On complex surfaces with 5 or 6 semistable singular fibers over $\mathbb{P}^1$ *

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Introduction

We denote by X a complex smooth projective surface, and by $f : X \to C$ a fibration over a curve C whose generic fiber is a curve of genus g. f is called *isotrivial* if all smooth fibers are isomorphic to a fixed curve. f is called *semistable* if all of the singular fibers are reduced nodal curves. If there is no (-1)-curve contained in the fibers, then we call f *relatively minimal*. The projection $X = F \times C \to C$ is called a trivial fibration. s is always the number of singular fibers of f.

In a well-known paper [3], Beauville proved that a non-isotrivial fibration $f : X \to \mathbb{P}^1$ admits at least 3 singular fibers, and if f is semistable, then $s \geq 4$. The first author proved in [11] that if f is a semistable fibration of genus $g \geq 2$, then it admits at least 5 singular fibers.

Beauville [4] gave a beautiful explicit classification of all semistable elliptic fibrations ($g = 1$) over $\mathbb{P}^1$ with 4 singular fibers. More precisely, there are exactly 6 non-isotrivial such families, and all of them are modular families of elliptic curves. A very interesting natural problem is to classify semistable fibrations over $\mathbb{P}^1$ of genus $g \geq 2$ with 5 singular fibers. In the survey ([13], §1 and Ex. 5.9), Viehweg considered this problem. The purpose of this note is to try to get some information on the structure of the surface X when f has 5 or 6 singular fibers. Our main results are the following two theorems.

Theorem 0.1. *Assume that $f : X \to \mathbb{P}^1$ is a non trivial semistable fibration of genus $g \geq 2$ with s singular fibers. Assume also that f is relatively minimal. Then we have:*

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(1) *If $s = 5$, then X is birationally rational or ruled.*

(2) *If $s = 6$ and $g = 2$, 3 or 4, then X is not of general type.*

(3) *If $s = 6$, $g = 5$ and X is of general type, then the minimal model S of X satisfies*

$$K_S^2 = 1, \quad p_g(S) = 2, \quad q(S) = 0.$$

The fibration f comes from a pencil on S with 5 simple base points (including infinitely near base points).

Thus if the Kodaira dimension of X is non negative, then $s \geq 6$. We have no example of case (3), and we conjecture that s is at least 7 when X is of general type. We shall give two examples to show that the two bounds are sharp.

A) $s = 6$, $g = 3$, $K_X^2 = -4$, and X is birationally a K3 surface.

B) $s = 7$, $g = 4$, $K_X^2 = -3$, $p_g(X) = 2$, $q(X) = 0$, and X is of general type containing 4 (-1)-curves.

Our next purpose is to give some inequalities for genus $g \geq 2$ fibrations over $\mathbb{P}^1$ (not necessarily semistable), which imply the lower bounds on s.

Theorem 0.2. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration of genus $g \geq 2$, and let $K_f = K_{X/\mathbb{P}^1} = K_X + 2F$ be its relative canonical divisor. Assume that f is not locally trivial. Then*

$$K_f^2 \geq \begin{cases} 4g-4, & \text{if } \kappa(X) = -\infty, \\ 6g-6, & \text{if } \kappa(X) = 0, \\ 6g-5, & \text{if } \kappa(X) = 1, \\ 6g-6+\dfrac{1}{2}\left(K_S^2 + \sqrt{K_S^2}\sqrt{K_S^2+8g-8}\right), & \text{if } \kappa(X) = 2, \end{cases}$$

where S is the unique minimal model of X when $\kappa(X) \geq 0$.

The bounds are optimal for infinitely many g. The example given in [11] satisfies $K_f^2 = 4(g-1)$, $g = 2$ and $s = 5$. The example A) above satisfies $K_f^2 = 6(g-1)$, $g = 3$ and $s = 6$, where X is a K3 surface. In Theorem 2.1, we will classify the fibrations $f : X \to \mathbb{P}^1$ with minimal K_f^2 according to its Kodaira dimension.

The proof of Theorem 0.1 is to use some inequalities, particularly the following strict canonical class inequality and its refinement for a non trivial semistable fibration $f : X \to C$ of genus $g \geq 2$ with $s \neq 0$ [11] (see also [9] for a differential geometric proof):

$$K_{X/C}^2 < (2g-2)(2g(C)-2+s). \tag{0.1}$$

We use Reider's method to prove Theorem 0.2. Note that the strict canonical class inequality and the inequality $K_f^2 \geq 6(g-1)$ (resp. $K_f^2 \geq 4(g-1)$) in Theorem 0.2 imply that $s \geq 6$ (resp. 5).

1 Preliminaries

Let $f : X \to C$ be a fibration of genus $g \geq 2$, namely X (resp. C) is a nonsingular complex surface (resp. curve) and the generic fiber F of f is a nonsingular curve of genus g. We always assume that f is relatively minimal, i.e., there is no (-1)-curve contained in the fibers. f is called *semistable* if all of the singular fibers are reduced nodal curves.

Denote by $K_f = K_{X/C} = K_X - f^*K_C$ the relative canonical divisor of f, and by $\omega_{X/C}$ its corresponding invertible sheaf. The relative invariants of f are defined as follows:

$$\begin{aligned}
\chi_f &= \deg f_*\omega_{X/C} = \chi(\mathcal{O}_X) - (g-1)(g(C)-1),\\
K_f^2 &= K_{X/C}^2 = K_X^2 - 8(g-1)(g(C)-1),\\
e_f &= \chi_{\rm top}(X) - 4(g-1)(g(C)-1) = \sum_{i=1}^{s}(\chi_{\rm top}(F_i) - (2-2g)).
\end{aligned}$$

These invariants are nonnegative, and $K_f^2 = 0$ (equivalently, $\chi_f = 0$) if and only if f is locally trivial. $e_f = 0$ iff f is smooth. Let

$$e_{F_i} = \chi_{\rm top}(F_i) - (2-2g).$$

If F_i is semistable, then e_{F_i} is equal to the number of nodes of F_i.

Arakelov [1] and Beauville [5] proved that K_f is a nef divisor and the curves E with $EK_f = 0$ are those (-2)-curves in the fibers. The map defined by $|nK_f|$ for large n is the contraction morphism $\sigma : X \to X^\#$ of the vertical (-2)-curves (cf. [12]), we get the relative canonical model $f^\# : X^\# \to C$ of f.

In what follows, we assume that $f : X \to C$ is semistable with s singular fibers. Denote by q a singular point of $X^\#$. Then $(X^\#, q)$ is a rational double point of type A_{μ_q}, here μ_q is the number of (-2)-curves in X over q. We also denote by q the singular point of the fibers on the smooth part of $X^\#$, in this case $\mu_q = 0$. Then

$$e_f = \sum_q (\mu_q + 1)$$

is the number of nodes in the singular fibers. For convenience, we let

$$r_f := \sum_q \frac{1}{\mu_q + 1}.$$

It is obvious that

$$r_f \le e_f. \tag{1.1}$$

In [11], we proved the strict canonical class inequality when $s \neq 0$:

$$K_f^2 < (2g-2)(2g(C) - 2 + s).$$

The inequality follows from the following

$$K_f^2 - (2g-2)(2g(C)-2+s) \le -\frac{(2g-2)s}{e} + \frac{3 \cdot r_f}{e^2}, \tag{1.2}$$

where $e \ge 2$ is any integer (see [11], p.594, (5)). Equivalently,

$$r_f \ge \frac{1}{3}e^2\left(K_X^2 - (2g-2)\left(6g(C) - 6 + s - \frac{s}{e}\right)\right). \tag{1.3}$$

Lemma 1.1. *Let $q_1, \cdots, q_r$ be the points such that $\mu_{q_i} \neq 0$. Let $\ell' = \sum_{\mu_q \neq 0} \mu_q$ be the number of (-2)-curves contained in the fibers of f. Then*

$$r_f \le e_f - \ell' - \frac{r}{2}. \tag{1.4}$$

Proof. Note that the number of points q such that $\mu_q = 0$ is $e_f - \sum_i (\mu_{q_i} + 1)$. Hence

$$\begin{aligned}\sum_q \frac{1}{\mu_q + 1} &= e_f - \sum_i (\mu_{q_i} + 1) + \sum_i \frac{1}{\mu_{q_i} + 1} \\ &\leq e_f - \ell' - r + \frac{r}{2} \\ &= e_f - \ell' - \frac{r}{2}.\end{aligned}$$

This completes the proof. □

Lemma 1.2. $e_{F_i} = g - 1 + c_i - g(\widetilde{F}_i)$, *where* c_i *is the number of components of* F_i *and* $\widetilde{F}_i$ *is the normalization of* F_i. *So*

$$e_f \leq s(g-1) + \sum_{i=1}^{s} c_i. \tag{1.5}$$

Proof. Let $\sigma : \widetilde{F}_i \to F_i$ be the normalization of F_i, and Δ be the subscheme of the singular locus of F_i. Then we have

$$0 \to \mathcal{O}_{F_i} \to \sigma_* \mathcal{O}_{\widetilde{F}_i} \to \mathcal{O}_\Delta \to 0.$$

Since F_i has only nodes as its singular points, we have $e_{F_i} = \deg \Delta = h^0(\mathcal{O}_\Delta) = \chi(\mathcal{O}_\Delta)$. Thus

$$\begin{aligned} e_{F_i} &= \chi(\mathcal{O}_{\widetilde{F}_i}) - \chi(\mathcal{O}_{F_i}) \\ &= c_i - g(\widetilde{F}_i) - (1-g) \\ &\leq g - 1 + c_i. \end{aligned}$$

This completes the proof. □

Corollary 1.3. *Denote by* $\ell = \sum_i c_i - \ell'$ *the number of curves in the singular fibers different from* (-2)*-curves. Then we have*

$$r_f \leq s(g-1) + \ell - \frac{r}{2}. \tag{1.6}$$

Proof. This follows from (1.4) and (1.5). □

Lemma 1.4. *If* $e^2 > \frac{6g+3}{g-1}$, *then*

$$K_f^2 \leq \frac{2e(g-1)^2\left((2g(C) - 2 + s)e - s\right)}{e^2(g-1) - 3(2g+1)}. \tag{1.7}$$

Proof. By Cornalba-Harris-Xiao's inequality [6, 14],

$$K_f^2 \geq \frac{4g-4}{g}\chi_f,$$

we get

$$e_f \leq \frac{2g+1}{g-1} K_f^2.$$

Then from (1.1) and (1.2), we obtain

$$K_f^2 - (2g-2)(2g(C) - 2 + s) \leq -\frac{(2g-2)s}{e} + \frac{3(2g+1)}{e^2(g-1)} K_f^2,$$

it implies (1.7). □

2 The proof of Theorem 0.2

In this section, we let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration of genus $g \geq 2$. If $\kappa(X) \neq -\infty$, we denote by S the unique minimal model of X.

Theorem 2.1. *Assume that f is not locally trivial. Then*

$$K_f^2 \geq \begin{cases} 4g-4, & \text{if } \kappa(X) = -\infty, \\ 6g-6, & \text{if } \kappa(X) = 0, \\ 6g-5, & \text{if } \kappa(X) = 1, \\ 6g-6+\frac{1}{2}\left(K_S^2+\sqrt{K_S^2}\sqrt{K_S^2+8g-8}\right), & \text{if } \kappa(X) = 2. \end{cases}$$

(1) *$K_f^2 = 4g-4$ if and only if X is the minimal resolution of the singularities of a double covering surface $Z \xrightarrow{\pi} \mathbb{P}^1 \times C$ ramified over a curve of numerical type $2F_1 + (2g+2-4g(C))F_2$, and the fibration f is induced by the first projection $\mathbb{P}^1 \times C \to \mathbb{P}^1$. Here F_i is a fiber of the i-th projection of $\mathbb{P}^1 \times C$.*

(2) *If $\kappa(X) = 0$, then $K_f^2 = 6g-6$ iff the fibration is induced by a pencil $\Lambda \subset |C|$ on its minimal model S with $C^2 = 2g-2$ simple base points (including infinitely near base points).*

(3) *If $\kappa(X) = 1$, then $K_f^2 = 6g-5$ iff the fibration is induced by a pencil $\Lambda \subset |C|$ of genus g on its minimal model S with $C^2 = 2g-3$ simple base points.*

(4) *If $\kappa(X) = 2$, then $K_S^2 \geq 1$. $K_f^2 = 6(g-1)+\frac{1}{2}\left(K_S^2+\sqrt{K_S^2}\sqrt{K_S^2+8g-8}\right)$ iff the fibration is induced by a pencil $\Lambda \subset |C|$ on S with only simple base points and $C \sim rK_S$. In particular, if $g=2$, then $K_f^2 \geq 8$, with equality iff $K_S^2 = 1$ and $C \sim K_S$. If $g = 3$, then $K_f^2 \geq 15$.*

Proof. (1) Let $A = K_X + F$. We claim that $|2A|$ (resp. $|3A|$) is base point free if $g \geq 3$ (resp. $g = 2$). In particular, A is nef and hence

$$K_f^2 - 4(g-1) = A^2 \geq 0.$$

Indeed, by Ramanujan's vanishing theorem ([2], p.131), $H^1(-F) = H^0(-F) = 0$. So

$$h^0(A) = h^2(-F) = \chi(-F) = \chi_f > 0.$$

Hence we can assume that A is an effective divisor. Since $AF = 2g-2 \geq 2$, A admits at least one horizontal component. Hence $AK_f \geq 1$.

Note that $|2A| = |K_X + K_f|$. If $g \geq 3$, $K_f^2 = AK_f + 2(g-1) \geq 1+4 = 5$. Suppose $|2A|$ has a base point p, by Reider's theorem [10], there is a curve E passing through p such that

(i) $K_fE = 0$ and $E^2 = -1$; or (ii) $K_fE = 1$ and $E^2 = 0$.

On the other hand, $K_fE = K_XE + 2FE \equiv K_XE \equiv E^2 \pmod 2$. Thus the two cases can not exist. This proves that $|2A|$ has no base point.

Now we consider the case $g = 2$. We first prove that A is nef.

Suppose Γ is an irreducible and reduced curve with $A\Gamma < 0$. It is easy to see that Γ must be a horizontal curve. Hence $(A+F)\Gamma = K_f\Gamma \geq 1$. Then we have $\Gamma F \geq 2$. On the other hand, $|A|$ is non empty, so Γ is the fixed part of $|A|$. Hence $2 = 2g-2 = AF \geq \Gamma F \geq 2$. We get $\Gamma F = 2$, and $A = \Gamma + E$, where E consists of vertical curves. It implies that $\Gamma^2 = A\Gamma - E\Gamma < 0$. Since $A\Gamma = K_X\Gamma + 2 < 0$, we have $K_X\Gamma \leq -3$. So $p_a(\Gamma) < 0$, impossible. Therefore, A is nef and $A^2 \geq 0$.

In this case, $|3A| = |K_X+2A+F|$. $L = 2A+F$ is nef and $L^2 = 4A^2+4AF \geq 8(g-1) > 4$. If $|3A|$ has a base point p, then there is a curve E passing through p such that
(i) $LE = 0$, $E^2 = -1$; or (ii) $LE = 1$, $E^2 = 0$.
Since A and F are nef, (i) implies $AE = FE = 0$. Note that $A = K_X + F$, we have $K_XE = 0$ and $K_XE \not\equiv E^2 \pmod 2$, a contradiction. So case (i) is impossible. In case (ii), we have $AE = 0$ and $FE = 1$, so $K_XE = -1$ which is also impossible since $E^2 = 0$. This proves that $|3A|$ is base point free.

Now we consider the case when $K_f^2 = 4g-4$, i.e., $A^2 = 0$.

In this case, the base point free linear system $|6A|$ is composed with a fibration $\varphi : X \to C$ over a smooth curve C . Denote by F' a generic fiber of φ. Then $AF' = 0$. Since $AF = 2g-2 \geq 2$, F' can not be the fiber of f. So $FF' \geq 1$. From $AF' = 0$, we get $K_XF' = -FF' \leq -1$. Since ${F'}^2 = 0$, we have $K_XF' = -2$ and $FF' = 2$. So $F' \cong \mathbb{P}^1$. Now we get a generally double cover $\pi : X \to \mathbb{P}^1 \times C$ defined by $\pi(x) = (f(x), \varphi(x))$. The pullback π^*F_2 of a generic fiber F_2 is still isomorphic to $\mathbb{P}^1$, and a generic π^*F_1 has genus g. By Hurwitz formula, we see that the branch curve B_π of π has numerical type $2F_1 + (2g+2-4g(C))F_2$.

Conversely, if the branch curve B_π has numerical type $2F_1 + (2g+2-4g(C))F_2$, then B_π is nonsingular or admits at most ADE singularities. Thus the canonical resolution is the minimal one, and the fibration induced is relatively minimal. By easy computations ([2], p.183, or [8]), we have $K_X^2 = -4(g-1)$ and hence $K_f^2 = 4g-4$. This completes the proof of (1).

(2) Suppose that the Kodaira dimension of X is non negative, and let S be the unique minimal model of X obtained by contracting (-1)-curves. So the fibers F of f are contracted to a pencil Λ in $|C|$ on S with base points $p_1, \cdots, p_m$ (including infinitely near base points). We consider a generic curve C in Λ, and let n_i be the multiplicity of C at p_i. Then $n_i \geq 1$. Note that $F^2 = 0$, $K_XF = 2g-2$. One can prove easily that

$$C^2 = F^2 + \sum_{i=1}^m n_i^2 = \sum_{i=1}^m n_i^2, \tag{2.1}$$

$$K_SC = K_XF - \sum_{i=1}^m n_i = 2g-2-\sum_{i=1}^m n_i. \tag{2.2}$$

Since S is neither ruled nor rational, K_S is nef and $K_S^2 \geq 0$. Hence $K_SC \geq 0$. Thus $m \leq \sum_i n_i \leq 2g-2-K_SC$.

$$K_f^2 = K_X^2 + 8(g-1) = K_S^2 - m + 8(g-1) \geq K_S^2 + K_SC + 6(g-1). \tag{2.3}$$

Hence $K_f^2 \geq 6g-6$. If $K_f^2 = 6g-6$, i.e., $K_X^2 = -(2g-2) < 0$, then $K_SC = 0$, $K_S^2 = 0$ and $n_i = 1$ for any i. In particular, $m = K_S^2 - K_X^2 = 2g-2 > 0$ and $C^2 > 0$. By Hodge index theorem, K_S is numerically trivial, thus $\kappa(X) = 0$. By (2.1), $C^2 = m = 2g-2$. This completes the proof of (2).

(3) We have proved that if $\kappa(X) \geq 0$ and $K_f^2 = 6g-6$, then $\kappa(X) = 0$. So if $\kappa(X) = 1$, we obtain $K_f^2 \geq 6g-5$ and $K_S^2 = 0$. Suppose $K_f^2 = 6g-5$, i.e., $K_X^2 = -(2g-3) < 0$, then $m = 2g-3 > 0$ and $C^2 > 0$. As in the proof of (2), $K_SC \neq 0$ (otherwise, by Hodge index theorem, $K_S \sim 0$ and $\kappa(X) = 0$). So $K_SC \geq 1$. From our assumption $K_f^2 = 6g-5$ and (2.3), we get $K_SC = 1$. Now (2.2) implies that $n_i = 1$ for any i, hence $C^2 = 2g-3$ and C is smooth. The fibration is induced by a pencil in $|C|$ with $2g-3$ simple base points.

(4) In the case when $\kappa(X) = 2$, we let $x = CK_S$, $y = C^2$. Then

$$x + y \geq 2g-2, \quad K_S^2 \cdot y \leq x^2. \tag{2.4}$$

It is easy to prove that

$$x \geq \frac{1}{2}\left(-K_S^2 + \sqrt{K_S^2}\sqrt{K_S^2+8g-8}\right).$$

Now from (2.3), we have

$$K_f^2 \geq 6g - 6 + \frac{1}{2}\left(K_S^2 + \sqrt{K_S^2}\sqrt{K_S^2 + 8g - 8}\right).$$

If the equality holds, then the equalities in (2.4) hold. We obtain easily the desired characterization. □

Corollary 2.2. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal semistable fibration of genus $g \geq 2$ with s singular fibers. If f is non trivial and $s = 5$, then X is birationally ruled or rational.*

Proof. By the strict canonical class inequality, we have $K_f^2 < 6g - 6$. The corollary follows from Theorem 2.1. □

Note that if $s = 5$, then S is either $\mathbb{P}^2$ or a geometrically ruled surface. So

$$m = K_S^2 - K_X^2 > K_S^2 + 2g - 2 = \begin{cases} 2g + 7, & \text{if } S \cong \mathbb{P}^2, \\ 2g + 6 - 8q(X), & \text{otherwise.} \end{cases}$$

3 The proof of Theorem 0.1 for $s = 6$

In this section, we assume that $f : X \to \mathbb{P}^1$ is semistable and $s = 6$. We use freely the notations in the previous sections, including those in the proof. Denote by $F_1, \cdots, F_6$ the 6 singular fibers.

We recall the inequalities in Sect. 1 for our f:

(A) $K_X^2 < 0$, by (0.1)

(B) $r_f \geq \frac{1}{3}e\left(K_X^2 e + 12g - 12\right)$, by (1.3)

(C) $r_f \leq 6g - 6 + \ell - \frac{r}{2}$, by (1.6)

(D) $K_X^2 \leq 4\frac{(g-1)^2(2e-3)e}{e^2(g-1) - 6g - 3} - 8(g-1)$, if $e^2 > \frac{6g+3}{g-1}$, by (1.7)

where $r_f = \sum_q \frac{1}{\mu_q + 1}$, and ℓ is the number of non (-2)-curves in the 6 singular fibers.

In what follows, we assume that X is of general type, thus $K_S^2 \geq 1$. Denote respectively by $C_1, \cdots, C_6 \subset S$ the image curves of $F_1, \cdots, F_6$. By (A),

$$m = K_S^2 - K_X^2 \geq 2. \tag{3.1}$$

We see that $K_S C \neq 1$. Otherwise, by Hodge index theorem, $K_S^2 C^2 \leq (K_S C)^2 = 1$, so $C^2 \leq 1$, which implies $m \leq 1$. Hence

$$2 \leq K_S C \leq 2g - 2 - m. \tag{3.2}$$

Theorem 3.1. *If $g \leq 4$, then X is not of general type.*

Proof. We shall rule out the cases $g = 2, 3$ and 4 under the assumption $\kappa(X) = 2$.

Case $g = 2$: (3.2) implies $m = 0$, which contradicts (3.1).

Case $g = 3$: By (2.2), (3.1) and (3.2), $m = K_S C = 2$, $K_S^2 = 1$, $K_X^2 = -1$, $n_1 = n_2 = 1$, hence $C^2 = 2$. By Noether inequality $K_S^2 \geq 2p_g(S) - 4$ and Debarre's inequality [7] $K_S^2 \geq 2p_g(S) \geq 2q(S)$ if $q(S) \geq 1$, we obtain

$$p_g(S) \leq 2, \quad q(S) = 0. \tag{3.3}$$

Note that

$$e_f = 12\chi_f - K_f^2 = 12(p_g(S)+1) - K_X^2 + 4(g-1) = 12p_g(S) + 21 \leq 45. \tag{3.4}$$

On the other hand, from (B) for $e = 12$, we get $r_f \geq 48$. This contradicts (1.1): $r_f \leq e_f$.

Case $g = 4$: (D) for $e = 12$ implies that $K_X^2 \leq -8/5$. Thus $K_X^2 \leq -2$ and $m \geq 3$. By (3.2), we see that $m \leq 4$.

If $m = 3$, then $K_S^2 = 1$ and $K_X^2 = -2$, we see that (3.3) holds. As in (3.4), we obtain $e_f \leq 50$. (1.3) for $e = 9$ implies that $r_f \geq 54$, which contradicts $r_f \leq e_f$.

If $m = 4$, then (2.2) and (3.2) imply that $K_SC = 2$, $n_1 = \cdots = n_4 = 1$, and $C^2 = 4$. By Hodge index theorem, we can see that $K_S^2 = 1$ and $K_S^2C^2 = (K_SC)^2$. Hence $C \sim 2K_S$. This means that any (-2)-curve E on S does not pass through any one of the 4 base points $p_1, \cdots, p_4$ because $CE = 2K_SE = 0$. Hence (-2)-curves in C_i must be (-2)-curves in F_i.

Note that $K_SC_i = 2$, so C_i has at most two components different from (-2)-curves. This implies that F_i has at most two components which are neither (-2)-curves nor the exceptional curves of the 4 base points.

On the other hand, among the 4 exceptional curves on X, at least one of them is a horizontal (-1)-curve. If the remaining three exceptional curves are vertical, then at least one of them is a (-2)-curve. Hence the total number ℓ of non (-2)-curves in the 6 singular fibers $\leq 6 \times 2 + 2 = 14$.

Now by (C), $r_f \leq 32$. On the other hand, (B) for $e = 6$ implies that $r_f \geq 36$, a contradiction. □

Theorem 3.2. *If $s = 6$, $g = 5$ and X is of general type, then the minimal model S of X satisfies*

$$K_S^2 = 1, \quad p_g(S) = 2, \quad q(S) = 0.$$

The fibration comes from a pencil on S with 5 *simple base points.*

Proof. In this case, (D) for $e = 10$ implies that $K_X^2 \leq -\frac{864}{367} < -2$, thus $K_X^2 \leq -3$ and $m \geq 4$. On the other hand, $m \leq 2g - 2 - 2 = 6$. If $m = 6$, then $C^2 \geq 6$ and $K_SC \leq 2$, by Hodge index theorem, $K_S^2C^2 \leq (K_SC)^2 \leq 4$, and hence $K_S^2 \leq 0$, a contradiction. Thus $m = 4$ or 5.

If $m = 4$, then $K_S^2 = 1$, $K_X^2 = -3$, $p_g(S) \leq 2$, $q(S) = 0$, $C^2 \geq 4$, and $K_SC \leq 4$. Thus

$$e_f = 12(p_g + 1) - K_X^2 + 4(g-1) \leq 55.$$

From (1.3) for $e = 8$, we have

$$r_f \geq \frac{1}{3}e(-3e + 48) = e(16 - e) = 64 > e_f.$$

It contradicts (1.1).

So $m = 5$. By (2.1) and (2.2), we have $C^2 \geq 5$, $K_SC \leq 3$. From (3.2), we get $K_SC \geq 2$. Since $K_S^2 \geq 1$, by Hodge index theorem, $K_S^2C^2 \leq (K_SC)^2$, we have $K_SC = 3$ and $K_S^2 = 1$. So $K_X^2 = -4$, $p_g \leq 2$ and $q = 0$. By (2.1) and (2.2), we obtain $n_1 = \cdots = n_5 = 1$ and $C^2 = 5$. Generic C in the pencil is smooth. Let $e = 6$. Then (B) gives us that $r_f \geq \frac{1}{3}e(-4e + 48) = 48$.

Now we prove that $p_g = 2$. Otherwise $p_g \leq 1$, by (3.4), $e_f \leq 12(1+1) - (-4) + 16 = 44 < r_f$, a contradiction. Then we have

$$K_S^2 = 1, \quad p_g = 2, \quad q = 0, \quad m = 5.$$

So $e_f = \sum_q (\mu_q + 1) = 56$, $r_f = \sum_q \frac{1}{\mu_q + 1} \geq 48$.

Let $m_i = \#\{\, q \mid \mu_q = i \,\}$. Then

$$e_f = m_0 + 2m_1 + 3m_2 + \cdots = 56,$$
$$r_f = m_0 + \frac{1}{2}m_1 + \frac{1}{3}m_2 + \cdots \geq 48.$$

We see that $e_f - m_0 \geq 4(r_f - m_0)$, hence $m_0 \geq 46$. If $m_0 = 46$, one can prove that $m_1 = 5$ and $m_i = 0$ for $i > 1$. So there are at least 46 points q such that $\mu_q = 0$, and there are at most $(56-46)/2 = 5$ points q with $\mu_q \neq 0$. By (1.4) we get

$$\ell' + \frac{r}{2} \leq 8.$$

It implies that $r \leq 5$ and $\ell' \leq 7$. □

4 Two examples

Note that for any two points p_1, p_2 on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, there exists a cyclic cover $\varphi_{p_1,p_2} : \mathbb{P}^1 \to \mathbb{P}^1$ ramified exactly over p_1 and p_2. For example, $\tau_n : \mathbb{P}^1 \to \mathbb{P}^1$ defined by $x \mapsto x^n$ is totally ramified over 0 and ∞. Now we are going to construct some covers of $\mathbb{P}^1$.

Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be the double cover ramified over 1 and ∞, and let $\psi_e : \mathbb{P}^1 \to \mathbb{P}^1$ be the cyclic cover of degree e totally ramified over the two points $\pi^{-1}(0)$. Then the composition

$$\varphi_{2e} : \mathbb{P}^1 \overset{\psi_e}{\to} \mathbb{P}^1 \overset{\pi}{\to} \mathbb{P}^1$$

is a covering of degree $2e$ ramified uniformly over $0, 1, \infty$. The map φ_{2e} is the quotient map of $\mathbb{P}^1$ for the standard action of the diedral group of order $2e$ on $\mathbb{P}^1$. The cover has two ramification points over 0 with ramification index e, and e ramification points over 1 (resp. ∞) with index 2. Now we consider the fiber product of $\tau_n : \mathbb{P}^1 \to \mathbb{P}^1$ and $\varphi_{2e} : \mathbb{P}^1 \to \mathbb{P}^1$, we get a curve

$$\Gamma = \{\, (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \tau_n(x) = \varphi_{2e}(y) \,\}.$$

This curve is of type $(2e, n)$. The first projection $p_1 : \Gamma \to \mathbb{P}^1$ can be viewed as the pullback of φ_{2e} under the base change τ_n. So the set Σ of critical points of p_1 is $\tau_n^{-1}\{0, 1, \infty\}$. Σ contains $n+2$ points. In fact, Γ is locally defined by $\varphi_{2e}(y) = \tau_n(x)$. Thus we can see that Γ admits two singular points defined by $y^e = x^n$ over $\tau_n^{-1}(0)$ and e singular points of type $y^2 = x^n$ over $\tau_n^{-1}(\infty)$. p_1 has e simple ramification points over each point of $\tau_n^{-1}(1)$.

$$\begin{array}{ccccc}
\Gamma & \subset & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\mathrm{pr}_2} & \mathbb{P}^1 \\
& p_1 \searrow & \mathrm{pr}_1 \downarrow & & 2e{:}1 \downarrow \varphi_{2e} \\
& & \mathbb{P}^1 & \xrightarrow[\tau_n]{n:1} & \mathbb{P}^1
\end{array}$$

Let $e = n = 4$. Then Γ is of type $(8, 4)$. Let X be the minimal resolution of the double cover $\Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$ ramified over Γ. X can be obtained by the canonical resolution $\sigma : Y \to \mathbb{P}^1 \times \mathbb{P}^1$, where σ is the blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the two singular points of Γ over $\tau_n^{-1}(0)$. Denote by E_1 and E_2 the two exceptional curves, and by $\bar{\Gamma}$ the strict transform of Γ on Y. Then $\bar{\Gamma} \equiv \sigma((8, 4)) - 4E_1 - 4E_2 = 2\bar{\delta}$, where $\bar{\delta} = \sigma^*((4, 2)) - 2E_1 - 2E_2$. Let $\bar{\pi} : \bar{X} \to Y$ be the double cover ramified over $\bar{\Gamma}$. Note that $\bar{\Gamma}$ admits only ADE singularities, so $\bar{X}$ admits at most rational double points, and X is the minimal resolution of $\bar{X}$. Now by the formulas ([2], p.183), we have

$$K_{\bar{X}} = \bar{\pi}^*(K_Y + \bar{\delta}) = \bar{\pi}^*(\sigma^*(2, 0) - E_1 - E_2).$$

Denote by C_1 and C_2 the two horizontal sections of $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ passing through the two points blown up by σ, and by $\bar{C}_i$ their strict transforms on Y. Since $\bar{C}_1$ and $\bar{C}_2$ are disjoint with $\bar{\Gamma}$, it is easy to see that their pullback on $\bar{X}$ are 4 (-1)-curves. Note that

$$\bar{C}_1 + \bar{C}_2 = \sigma^*(2,0) - E_1 - E_2,$$

so $K_{\bar{X}} = \bar{\pi}^*(\bar{C}_1 + \bar{C}_2)$ is the sum of the 4 (-1)-curves. Let S be the surfaces obtained by contracting the 4 horizontal (-1)-curves. Then we see that $K_S \equiv 0$. By easy computation, we have

$$\begin{aligned}\chi(\mathcal{O}_S) &= \chi(\mathcal{O}_{\bar{X}}) \\ &= 2\chi(\mathcal{O}_Y) + \frac{1}{2}(\bar{\delta}^2 + K_Y\bar{\delta}) \\ &= 2 + 0 = 2.\end{aligned}$$

So S is a $K3$ surface. The first projection of $\mathbb{P}^1 \times \mathbb{P}^1$ induces a semistable fibration $f : X \to \mathbb{P}^1$ of genus 3 with $n+2 = 6$ singular fibers. Because $K_X^2 = -4$, $K_f^2 = 12 = 6g-6$.

Similarly, let $e = n = 5$, and let F_∞ be the vertical fiber of $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the two singular points of type $x^5 = y^5$. Then consider the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over $F_\infty + \Gamma$. The canonical resolution is similar to the above case. We have $\bar{\delta} \equiv \sigma^*((5,3)) - 3E_1 - 3E_2$ and

$$\begin{aligned}K_Y + \bar{\delta} &= \sigma^*((3,1)) - 2E_1 - 2E_2 \\ &= \sigma^*((1,1)) - E_1 - E_1 + \bar{C}_1 + \bar{C}_2 \\ &= \sigma^*((1,0)) + \bar{F}_\infty + \bar{C}_1 + \bar{C}_2,\end{aligned}$$

where $\bar{F}_\infty \equiv \sigma^*((0,1)) - E_1 - E_2$ is the strict transform of F_∞ on Y. The pullback of $\bar{F}_\infty + \bar{C}_1 + \bar{C}_2$ on $\bar{X}$ consists 5 (-1)-curves, the one from $\bar{F}_\infty$ is a vertical (-1)-curve. After blowing-down this vertical (-1)-curve, we get a relatively minimal semistable fibration $f : X \to \mathbb{P}^1$ of genus $g = 4$ with $n + 2 = 7$ singular fibers. We see easily that $|K_S|$ is a genus 2 pencil with one simple base point. The pencil is in fact induced by pr_2. So X is of general type. By easy computation, we have $\chi(\mathcal{O}_S) = 3$, hence

$$K_S^2 = 1,\ p_g(S) = 2,\ q(S) = 0.$$

In this case, $K_X^2 = -3$ and $K_f^2 = 6g - 3$.

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