilberger from basic properties of Koszul algebras which fairly effortlessly lead to a generalized MacMahon identity. This identity is stated as equation (8) below; the quantum MacMahon Master Theorem is the special case of (8) where the Koszul algebra in question is the so-called quantum affine n -space. Thus, neither the main result of this note nor the methods employed are ours but we believe that the connection between Koszul algebras and the quantum Master Theorem deserves to be explicitly stated and fully exploited. This connection was in fact already briefly mentioned in the last section of [3], but the proof given here appears to be new.

We have tried to keep this note reasonably self-contained and accessible to readers unfamiliar with Koszul algebras. Sections 2 and 3 serve to deploy the pertinent background material concerning Koszul algebras and cocharacters, respectively, in some detail. The operative technicalities for our proof are collected in Lemmas 1 and 2 below; they are presented here with full proofs for lack of a suitable reference. The quantum MacMahon Master Theorem [3, Theorem 1] is then stated and proved in Section 4. The short final Section 5 discusses certain modifications of the MacMahon identity.

Our basic reference for Koszul algebras are Manin’s notes [7]; for bialgebras, our terminology follows Kassel [5]. We work over a commutative base field \mathbb{k} except in 3.1 and 3.2 where \mathbb{k} can be any commutative ring at no extra cost. Throughout, \otimes will stand for $\otimes_{\mathbb{k}}$.

2. KOSZUL ALGEBRAS

2.1. Quadratic algebras. A *quadratic algebra* is a factor of the tensor algebra $T(V)$ of some finite-dimensional \mathbb{k} -vector space V modulo the ideal generated by some subspace $R(A) \subseteq T(V)_2 = V^{\otimes 2}$. Thus, $A \cong T(V)/(R(A))$. The natural grading of $T(V)$ descends to a grading $A = \bigoplus_{d \geq 0} A_d$ of A with $A_0 = \mathbb{k}$ and $A_1 \cong V$. In practise, one often fixes a \mathbb{k} -basis $\tilde{x}_1, \dots, \tilde{x}_n$ of V . Then $T(V)$ can be viewed as the free algebra $\mathbb{k}\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle$ and the images $x_i = \tilde{x}_i \bmod R(A)$ of the elements \tilde{x}_i in A form a set of generators for A .

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Example (Quantum affine n -space). This is the quadratic algebra $A = A_q^{n|0}$ that is defined, for a fixed $0 \neq q \in \mathbb{k}$, as the factor of $\mathbb{k}\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle$ modulo the ideal generated by the 2-homogeneous elements $\tilde{x}_j \tilde{x}_i - q \tilde{x}_i \tilde{x}_j$ for $1 \leq i < j \leq n$. Thus, the algebra $A_q^{n|0}$ is generated by elements x_1, \dots, x_n satisfying the relations

$$(1) \quad x_j x_i = q x_i x_j \quad \text{for } i < j.$$

2.2. **Quadratic dual.** Given a quadratic algebra $A \cong \mathbb{T}(V)/(R(A))$, consider the subspace $R(A)^\perp$ of the linear dual $(V^{\otimes 2})^*$ consisting of all linear form on $V^{\otimes 2}$ that vanish on $R(A)$. Identifying $(V^{\otimes 2})^*$ with $V^* \otimes V^*$ in the usual way, we may view $R(A)^\perp \subseteq V^* \otimes V^*$ and hence define a new quadratic algebra by

$$A^1 = \mathbb{T}(V^*) / (R(A)^\perp).$$

The algebra A^1 is called the *quadratic dual* of A . If $\tilde{x}_1, \dots, \tilde{x}_n$ is a fixed \mathbb{k} -basis of V , as above, then one usually chooses the dual basis $\tilde{x}^1, \dots, \tilde{x}^n$ for V^* : $\langle \tilde{x}^i, \tilde{x}_j \rangle = \delta_{i,j}$, where $\langle \cdot, \cdot \rangle$ denotes evaluation. This yields algebra generators $x^i = \tilde{x}^i \bmod R(A)^\perp$ for A^1 .

Example (Quantum exterior algebra). Consider the algebra $A = A_q^{n|0}$ as above. The quadratic dual A^1 is denoted $A_q^{0|n}$ in [7]. The above procedure yields algebra generators x^1, \dots, x^n for $A_q^{0|n}$ satisfying the defining relations $x^\ell x^\ell = 0$ for all ℓ and $x^i x^j + q x^j x^i = 0$ for $i < j$.

2.3. **The bialgebra $\text{end } A$.** Given quadratic algebras $A \cong \mathbb{T}(A_1)/(R(A))$ and $B \cong \mathbb{T}(B_1)/(R(B))$, one defines the quadratic algebra $A \bullet B = \mathbb{T}(A_1 \otimes B_1)/(S_{23}(R(A) \otimes R(B)))$. Here, $S_{23}: A_1^{\otimes 2} \otimes B_1^{\otimes 2} \rightarrow (A_1 \otimes B_1)^{\otimes 2}$ switches the second and third factors.

The algebra $\text{end } A = A^1 \bullet A$ is particularly important. Identifying A_1 with V as above, we have

$$\text{end } A = \mathbb{T}(V^* \otimes V) / (S_{23}(R(A)^\perp \otimes R(A))).$$

Fix generators $x_i = \tilde{x}_i \bmod R(A)$ for A and $x^i = \tilde{x}^i \bmod R(A)^\perp$ for A^1 as in 2.1 and 2.2. Then the elements $\tilde{z}_i^j = \tilde{x}^j \otimes \tilde{x}_i$ form a basis of $V^* \otimes V$ and their images $z_i^j = \tilde{z}_i^j \bmod R(\text{end } A)$ form algebra generators for $\text{end } A$. The algebra $\text{end } A$ is endowed with a comultiplication

$$(2) \quad \Delta: \text{end } A \rightarrow \text{end } A \otimes \text{end } A, \quad \Delta(z_i^j) = \sum_{\ell} z_i^\ell \otimes z_\ell^j$$

and a counit

$$\epsilon: \text{end } A \rightarrow \mathbb{k}, \quad \epsilon(z_i^j) = \delta_{i,j} \text{ (Kronecker delta)}$$

which make it a bialgebra over \mathbb{k} ; see [7, 5.7 and 5.8]. Furthermore, defining a coaction

$$(3) \quad \delta_A: A \rightarrow \text{end } A \otimes A, \quad \delta_A(x_i) = \sum_j z_i^j \otimes x_j,$$

the algebra A becomes a left $\text{end } A$ -comodule algebra: δ_A is a \mathbb{k} -algebra map that makes A a comodule for $\text{end } A$; see [7, 5.4].

Example (Right quantum matrices). Returning to the case $A = A_q^{n|0}$, we describe the algebra $\text{end } A_q^{n|0}$. Using the notation and the relations of the Examples in 2.1 and 2.2, the foregoing leads to the following generators for $R(\text{end } A_q^{n|0}) \subseteq (V^* \otimes V)^{\otimes 2}$: for all ℓ and $i < j$ we have a generator $\tilde{z}_j^\ell \otimes \tilde{z}_i^\ell - q \tilde{z}_i^\ell \otimes \tilde{z}_j^\ell$, and for all $i < j$ and $k < \ell$ there is $\tilde{z}_j^k \otimes \tilde{z}_i^\ell - q \tilde{z}_i^k \otimes \tilde{z}_j^\ell + q \tilde{z}_j^\ell \otimes \tilde{z}_i^k - q^2 \tilde{z}_i^\ell \otimes \tilde{z}_j^k$. After multiplying the second set of generators with q^{-1} , we obtain the following relations between the generators z_i^j of $\text{end } A_q^{n|0}$:

$$(4) \quad z_j^\ell z_i^\ell = q z_i^\ell z_j^\ell \quad \text{for all } \ell \text{ and } i < j \quad \text{(column commutation relations)}$$

$$(5) \quad z_i^k z_j^\ell - z_j^\ell z_i^k = q^{-1} z_j^k z_i^\ell - q z_i^\ell z_j^k \quad \text{for } i < j \text{ and } k < \ell \quad \text{(cross commutation relations).}$$

Thus, using the terminology of [3], the $n \times n$ -matrix $Z = (z_i^j)$ is a *right-quantum matrix*.

2.4. The Koszul complex. For any quadratic algebra A , one can define Koszul complexes

$$K^{\ell, \bullet}(A): \quad 0 \rightarrow A_{\ell}^{\bullet} \rightarrow A_{\ell-1}^{\bullet} \otimes A_1 \rightarrow \cdots \rightarrow A_1^{\bullet} \otimes A_{\ell-1} \rightarrow A_{\ell} \rightarrow 0$$

as in [7, 9.6]. The quadratic algebra A is called a *Koszul algebra* if all complexes $K^{\ell, \bullet}(A)$ for $\ell > 0$ are exact. It is known that if A and B are Koszul then so are A^{\bullet} and $A \bullet B$. Moreover, if A has a so-called PBW-basis consisting of certain standard monomials, then A is Koszul; see, e.g., [8, Theorem 4.3.1]. This applies in particular to the algebras $A = A_q^{n|0}$ [8, 4.2 Example 1].

Lemma 1. *Let A be a quadratic algebra. Then all A_i and all A_j^{\bullet} are (left) comodules over $\underline{\text{end}} A$, and hence so are the components $K^{\ell, i}(A) = A_{\ell-i}^{\bullet} \otimes A_i$ of the Koszul complex. Moreover, the Koszul differential is an $\underline{\text{end}} A$ -comodule map.*

Proof. We will write $B = \underline{\text{end}} A$ for brevity; so $B_1 = V^* \otimes V$. Equation (3) shows that δ_A sends $V = A_1$ to $B_1 \otimes V$. Thus, A_i is mapped to $B_i \otimes A_i \subseteq B \otimes A_i$, and so each A_i is a B -comodule. Moreover, as is shown in [7, 5.5], the map δ_A comes from a map $\tilde{\delta}: \mathbb{T}(V) \rightarrow \mathbb{T}(B_1) \otimes \mathbb{T}(V)$, $\tilde{\delta}(\tilde{x}_i) = \sum_j \tilde{z}_i^j \otimes \tilde{x}_j$, which satisfies $\tilde{\delta}(R(A)) \subseteq R(B) \otimes \mathbb{T}(V) + \mathbb{T}(B_1) \otimes R(A)$. Following $\tilde{\delta}$ by the canonical map $\mathbb{T}(B_1) \rightarrow B = \mathbb{T}(B_1)/R(B)$ tensored with $\text{Id}_{\mathbb{T}(V)}$, we obtain a map

$$(6) \quad \delta': \mathbb{T}(V) \rightarrow B \otimes \mathbb{T}(V), \quad \delta'(\tilde{x}_i) = \sum_j z_i^j \otimes \tilde{x}_j$$

satisfying $\delta'(V^{\otimes i}) \subseteq B_i \otimes V^{\otimes i}$ and $\delta'(R(A)) \subseteq B \otimes R(A)$. Therefore, all $V^{\otimes i}$ are B -comodules and $R(A)$ a B -subcomodule of $V^{\otimes 2}$. More generally, the subspaces $V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i}$ for $0 \leq i \leq m-2$ are B -subcomodules of $V^{\otimes m}$, and hence so are the subspaces $\bigcap_{i=0}^{m-2} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i}$ and $R_m(A) := \sum_{i=0}^{m-2} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i}$. Following [7, 9.6], we identify the former with the linear dual of $A_m^{\bullet} = V^* \otimes^m / \sum_{i=0}^{m-2} V^* \otimes^i \otimes R(A)^{\perp} \otimes V^* \otimes^{m-2-i}$, so

$$(7) \quad A_m^{\bullet} = \bigcap_{i=0}^{m-2} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i}.$$

Also according to [ibid, 9.6], the diagram

$$\begin{array}{ccc} A_{m+1}^{\bullet} \otimes V^{\otimes n-1} & \hookrightarrow & A_m^{\bullet} \otimes V^{\otimes n} \\ \cup & & \cup \\ A_{m+1}^{\bullet} \otimes R_{n-1}(A) & \hookrightarrow & A_m^{\bullet} \otimes R_n(A) \end{array}$$

induces the Koszul differential

$$A_{m+1}^{\bullet} \otimes A_{n-1} = A_{m+1}^{\bullet} \otimes \frac{V^{\otimes n-1}}{R_{n-1}(A)} \longrightarrow A_m^{\bullet} \otimes \frac{V^{\otimes n}}{R_n(A)} = A_m^{\bullet} \otimes A_n$$

Since all spaces here are B -comodules, the differential is a B -comodule map. \square

3. BIALGEBRAS AND COCHARACTERS

3.1. The Grothendieck ring. For now, let B denote an arbitrary bialgebra over some commutative base ring \mathbb{k} . We let \mathcal{R}_B denote the Grothendieck ring of all (left) B -comodules that are finitely generated (f.g.) projective over \mathbb{k} . Thus, for each such B -comodule, V , there is an element $[V] \in \mathcal{R}_B$ and any short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of B -comodules (f.g. projective over \mathbb{k}) gives rise to an equation $[V] = [U] + [W]$ in \mathcal{R}_B . Multiplication in \mathcal{R}_B is given by the tensor product of B -comodules; see [5, III.6].

3.2. Cocharacters. Continuing with the notation of 3.1, let V be a B -comodule that is f.g. projective over \mathbb{k} . The structure map $\delta_V: V \rightarrow B \otimes V$ is an element of $\text{Hom}_{\mathbb{k}}(V, B \otimes V)$. Using the standard isomorphisms $\text{Hom}_{\mathbb{k}}(V, B \otimes V) \cong B \otimes V \otimes V^*$ (see, e.g., [2, II.4.2]) and letting $\langle \cdot, \cdot \rangle: V \otimes V^* \rightarrow \mathbb{k}$ denote the evaluation map, we have a homomorphism

$$\text{Hom}_{\mathbb{k}}(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\text{Id}_B \otimes \langle \cdot, \cdot \rangle} B \otimes \mathbb{k} \cong B.$$

The image of δ_V under this map will be denoted by χ_V . If V is free over \mathbb{k} , with basis $\{v_i\}$, and $\delta_V(v_j) = \sum_i b_{i,j} \otimes v_i$ then

$$\chi_V = \sum_i b_{i,i}.$$

Lemma 2. *The map $[V] \mapsto \chi_V$ yields a well-defined ring homomorphism $\chi: \mathcal{R}_B \rightarrow B$.*

Proof. The assertion is equivalent to the identities $\chi_{V \otimes W} = \chi_V \chi_W$ for any two B -comodule V and W (f.g. projective over \mathbb{k}) and $\chi_V = \chi_U + \chi_W$ for any short exact sequence $0 \rightarrow U \xrightarrow{\mu} V \xrightarrow{\pi} W \rightarrow 0$. Both are easy to check; we sketch the proof of the second identity. Fix \mathbb{k} -linear splittings $\sigma: W \rightarrow V$ and $\tau: V \rightarrow U$ with $\tau \circ \mu = \text{Id}_U$, $\pi \circ \sigma = \text{Id}_W$ and $\sigma \circ \pi + \mu \circ \tau = \text{Id}_V$. Under the canonical isomorphism $V \otimes V^* \cong \text{End}_{\mathbb{k}}(V)$, evaluation $\langle \cdot, \cdot \rangle$ becomes the trace map $\text{trace}_V: \text{End}_{\mathbb{k}}(V) \rightarrow \mathbb{k}$. For any $\varphi \in \text{End}_{\mathbb{k}}(V)$, we have $\text{trace}_V(\varphi) = \text{trace}_U(\tau \circ \varphi \circ \mu) + \text{trace}_W(\pi \circ \varphi \circ \sigma)$. The equation $\delta_V \circ \mu = (\text{Id}_B \otimes \mu) \circ \delta_U$ gives $\delta_U = (\text{Id}_B \otimes \tau) \circ \delta_V \circ \mu$. Similarly, $\delta_W = (\text{Id}_B \otimes \pi) \circ \delta_V \circ \sigma$. Therefore, $\chi_V = (\text{Id}_B \otimes \text{trace}_V)(\delta_V) = (\text{Id}_B \otimes \text{trace}_U)(\delta_U) + (\text{Id}_B \otimes \text{trace}_W)(\delta_W) = \chi_U + \chi_W$, as desired. \square

3.3. Application to Koszul algebras. Returning to the case of a Koszul algebra A over a field \mathbb{k} , we apply the foregoing to the bialgebra $B = \underline{\text{end}} A$. By Lemma 1, the (exact) Koszul complex $K^{\ell, \bullet}(A)$ for $\ell > 0$ gives an equation in \mathcal{R}_B :

$$\sum_i (-1)^i [A_i^! *] [A_{\ell-i}] = 0.$$

Equivalently, defining the Poincaré series $P_A(t) = \sum_i [A_i] t^i$ and $P_{A^! *}(t) = \sum_i [A_i^! *] t^i$, we have

$$P_A(t) P_{A^! *}(-t) = 1$$

in the power series ring $\mathcal{R}_B[[t]]$. Applying the ring homomorphism $\chi[[t]: \mathcal{R}_B[[t]] \rightarrow B[[t]]$ (Lemma 2), this equation becomes the following equation in $B[[t]]$:

$$(8) \quad \left(\sum_{\ell \geq 0} \chi_{A_\ell} t^\ell \right) \cdot \left(\sum_{m \geq 0} (-1)^m \chi_{A_m^! *} t^m \right) = 1.$$

Since the coactions δ_{A_ℓ} and $\delta_{A_m^! *}$ in (3) and (6) respect the grading, both factors actually belong to the subring $\prod_{n \geq 0} B_n t^n$ of $B[[t]]$.

4. PROOF OF THE QUANTUM MACMAHON MASTER THEOREM

The quantum MacMahon Master Theorem [3, Theorem 1] is the case $A = A_q^{n|0}$ of equation (8). In detail, choose generators x_1, \dots, x_n for A as in 2.1. For each n -tuple $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$, put $x^{\mathbf{m}} := x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \in A$. Then the homogeneous component A_ℓ has a \mathbb{k} -basis consisting of elements $x^{\mathbf{m}}$ with $|\mathbf{m}| = m_1 + m_2 + \dots + m_n = \ell$. (This is the PBW-basis of $A_q^{n|0}$ referred to earlier.) With respect to this basis, the coaction δ_A of $B = \underline{\text{end}} A$ on A_ℓ in (3) has the form

$$\delta_A(x^{\mathbf{m}}) = \delta_A(x_1)^{m_1} \delta_A(x_2)^{m_2} \dots \delta_A(x_n)^{m_n} = \sum_{\substack{\mathbf{r} \in \mathbb{Z}_{\geq 0}^n \\ |\mathbf{r}| = \ell}} b_{\mathbf{r}, \mathbf{m}} \otimes x^{\mathbf{r}}$$

for uniquely determined $b_{\mathbf{r}, \mathbf{m}} \in B_\ell$. In particular, $G(\mathbf{m}) := b_{\mathbf{m}, \mathbf{m}} \in B_{|\mathbf{m}|}$ has the same meaning as in [3]. Taking the trace we obtain the equality

$$(9) \quad \chi_{A_\ell} = \sum_{\mathbf{m}: |\mathbf{m}|=\ell} G(\mathbf{m}) .$$

In order to deal with the second factor in equation (8), we identify $A_m^!*$ with a subspace of $\mathbb{T}(V)_m = V^{\otimes m}$ as in (7), and $\mathbb{T}(V)$ with the free algebra $\mathbb{k}\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle$ as in 2.1. Then $A_m^!*$ has a basis consisting of the elements

$$\wedge \tilde{x}_J := \sum_{\pi \in \mathfrak{S}_m} (-q)^{-l(\pi)} \tilde{x}_{j_{\pi_1}} \otimes \tilde{x}_{j_{\pi_2}} \otimes \dots \otimes \tilde{x}_{j_{\pi_m}} ,$$

where $J = (j_1 < j_2 < \dots < j_m)$ is an m -element subset of $\{1, \dots, n\}$, \mathfrak{S}_m is the symmetric group and $l(\pi)$ denotes the number of pairs (i, j) with $1 \leq i < j \leq m$ and $\pi i > \pi j$. Indeed, recalling the fact that $R(A^!) = R(A)^\perp$ is generated by the elements $\tilde{x}^\ell \tilde{x}^\ell$ for all ℓ and $\tilde{x}^i \tilde{x}^j + q \tilde{x}^j \tilde{x}^i = 0$ for $i < j$ (see 2.2), it is straightforward to check that $\wedge \tilde{x}_J$ vanishes on $R_m(A^!)$. Hence, $\wedge \tilde{x}_J$ belongs to $A_m^!*$. Furthermore, the elements $\wedge \tilde{x}_J$ are obviously linear independent over \mathbb{k} and their number is $\binom{n}{m}$ which is equal to the dimension of $A_m^!*$. We claim that the coaction δ' of $B = \text{end } A$ on $A_m^!*$ in equation (6) has the form

$$(10) \quad \delta'(\wedge \tilde{x}_J) = \det_q(Z_J) \otimes \wedge \tilde{x}_J + \text{other terms} ,$$

where

$$(11) \quad \det_q(Z_J) = \sum_{\pi \in \mathfrak{S}_m} (-q)^{-l(\pi)} z_{j_{\pi_1}}^{j_1} z_{j_{\pi_2}}^{j_2} \dots z_{j_{\pi_m}}^{j_m}$$

denotes the quantum determinant (as defined in [9]) of the submatrix $Z_J = (z_i^j)_{i,j \in J}$ of the generic $n \times n$ right-quantum matrix $Z = (z_i^j)$ as in 2.3. In order to prove (10), it suffices to show that, for $J = (1, 2, \dots, n) = [1, n]$, we have

$$(12) \quad \delta'(\wedge \tilde{x}_{[1, n]}) = \det_q(Z) \otimes \wedge \tilde{x}_{[1, n]} .$$

To this end, note that the element $\wedge \tilde{x}_{[1, n]}$ spans the one-dimensional B -subcomodule $A_n^!*$ of $V^{\otimes n}$. Hence, $\delta'(\wedge \tilde{x}_{[1, n]}) = D \otimes \wedge \tilde{x}_{[1, n]}$ for some group-like element $D \in B$; cf. [7, 8.2]. We need to show that $D = \det_q(Z) = \sum_{\pi \in \mathfrak{S}_n} (-q)^{-l(\pi)} z_{\pi_1}^1 z_{\pi_2}^2 \dots z_{\pi_n}^n$. But equation (6) readily implies that

$$\delta'(\tilde{x}_{\pi_1} \otimes \dots \otimes \tilde{x}_{\pi_n}) = z_{\pi_1}^1 \dots z_{\pi_n}^n \otimes \tilde{x}_1 \otimes \dots \otimes \tilde{x}_n + \text{other terms} .$$

Multiplying with $(-q)^{-l(\pi)}$ and summing up on π we see that $\delta'(\wedge \tilde{x}_{[1, n]}) = \det_q(Z) \otimes \tilde{x}_1 \otimes \dots \otimes \tilde{x}_n + \text{other terms}$. This implies (12), thereby completing the proof of (10). Therefore, the cocharacter of $A_m^!*$ is given by

$$(13) \quad \chi_{A_m^!} = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=m}} \det_q(Z_J) .$$

To summarize, we rephrase equation (8) with (9) and (13) using the notation of [3]:

Theorem 3 (Garoufalidis, Le and Zeilberger). *Let A denote the \mathbb{k} -algebra generated by x_1, \dots, x_n subject to the relations (1), B the \mathbb{k} -algebra generated by z_1^1, \dots, z_n^n subject to (4) and (5), and let $Z = (z_i^j)$ denote the generic right-quantum $n \times n$ -matrix. Consider the elements $X_i = \sum_j z_i^j \otimes x_j \in B \otimes A = \bigoplus_{\mathbf{m}} B \otimes x^{\mathbf{m}}$, where \mathbf{m} runs over the n -tuples $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$ and $x^{\mathbf{m}} := x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$. Define power series in $B[[t]]$ by*

$$\text{Bos}(Z) := \sum_{\ell \geq 0} \sum_{\mathbf{m}: |\mathbf{m}|=\ell} G(\mathbf{m}) t^\ell ,$$

where $G(\mathbf{m})$ is the B -coefficient of $x^{\mathbf{m}}$ in $X^{\mathbf{m}} = X_1^{m_1} X_2^{m_2} \dots X_n^{m_n}$, and

$$\text{Ferm}(Z) := \sum_{m \geq 0} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=m}} (-1)^m \det_q(Z_J) t^m$$

with $\det_q(Z_J)$ as in (11). Then:

$$\text{Bos}(Z) \cdot \text{Ferm}(Z) = 1 .$$

5. MODIFYING THE MACMAHON IDENTITY

Applying endomorphisms of $B[[t]]$ to the generalized MacMahon identity (8), we obtain new versions of this identity. In this section, we discuss a particular example for the case $A = A_q^{n|0}$. As usual, we let $B = \underline{\text{end}} A$.

The algebraic torus $T = (\mathbb{k}^*)^n$ acts on B via

$$(14) \quad \tau(z_i^j) = c_j z_i^j$$

for $\tau = (c_1, \dots, c_n) \in T$. Indeed, τ respects the relations (4) and (5) of B , and hence τ defines a graded algebra automorphism of B . Note however that, unless $\tau = 1$, the automorphism τ does not preserve the comultiplication Δ of B in (2) but rather is a homomorphism of left B -comodules, that is, $\Delta \circ \tau = (\text{Id}_B \otimes \tau) \circ \Delta$. Applying $\tau = (c_1, \dots, c_n)$ to the cocharacters of A_ℓ and $A_m^!*$ as determined in (9) and (13), we obtain

$$(15) \quad \tau(\chi_{A_m^!}) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=m}} c_J(\tau) \det_q(Z_J) \quad \text{with } c_J(\tau) = \prod_{j \in J} c_j$$

$$(16) \quad \tau(\chi_{A_\ell}) = \sum_{|\mathbf{m}|=\ell} c_{\mathbf{m}}(\tau) G(\mathbf{m}) \quad \text{with } c_{\mathbf{m}}(\tau) = \prod_i c_i^{m_i} .$$

The particular choice

$$(17) \quad \tau = (q^{n-1}, q^{n-3}, \dots, q^{1-n})$$

leads to the following version of quantum MacMahon Master theorem:

$$(18) \quad \widetilde{\text{Bos}}(Z) \cdot \widetilde{\text{Ferm}}(Z) = 1 ,$$

where

$$(19) \quad \widetilde{\text{Bos}}(Z) := \sum_{\ell \geq 0} \sum_{\mathbf{m}: |\mathbf{m}|=\ell} q^{\ell(n+1)-2 \sum im_i} G(\mathbf{m}) t^\ell$$

$$(20) \quad \widetilde{\text{Ferm}}(Z) := \sum_{m \geq 0} \sum_{J=(j_1 < \dots < j_m)} (-1)^m q^{m(n+1)-2(j_1+j_2+\dots+j_m)} \det_q(Z_J) t^m .$$

Remark. Let $H := \underline{\text{gl}}(A)$ be the coordinate ring on the quantum general linear group (cf. [7, 8.5]). Then H is a Hopf algebra which is in fact coquasitriangular or cobraided (cf. [5, VIII.5]). Thus for any finite dimensional comodule X , there exists a canonical isomorphism $X \rightarrow X^{**}$ of H -comodules, given in terms of the braiding. This isomorphism is in general not compatible with the tensor product. We note that the category of H -comodules also possesses a ribbon [5, XIV.6]. By composing the above canonical isomorphism with the ribbon, one obtains a functorial isomorphism $\tau_X : X \rightarrow X^{**}$ which is compatible with the tensor product in the sense that

$$\tau_{X \otimes Y} = \tau_X \otimes \tau_Y .$$

Using τ_X we can define a new type of cocharacter of X , called quantum cocharacter, as follows (cf. [4]). $\chi_{q,X}$ is the image of $1 \in \mathbb{k}$ under the map

$$\mathbb{k} \rightarrow X^* \otimes X \xrightarrow{\text{Id} \otimes \tau_X} X^* \otimes X^{**} \rightarrow H \otimes X^* \otimes X^{**} \xrightarrow{\text{ev}} H$$

where db is the “dual base” map $1 \mapsto \sum_i x^i \otimes x_i$. As for the ordinary cocharacter, one can show that the quantum cocharacter is multiplicative with respect to the tensor product and additive with respect to exact sequences. Applying the quantum cocharacter to the Koszul complex in 2.4 we obtain an identity analogous to (8) for χ_{q,A_ℓ} and $\chi_{q,A_m^!}$.

Explicit computation shows that, for $X = V = A_1$, $\tau_V = \text{diag}(q^{n-1}, q^{n-3}, \dots, q^{1-n})$. Further, one can check that

$$\chi_{q,A_\ell} = \tau(\chi_{A_\ell}); \quad \chi_{q,A_m^!} = \tau(\chi_{A_m^!})$$

This explains the origin of the choice of τ in (17).

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