

SYMBOLIC POWERS OF MONOMIAL IDEALS AND VERTEX COVER ALGEBRAS

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ABSTRACT. We introduce and study vertex cover algebras of weighted simplicial complexes. These algebras are special classes of symbolic Rees algebras. We show that symbolic Rees algebras of monomial ideals are finitely generated and that such an algebra is normal and Cohen-Macaulay if the monomial ideal is squarefree. For a simple graph, the vertex cover algebra is generated by elements of degree 2, and it is standard graded if and only if the graph is bipartite. We also give a general upper bound for the maximal degree of the generators of vertex cover algebras.

Dedicated to Winfried Bruns on the occasion of his sixtieth birthday

INTRODUCTION

Let P be a prime ideal in a Noetherian ring. Then the unique P -primary component of P^n is called the n th symbolic power of P , and is denoted $P^{(n)}$. It is clear that $P^n \subset P^{(n)}$ and that equality holds if and only if P is the only associated prime ideal of P^n . Symbolic powers of a prime ideal have an interesting interpretation, due to Zariski and Nagata: if K is an algebraically closed field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, $P \subset S$ a prime ideal, $X \subset K^n$ the irreducible algebraic set corresponding to P , and $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$ the maximal ideal corresponding to the point $a \in K^n$, then $P^{(n)} = \bigcap_{a \in X} \mathfrak{m}_a^n$. This result has been widely generalized by Eisenbud and Hochster [12].

In 1985 Cowsik [8] showed that if a prime ideal P in a regular local ring R with $\dim R/P = 1$ has the property that its symbolic Rees algebra $\bigoplus_{n \geq 0} P^{(n)}t^n \subset R[t]$ is Noetherian, then P is a set theoretic complete intersection, and he raised the question whether under the given conditions, $\bigoplus_{n \geq 0} P^{(n)}t^n$ is always Noetherian. P. Roberts [23] was the first to find a counterexample based on the counterexamples of Nagata [21] to the 14th problem of Hilbert. Many other remarkable positive and negative results concerning Cowsik's question, especially for monomial space curves, have been obtained, see for example [9], [19] and [15].

Our motivation to study symbolic powers comes from combinatorics. Suppose G is a finite graph on the vertex set $[n] = \{1, \dots, n\}$ without loops and multiple edges. We say that an integer vector $a = (a(1), \dots, a(n))$ with $a(i) \geq 0$ for $i = 1, \dots, n$ is a *vertex cover of G of order k* if $a(i) + a(j) \geq k$ for all edges $\{i, j\}$ of G . It is common to call a subset $V \subset [n]$ a vertex cover of G if $V \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\}$

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of G . Thus, an ordinary vertex cover corresponds in our terminology to a $(0, 1)$ -vector which is a vertex cover of order 1. We say that a vertex cover a of degree k is *decomposable* if there exists a vertex cover b of degree i and a vertex cover c of degree j such that $a = b + c$ and $k = i + j$. The fundamental problem arises whether each graph has only finitely many indecomposable vertex covers. This problem can be translated into an algebraic problem. Given a graph G , we fix a field K and let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K . Then we define the K -subalgebra of the polynomial ring $S[t]$ generated by all monomials $x_1^{a(1)} \cdots x_n^{a(n)} t^k$ where $a = (a(1), \dots, a(n))$ is a vertex cover of G of order k , and call this the *vertex cover algebra of G* .

The concept of vertex covers can be easily extended to weighted simplicial complexes. A *weight* on a simplicial complex Δ is a numerical function w that assigns to each facet of Δ a non-negative integer. Then the vertex cover algebra $A(\Delta, w)$ is defined similarly as in the case of a graph, described above. The crucial fact now is that these vertex cover algebras may be viewed as symbolic Rees algebras of suitable monomial ideals.

For this it is needed to extend the definition of symbolic powers to arbitrary ideals. The most general such definition is as follows: given ideals I and J in a Noetherian ring R . Then the n th symbolic power of I with respect to J is defined to be

$$I^n : J^\infty = \bigcup_{k \geq 0} I^n : J^k = \{a \in R \mid aJ^k \in I^n \text{ for some } k\}.$$

If $I = P$ happens to be a prime ideal, then one recovers the old definition of the n th symbolic power by choosing J as the intersection of all associated prime ideals of P^n which are different from P . In case I and J are monomial ideals, there exist monomial ideals I_1, \dots, I_r such that for all n , the n th symbolic power of I with respect to J is of the form $I_1^n \cap I_2^n \cap \dots \cap I_r^n$, so that the corresponding symbolic Rees algebra of I has the form $\bigoplus_{n \geq 0} (\bigcap_{i=1}^r I_i^n) t^n$. The vertex cover algebra of a simplicial complex is a particular case of such an algebra. Conversely, the symbolic Rees algebra of any squarefree monomial ideal can be viewed as the vertex cover algebra of a simplicial complex.

In Section 1 we prove that for any set $\{I_1, \dots, I_r\}$ of monomial ideals the algebra $\bigoplus_{n \geq 0} (\bigcap_{i=1}^r I_i^n) t^n$ is finitely generated. Lyubeznik [20] proved already in 1988 such a result for squarefree monomial ideals.

In Section 2 we recall a well-known finiteness criterion for algebras which in our case implies the following surprising fact: let $\{I_1, \dots, I_r\}$ be a set of monomial ideals. Then there exists an integer d such that

$$\left(\bigcap_{i=1}^r I_i^d\right)^k = \bigcap_{i=1}^r I_i^{dk}$$

for all $k \geq 0$.

In Section 3 we show that all kinds of symbolic Rees algebras of monomial ideals (which also includes the saturated Rees algebras) are finitely generated. From this it follows that the regularity of symbolic powers of monomials is asymptotically a

quasi-linear function. This generalizes a result of [17] on the existence of a linear bound for the regularity of such symbolic powers.

Section 4 is devoted to explain in detail the vertex cover algebra. We show that the vertex cover algebra of any simplicial complex is a finitely generated normal Cohen-Macaulay algebra. This algebra is Gorenstein if and only if the simplicial complex is a graph. The generators of a vertex cover algebra over S correspond to the indecomposable vertex covers of the simplicial complex. As an example we will describe the generation of the vertex cover algebras of skeletons of an arbitrary simplex.

Section 5 studies the maximal degree of the generators of vertex cover algebras over S . One of the surprising results in this section is that vertex cover algebras of finite graphs are always generated in degree ≤ 2 and they are standard graded over S if and only if the underlying graph is bipartite. Note that a vertex cover algebra is standard graded over S if and only if the ordinary and symbolic powers of the corresponding monomial ideal are equal. For bipartite graphs, this equality was already established by Gitler, Reyes and Villarreal [14].

The generation of symbolic Rees algebra of monomial ideals have been studied first for edge ideals of graphs. Simis, Vasconcelos and Villarreal [24] showed that if I is the edge ideal of a graph G , then $I^{(n)} = I^n$ for all $n \geq 0$ if and only if G is bipartite. Due to a result of Bahiano [2], the symbolic Rees algebra of the edge ideal I of a graph is generated by elements of degree at most $(n-1)(n-h)$, where n is the number of vertices and $h = \text{height}(I)$. Vasconcelos has asked whether this bound can be improved to $n-1$. We find a family of counter-examples to this question. On the other hand, we give a (rather large) upper bound for the maximal degree of the generators of vertex cover algebras which only depends on the number of vertices. Our estimation uses upper bounds for the absolute value of the determinant of $(0,1)$ -matrices given by Faddeev and Sominskii [13], and is related to Hadamard's Maximum Determinant Problem.

1. FINITE GENERATION OF AFFINE SEMIGROUPS

We call a submonoid of \mathbb{Z}^n an *affine semigroup*. In general, H need not be finitely generated. For instance, if H is the affine semigroup of all elements $(i, j) \in \mathbb{N}^2$ with $(i, j) = (0, 0)$ or $ij \neq 0$, then H is not finitely generated.

For any affine subset $H \in \mathbb{Z}^n$ we will denote by \mathbb{Q}_+H the set of all linear combinations of elements in H with non-negative rational numbers as coefficients. It is obvious that if H is a finitely generated affine semigroup, then \mathbb{Q}_+H is a *rational cone*, that is, the intersection of finitely many rational halfspaces. This turns out to be a necessary and sufficient condition for the finite generation of H .

Theorem 1.1. *Let H be an affine semigroup in \mathbb{Z}^n such that \mathbb{Q}_+H is a rational cone. Then H is finitely generated.*

Proof. Let $\mathbb{Z}H$ be the smallest subgroup of \mathbb{Z}^n containing H and $r = \text{rank } \mathbb{Z}H$. Since \mathbb{Q}_+H is the intersection of finitely many rational half-spaces, there exist finitely

many elements $q_1, \dots, q_s \in H$ such that $\mathbb{Q}_+H = \mathbb{Q}_+\{q_1, \dots, q_s\}$. Consider all rational cones of the form $\mathbb{Q}_+\{q_{i_1}, \dots, q_{i_r}\}$, where $\{q_{i_1}, \dots, q_{i_r}\}$ are r linearly independent elements. These cones cover \mathbb{Q}_+H . Thus, it suffices to show that each of the affine semigroups $H \cap \mathbb{Q}_+\{q_{i_1}, \dots, q_{i_r}\}$ is finitely generated. In other words, we may assume that $\mathbb{Q}_+H = \mathbb{Q}_+\{q_1, \dots, q_r\}$ with linearly independent elements $q_1, \dots, q_r \in H$.

Let $B = \{\sum_{i=1}^r c_i q_i \mid 0 \leq c_i \leq 1\} \cap \mathbb{Z}H$. Then $\mathbb{Z}H$ is the disjoint union of the set $L_p = p + \mathbb{Z}\{q_1, \dots, q_r\}$ with $p \in B$. We set $H_p = H \cap L_p$ for all $p \in B$. Then H is the finite disjoint union of the sets H_p . To each H_p we attach a monomial ideal I_p in $K[x_1, \dots, x_r]$ consisting of those monomials $u = x_1^{c_1} \cdots x_r^{c_r}$ for which $p + c_1 q_1 + \cdots + c_r q_r \in H_p$. Let $G(I_p)$ denote the finite minimal set of monomial generators of I_p . Then every element of H_p is the sum of an element of the set $\{p + c_1 q_1 + \cdots + c_r q_r \mid x_1^{c_1} \cdots x_r^{c_r} \in G(I_p)\}$ with an element of the set $\mathbb{Z}_+\{q_1, \dots, q_r\}$. Therefore, the finite set

$$\bigcup_{p \in B} \{p + c_1 q_1 + \cdots + c_r q_r \mid x_1^{c_1} \cdots x_r^{c_r} \in G(I_p)\}$$

is a set of generators of H . \square

A different proof of the following consequence of Theorem 1.1 was given by Bruns and Gubeladze in [5, Corollary 7.2].

Corollary 1.2. *Let $H_1, \dots, H_r \subset \mathbb{Z}^n$ be finitely generated affine semigroups in \mathbb{Z}^n . Then $H = \bigcap_{i=1}^r H_i$ is again a finitely generated affine semigroup.*

Proof. We claim that $\mathbb{Q}_+H = \bigcap_{i=1}^r \mathbb{Q}_+H_i$. Obviously, we have $\mathbb{Q}_+H \subset \mathbb{Q}_+H_i$ for all i , so that $\mathbb{Q}_+H \subset \bigcap_{i=1}^r \mathbb{Q}_+H_i$. Conversely, let $p \in \bigcap_{i=1}^r \mathbb{Q}_+H_i$. Then for each i there exists an integer n_i with $n_i p \in H_i$. Set $n = n_1 \cdots n_r$. Then $np \in H_i$ for all i and hence $np \in H$. This implies that $p \in \mathbb{Q}_+H$.

Since the intersection of finitely many rational cones is again a rational cone, \mathbb{Q}_+H is a rational cone, and the assertion follows from Theorem 1.1. \square

As an important consequence of Corollary 1.2 we have

Corollary 1.3. *Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring over K in n variables and $I_1, \dots, I_r \subset S$ monomial ideals. Then the algebra*

$$A = \bigoplus_{k \geq 0} \left(\bigcap_{j=1}^r I_j^k \right) t^k$$

is a finitely generated S -algebra.

Proof. Note that $A = \bigcap_{j=1}^r A_j$ where for each j , $A_j = \bigoplus_{k \geq 0} I_j^k t^k$ is the Rees algebra of the ideal I_j .

Since the ideals I_j are monomial ideals, we may view A and the algebras A_j as semigroup rings, say $A = K[H]$ and $A_j = K[H_j]$ for $j = 1, \dots, r$. The semigroups $H_j \subset \mathbb{Z}^{n+1}$ are finitely generated since the Rees algebras A_j are finitely generated, and $H = \bigcap_{j=1}^r H_j$ since $A = \bigcap_{j=1}^r A_j$. Thus, Corollary 1.2 implies that H is finitely generated, and consequently A is a finitely generated S -algebra, as desired. \square

Theorem 1.1 is a generalization of Gordan's lemma which says that the set of all lattice points in a rational cone is a finitely generated affine semigroup (cf. [6, Proposition 6.1.2]). We can also deduce Theorem 1.1 from Gordan's lemma as follows.

The *normal closure* of H is the subsemigroup of \mathbb{Z}^n with

$$\bar{H} = \{c \in \mathbb{Z}H \mid mc \in H \text{ for some } m \in \mathbb{N}\}$$

The semigroup H is called *normal* if $H = \bar{H}$. In fact, $K[\bar{H}]$ is the normalization of $K[H]$.

It is obvious that $\bar{H} = \mathbb{Z}H \cap \mathbb{Q}_+H$. By Gordan's lemma, $K[\bar{H}]$ is finitely generated if \mathbb{Q}_+H is a rational cone. Therefore, Theorem 1.1 follows from the following result of Artin and Tate, see [1] and [11, Exercise 4.32].

Proposition 1.4. *Let K be a field and A a finitely generated K -algebra. Let B be a K -subalgebra of A , and suppose that A is integral over B . Then B is also a finitely generated K -algebra.*

Proof. Let $A = K[a_1, \dots, a_n]$. For each i , since a_i is integral over B , there exists a monic polynomial $f_i \in B[x]$ such that $f_i(a_i) = 0$. Let C be the K -subalgebra of B which is obtained by adjoining all the coefficients of the polynomials f_i to K . Then A is a finitely generated C -algebra which is integral over C , and hence A is a finitely generated C -module. As C is a finitely generated K -algebra, C is Noetherian, so that the C -submodule B of A is also a finitely generated C -module. This implies that B is a finitely generated K -algebra. \square

2. A FINITENESS CRITERION FOR ALGEBRAS

As before let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K , and let I_j , $j = 0, 1, 2, \dots$ be a family of graded ideals of S with $I_0 = S$ and $I_j I_k \subset I_{j+k}$ for all j and k . Then $A = \bigoplus_{j \geq 0} I_j t^j \subset S[t]$ is a positively graded S -algebra with $A_0 = S$.

In this section we formulate a well-known finiteness criterion for A which is adapted to our situation (see e.g. [22]). For any positive integer d we denote by $A^{(d)} = \bigoplus_{j \geq 0} A_{jd}$ the d th Veronese subalgebra of A .

Theorem 2.1. *The following conditions are equivalent:*

- (a) *A is a finitely generated S -algebra.*
- (b) *There exists an integer d such that $A^{(d)}$ is a standard graded S -algebra.*
- (c) *There exists a number d such that $A^{(d)}$ is a finitely generated S -algebra.*

Proof. (a) \Rightarrow (b): Let $A = S[f_1 t^{c_1}, \dots, f_m t^{c_m}]$ with homogeneous $f_i \in I_{c_i}$ for $i = 1, \dots, m$, and let c be the least common multiple of the numbers c_i .

We set $B = S[g_1, \dots, g_m]$ with $g_i = f_i^{c/c_i} t^c$. Then all g_i belong to A_c , so that $B_i \subset A_{ic}$ for all i . Since each $f_i t^{c_i}$ is integral over B it follows that A is a finitely generated B -module. Therefore the B -submodule $A^{(c)}$ is also finitely generated. Let $\{a_i \in A_{s_i c} : i = 1, \dots, m\}$ be a system of generators of the B -module $A^{(c)}$, and let $d = sc$ where $s = \max\{s_1, \dots, s_m\}$. We claim that $A_{id} = A_d^i$ for all $i \geq 0$. This then shows that $A^{(d)}$ is standard graded.

We proceed by induction on i . The assertion is trivial for $i = 0, 1$. So now let $i > 1$. Since $B_i A_{jc} = A_{(i+j)c}$ for $i \geq 0$ and $j \geq s$ it follows that $B_{is} A_{jd} = A_{(i+j)d}$ for all integer $i, j \geq 0$. Therefore, using the induction hypothesis and the fact that B is standard graded, we get

$$A_{(i+1)d} = B_{is} A_d = B_s(B_{(i-1)s} A_d) = B_s A_{id} = B_s A_d^i \subset A_d^{i+1}.$$

Since the opposite inclusion $A_d^{i+1} \subset A_{(i+1)d}$ is obvious, the assertion follows.

(b) \Rightarrow (c) is trivial.

(c) \Rightarrow (a): Note that $A^{(d;j)} = \bigoplus_{i \geq 0} A_{id+j}$ is an $A^{(d)}$ -module of rank 1 and hence isomorphic to an ideal of $A^{(d)}$. It follows that $A^{(d;j)}$ is a finitely generated $A^{(d)}$ -module. Since $A = \bigoplus_{j=0}^{d-1} A^{(d;j)}$ as an $A^{(d)}$ -module, we see that A is a finitely generated $A^{(d)}$ -module. This implies the assertion. \square

As an immediate consequence of Theorem 2.1 and Corollary 1.3 we obtain

Corollary 2.2. *Let $I_1, \dots, I_r \subset S$ be monomial ideals. Then there exists an integer d such that*

$$\left(\bigcap_{j=1}^r I_j^d \right)^k = \bigcap_{j=1}^r I_j^{dk} \quad \text{for all } k \geq 1.$$

Remark 2.3. The number d in statement (b) and (c) of Theorem 2.1 is not a bound for the degree of the minimal generators of A . Consider for example for an odd number $n > 1$ the S -algebra $A = S[x^2t, x^2t^2, x^nt^n] \subset S[t]$ where $S = K[x]$ is the polynomial ring over the field K and $S[t]$ is a polynomial ring over S . The grading of A is given by the powers of t . Then $A^{(2)} = S[x^2t^2]$ is standard graded, but $x^nt^n \in A_n$ is a minimal generator of the S -algebra A .

Remark 2.4. There does not exist a number d_0 such that $A^{(d)}$ is standard graded for all $d \geq d_0$. Consider for example the ideal $I = (x, y) \cap (y, z) \cap (x, z)$ and let $I_j = I^{(j)} = (x, y)^j \cap (y, z)^j \cap (x, z)^j$ for all $j \geq 1$. Then I_{2k} is generated by the elements

$$x^i y^{2k-i} z^{2k-i}, x^{2k-i} y^i z^{2k-i}, x^{2k-i} y^{2k-i} z^i, \quad i = 0, \dots, k.$$

From this it follows that $I_{2(k+h)} \subseteq I_{2k} I_{2h}$ for all $k, h \geq 1$. Since $I_{2k} I_{2h}$ is obviously contained in $I_{2(k+h)}$, we obtain $I_{2(k+h)} = I_{2k} I_{2h}$. As a consequence, $A^{(2k)}$ is standard graded for $k \geq 1$. On the other hand, I_{2k+1} is generated by the elements

$$x^i y^{2k-i+1} z^{2k-i+1}, x^{2k-i+1} y^i z^{2k-i+1}, x^{2k-i+1} y^{2k-i+1} z^i, \quad i = 0, \dots, k.$$

It is easy to check that $\deg f \geq 6k + 4$ for all $f \in (I_{2k+1})^2$. Therefore, $(I_{2k+1})^2$ does not contain the element $(xyz)^{2k+1}$ of $I_{2(2k+1)}$. Hence $A^{(2k+1)}$ is not standard graded for $k \geq 0$. This is a counter-example to a result of Bishop [3, Proof of Theorem 2.17 and Corollary 2.21] (see also [22, Remark 2.3]) which claimed that if $A = \bigoplus_{j \geq 0} I_j t^j$ is finitely generated, then $A^{(d)}$ is standard graded for all large d .

Let B be a finitely generated S -algebra. We denote by $d(B)$ the maximal degree of a generator of B . If A is an S -algebra as above, then by Theorem 2.1 we have $d(A^{(c)}) = 1$ for some c , but by Remark 2.4 there does not necessarily exist an integer c_0 such that $d(A^{(c)}) = 1$ for all $c \geq c_0$. However, we have

Proposition 2.5. *Let A be a finitely generated graded S -algebra, Then*

$$d(A^{(c)}) \leq d(A) \text{ for all } c \geq 1.$$

Proof. Let $d = d(A)$ and let $k \geq 1$ be any integer. By assumption A_{kc} is the sum of S -modules of the form $A_{t_1} \cdots A_{t_r}$ where $t_i \leq d$ for $i = 1, \dots, r$ and $\sum_{i=1}^r t_i = kc$.

We show by induction on k that $A_{t_1} \cdots A_{t_r} \subset S[A_c, A_{2c}, \dots, A_{dc}]$ for each of the summands $A_{t_1} \cdots A_{t_r}$ of A_{kc} . If $k \leq d$, $A_{t_1} \cdots A_{t_r} \subset S[A_{kc}] \subseteq S[A_c, A_{2c}, \dots, A_{dc}]$. If $k > d$, then $\sum_{i=1}^r t_i > dc$. Hence $r > c$. Consider the residue classes modulo c of all the partial sums $t_1 + \cdots + t_l$ for $l = 1, \dots, r$. Then there exist integers $1 \leq l_1 < l_2 \leq r$ such that $t_1 + \cdots + t_{l_1}$ and $t_1 + \cdots + t_{l_2}$ belong to the same residue class. It follows that $t_{l_1+1} + \cdots + t_{l_2}$ is divisible by c . Now let $t_{l_1+1} + \cdots + t_{l_2} = uc$. Then $0 < u < k$ and $A_{t_1} \cdots A_{t_r} \subset A_{uc}A_{(k-u)c}$. By induction we may assume that $A_{uc}, A_{(k-u)c} \subset S[A_c, A_{2c}, \dots, A_{dc}]$. Therefore, $A_{t_1} \cdots A_{t_r} \subset S[A_c, A_{2c}, \dots, A_{dc}]$. \square

3. SYMBOLIC POWERS

Let $I, J \subset S = K[x_1, \dots, x_n]$ be graded ideals. The n th symbolic power of I with respect to J is defined to be the ideal

$$I^n : J^\infty = \{f \in S \mid fJ^k \subset I^n \text{ for some } k\}$$

Let $I^n = \bigcap_i Q_i$ be a primary decomposition of I^n where Q_i is P_i -primary. It is easy to see that $I^n : J^\infty$ is the intersection of those Q_i for which $J \not\subseteq P_i$, cf. [11, Proposition 3.13 a]. Two special cases are of interest.

First, if $J = \mathfrak{m} = (x_1, \dots, x_n)$, then $I^n : J^\infty$ is the n th saturated power \tilde{I}^n of I^n .

Second, let $\text{Min}(I)$ denote the set of minimal prime ideals of I and $\text{Ass}^*(I)$ the union of the associated prime ideals of I^n for all $n \geq 0$. It is known that $\text{Ass}^*(I)$ is a finite set [4]. Obviously, one has $\text{Min}(I) \subseteq \text{Ass}^*(I)$. Let

$$J = \bigcap_{P \in \text{Ass}^*(I) \setminus \text{Min}(I)} P.$$

For this choice of J , the symbolic powers $I^n : J^\infty$ of I^n with respect to J are the ordinary symbolic power of I , denoted by $I^{(n)}$. The symbolic Rees algebra of I with respect to J is defined to be the graded S -algebra

$$\bigoplus_{k \geq 0} (I^k : J^\infty)t^k.$$

We shall see that for monomial ideals, this algebra is finitely generated.

Lemma 3.1. *Let I and J be monomial ideals of S . Assume that $\text{Ass}(I : J^\infty) = \text{Ass}(I^k : J^\infty)$ for all k . Let $\mathcal{A} = \{P \in \text{Ass}(I) \mid J \not\subseteq P\}$, and P_1, \dots, P_r be the maximal elements in \mathcal{A} (with respect to inclusion). Furthermore let $I = \bigcap_{P \in \text{Ass}(I)} Q(P)$ be a primary decomposition of I and set*

$$Q_i = \bigcap_{P \in \text{Ass}(I), P \subseteq P_i} Q(P)$$

for $i = 1, \dots, r$. Then

$$I^k : J^\infty = \bigcap_{i=1}^r Q_i^k \quad \text{for all } k$$

Proof. Let $I^k : J^\infty = \bigcap_{P \in \text{Ass}(I : J^\infty)} Q_k(P)$ be a primary decomposition of $I^k : J^\infty$. (Here we use that $\text{Ass}(I : J^\infty) = \text{Ass}(I^k : J^\infty)$ for all k). If we set

$$Q_{k,i} = \bigcap_{P \in \text{Ass}(I), P \subseteq P_i} Q_k(P)$$

for $i = 1, \dots, r$, then $I^k : J^\infty = \bigcap_{i=1}^r Q_{k,i}$. Thus, it remains to show that $Q_{k,i} = Q_i^k$ for $i = 1, \dots, r$.

Assume that $P_i = (x_1, \dots, x_s)$. Let $R = k(x_{s+1}, \dots, x_n)[x_1, \dots, x_s]$. For each monomial f we denote by f^* the largest divisor of f involving only the variables x_1, \dots, x_s . Then

$$Q_i = IR \cap S = (f^*)_{f \in I}, \quad Q_{k,i} = I^k R \cap S = (g^*)_{g \in I^k}.$$

Since every monomial g^* with $g \in I^k$ can be written as a product of k elements of the form f^* with $f \in I$, we get $Q_{k,i} = Q_i^k$, as desired. \square

Theorem 3.2. *Let I and J be monomial ideals in S . Then the symbolic Rees algebra of I with respect to J is finitely generated. In particular, the ordinary symbolic Rees algebra and the saturated Rees algebra of I are finitely generated.*

Proof. Let $A = \bigoplus_{k \geq 0} (I^k : J^\infty)t^k$ be the symbolic Rees algebra of I with respect to J . By Theorem 2.1 it suffices to show that $A^{(d)}$ is finitely generated for some d , or equivalently that the symbolic Rees algebra of I^d with respect to J is finitely generated.

It is known that there exists an integer d such that $\text{Ass}(I^k) = \text{Ass}(I^d)$ for all $k \geq d$ [4]. Thus, if we replace I by I^d we may assume that $\text{Ass}(I) = \text{Ass}(I^k)$ for all $k \geq 1$. From this it follows that $\text{Ass}(I^k : J^\infty) = \text{Ass}(I : J^\infty)$ for all $k \geq 1$. Let $I = \bigcap_{P \in \text{Ass}(I)} Q(P)$ be a primary decomposition of I . Since I is a monomial ideal, this decomposition can be chosen such that $Q(P)$ is a monomial ideal for all $P \in \text{Ass}(I)$. Defining the ideals Q_i as in the previous lemma, it follows that $A = \bigoplus_{k \geq 0} \bigcap_{i=1}^r Q_i^k t^k$ and that all Q_i are monomial ideals. Thus, the assertion follows from Corollary 2.2. \square

It has been shown in [17, Theorem 2.9] that there is a linear bound for the regularity of the symbolic powers of a monomial ideal. Now, as we know that this algebra is finitely generated we obtain together with [10, Theorem 3.4] the following result.

Corollary 3.3. *Let I and J be monomial ideals in S . Then there exist an integer d and integers a_i and c_i for $i = 1, \dots, d$ such that*

$$\text{reg}(I^k : J^\infty) = a_i k + c_i \quad \text{for all } k \gg 0 \text{ where } i \equiv k \pmod{d}.$$

4. VERTEX COVER ALGEBRAS

Let G be a finite graph on the vertex set $[n]$ without loops and multiple edges. We denote by $E(G)$ the set of edges of G . A subset $C \subset [n]$ is called a *vertex cover* of G if $V \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\} \in E(G)$.

We extend this definition in various directions: a graph may be viewed as a one-dimensional simplicial complex. We replace G by a simplicial complex Δ on the vertex set $[n]$, and denote by $\mathcal{F}(\Delta)$ the set of facets of Δ . A subset $C \subset [n]$ is called a *vertex cover* of Δ if $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}(\Delta)$.

A subset $C \subset [n]$ may be identified with the $(0, 1)$ -vector $a_C \in \mathbb{N}^n$, where

$$a_C(i) = \begin{cases} 1, & \text{if } i \in C, \\ 0, & \text{if } i \notin C. \end{cases}$$

Here $a(i)$ denotes the i th component of a vector $a \in \mathbb{Q}^n$.

It is clear that a $(0, 1)$ -vector $a \in \mathbb{N}^n$ corresponds to a vertex cover of Δ if and only if $\sum_{i \in F} a(i) \geq 1$ for all $F \in \mathcal{F}(\Delta)$.

Consider a function

$$w: \mathcal{F}(\Delta) \rightarrow \mathbb{N} \setminus \{0\}, \quad F \mapsto w_F$$

that assigns to each facet a positive integer. In this case, Δ is called a *weighted simplicial complex*, denoted by (Δ, w) . We call $a \in \mathbb{N}^n$ a *vertex cover* of (Δ, w) of order k if $\sum_{i \in F} a(i) \geq kw_F$ for all $F \in \Delta$.

The *canonical weight function* on a simplicial complex Δ is the weight function $w_0(F) = 1$ for all facets $F \in \mathcal{F}(\Delta)$. The vertex covers of (Δ, w_0) of order 1 are just the usual vertex covers of Δ .

Let K be a field and $S = K[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates over K . Let $S[t]$ be a polynomial ring over S in the indeterminate t , and consider the K -vector space $A(\Delta, w) \subset S[t]$ generated by all monomials $x_1^{a(1)} \cdots x_n^{a(n)} t^k$ such that $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$ is a vertex cover of Δ of order k . We have

$$A(\Delta, w) = \bigoplus_{k \geq 0} A_k(\Delta, w) \quad \text{with } A_0(\Delta, w) = S,$$

where $A_k(\Delta, w)$ is spanned by the monomials $ut^k \in A(\Delta, w)$ and u is a monomial in S . If a is vertex cover of order k , and b a vertex cover of order ℓ , then $a + b$ is a vertex cover of order $k + \ell$. This implies that

$$A_k(\Delta, w)A_\ell(\Delta, w) \subset A_{k+\ell}(\Delta, w).$$

Therefore $A(\Delta, w)$ is a graded S -algebra. We call it the *vertex cover algebra* of the weighted simplicial complex (Δ, w) . For simplicity, we will use the notation $A(\Delta)$ instead of $A(\Delta, w_0)$.

For any subset $F \subset [n]$ we denote by P_F the prime ideal generated by the variables x_i with $i \in F$. Set

$$I^*(\Delta, w) := \bigcap_{F \in \mathcal{F}(\Delta)} P_F^{w_F}.$$

Having introduced this notation the vertex cover algebra has the following interpretation.

Lemma 4.1. *Let (Δ, w) be a weighted simplicial complex on the vertex set $[n]$. Then $A(\Delta, w)$ is the symbolic Rees algebra of the ideal $I^*(\Delta, w)$.*

Proof. It is immediate from the definition of $A_k(\Delta, w)$ that ut^k belongs to $A_k(\Delta, w)$ if and only if

$$u \in \bigcap_{F \in \mathcal{F}(\Delta)} P_F^{kw_F} = \bigcap_{F \in \mathcal{F}(\Delta)} (P_F^{w_F})^k.$$

Hence the assertion follows from Lemma 3.1. \square

We would like to emphasize that $I^*(\Delta, w)$ is the ideal generated by the monomials $x_1^{a(1)} \cdots x_n^{a(n)}$ such that $a = (a(1), \dots, a(n))$ is a vertex cover of Δ of order 1. If w is the canonical weight function, we will use the notation $I^*(\Delta)$ instead of $I^*(\Delta, w)$. In this case, $I^*(\Delta)$ is a squarefree monomial ideal.

Conversely, the symbolic Rees algebra of any squarefree monomial ideal I is the vertex cover algebra of a simplicial complex. To see this we first consider the simplicial complex Σ whose facets correspond to the exponent vectors of the generators of I . In other word, I is the facet ideal $I(\Sigma)$ of Σ . Then $I = I^*(\Delta)$, where the facets of Δ are exactly the minimal vertex covers of Σ .

The structure of a vertex cover algebra is described in the next result.

Theorem 4.2. *The vertex cover algebra $A(\Delta, w)$ is a finitely generated, graded and normal Cohen-Macaulay S -algebra.*

Proof. The finite generation follows from Theorem 3.2.

Next we show that $A(\Delta, w)$ is normal. Let $F \in \mathcal{F}(\Delta)$ and $u \in S$ a monomial. We define $\nu_F(u)$ to be the least integer k such that $u \in P_F^k$. Then $ut^k \in S[t]$ in $A_k(\Delta, w)$ if and only if $\nu_F(u) \geq kw_F$ for all $F \in \mathcal{F}(\Delta)$.

Now let $ut^k \in S[t]$ and suppose that $(ut^k)^m \in A_k(\Delta, w)$ for some integer $m \geq 1$. Then

$$m\nu_{F_j}(u) = \nu_{F_j}(u^m) \geq mkw_{F_j} \quad \text{for } j = 1, \dots, s.$$

It follows that $\nu_{F_j}(u) \geq kw_{F_j}$ for $j = 1, \dots, s$, and hence that $ut^k \in A(\Delta, w)$. Since $A(\Delta, w)$ is a toric ring, this implies that $A(\Delta, w)$ is normal.

Finally, by a theorem of Hochster [18], any normal toric ring is Cohen-Macaulay.

\square

In general, we may restrict the study on vertex cover algebras to the case Δ has no zero-dimensional facet (which consists of only one vertex).

Indeed, let Δ_0 denote the simplicial complex of the zero-dimensional facets and Δ_1 the simplicial complex of the higher dimensional facets of Δ . Then $I^*(\Delta_0, w)$ is a principal ideal and $I^*(\Delta, w) = I^*(\Delta_0, w) \cdot I^*(\Delta_1, w)$. Therefore,

$$I^*(\Delta, w)^{(n)} = I^*(\Delta_0, w)^n \cdot I^*(\Delta_1, w)^{(n)}.$$

Hence the symbolic Rees algebra of $I^*(\Delta, w)$ is isomorphic to the symbolic Rees algebra of $I^*(\Delta_1, w)$. By Lemma 4.1, this implies $A(\Delta, w) \cong A(\Delta_1, w)$.

Theorem 4.3. *Assume that Δ has no zero-dimensional facet. Then $A(\Delta, w)$ is a Gorenstein ring if and only if $w_F = |F| - 1$ for all facets F of Δ .*

Proof. Let $A(\Delta, w)$ be a Gorenstein ring. By Lemma 4.1,

$$A(\Delta, w) = \bigoplus_{h \geq 0} \left(\bigcap_{F \in \mathcal{F}(\Delta)} P_F^{hw_F} \right) t^h.$$

For every facet $F \in \mathcal{F}(\Delta)$ let $K^* = K(x_i \mid i \notin F)$ and $S^* = K^*[x_i \mid i \in F]$. Let

$$B = \bigoplus_{h \geq 0} (P_F^{hw_F} S^*) t^h$$

be the Rees algebra of the ideal $P_F^{w_F} S^*$. Then B is a localization of $A(\Delta, w)$. Since $P_F S^*$ is the maximal graded ideal of S^* , the Gorensteiness of B implies that $w_F = |F| - 1$ (see e.g. [16, 3.2.13]).

For the converse, let $w_F = |F| - 1$ for all facets F of Δ . Let H denote the affine semigroup of all vectors $a = (a(1), \dots, a(n+1)) \in \mathbb{N}^{n+1}$ such that $(a(1), \dots, a(n))$ is a vertex cover of order $a(n+1)$ of Δ . Then $A(\Delta, w) = K[H]$. Note that $a \in H$ if and only if

$$\sum_{i \in F} a(i) \geq w_F a(n+1)$$

for all facets F of Δ . Then the relative interior $\text{relint}(H)$ of H consists of the vectors $a \in H$ with $a(i) > 0$ for all i and

$$\sum_{i \in F} a(i) > w_F a(n+1)$$

for all facets F of Δ . Obviously, $(1, \dots, 1) \in \text{relint}(H)$. Moreover, the last inequality implies

$$\sum_{i \in F} (a(i) - 1) = \sum_{i \in F} a(i) - |F| \geq a(n+1)w_F - w_F = (a(n+1) - 1)w_F.$$

Hence $a - (1, \dots, 1) \in H$. So $\text{relint}(H) = (1, \dots, 1) + H$. It is well-known that this relation implies the Gorensteiness of $k[H]$ (see e.g. [6, Corollary 6.3.8]). \square

Corollary 4.4. $A(\Delta)$ is a Gorenstein ring if and only if Δ is a graph.

Proof. Without restriction we may assume that Δ has no zero-dimensional facet. Since w is the canonical weight function, $w_F = 1$ for all facets F of Δ . By the above theorem, $A(\Delta)$ is a Gorenstein ring if and only if $|F| = 2$, which just means that Δ is a graph. \square

The Gorensteiness of $A(\Delta)$ was known before only for a very particular case of bipartite graphs by Gitler, Reyes and Villarreal [14, Corollary 2.8].

A vertex cover $a \in \mathbb{N}^n$ of order k of a weighted simplicial complex is *decomposable* if there exists a vertex cover b of order i and a cover c of order j such that $a = b + c$ and $k = i + j$. If a is not decomposable, we call it *indecomposable*.

Example 4.5. Consider the graph of a triangle with the canonical weight function. Then $a = (1, 1, 1)$ is a vertex cover of order 2. Even though a can be written as a sum of a vertex cover of order 0 and a vertex cover of order 1, the vertex cover a is indecomposable, as it can be easily checked.

It is clear that the indecomposable vertex covers of order > 0 correspond to a minimal homogeneous set of generators of the S -algebra $A(\Delta, w)$. The indecomposable vertex covers of order 0 are just the canonical unit vectors. Thus, all the indecomposable vertex covers correspond to a minimal homogeneous set of generators of $A(\Delta, w)$, viewed as a K -algebra. The preceding result implies that any weighted simplicial complex has only finitely many indecomposable vertex covers.

As an example we consider the skeletons of a simplex. Let Δ be a simplicial complex on $[n]$. The j th skeleton of Δ is the simplicial complex $\Delta^{(j)}$ on $[n]$ whose faces are those faces F of Δ with $|F| \leq j + 1$.

Proposition 4.6. *Let Σ_n denote the simplex of all subsets of $[n]$. Let $0 \leq j \leq n - 2$. Then the minimal system of monomial generators of the S -algebra $A(\Sigma_n^{(j)})$ consists of the monomials $x_{i_1}x_{i_2} \cdots x_{i_{n-j+q-1}}t^q$, where $1 \leq q \leq j + 1$ and where $1 \leq i_1 < i_2 < \cdots < i_{n-j+q-1} \leq n$.*

Proof. Let $A_{n,j}$ denote the S -subalgebra of $A(\Sigma_n^{(j)})$ generated by those monomials $x_{i_1}x_{i_2} \cdots x_{i_{n-j+q-1}}t^q$ with $1 \leq q \leq j + 1$, where $1 \leq i_1 < i_2 < \cdots < i_{n-j+q-1} \leq n$. Since $n - j \geq 2$, it follows that such the monomials form the minimal system of monomial generators of the S -algebra $A_{n,j}$. By using induction on n , we show that $A_{n,j} = A(\Sigma_n^{(j)})$. Let $n = 2$ and $j = 0$. Then the K -vector space $A_k(\Sigma_2^{(0)})$ is spanned by the monomials $x_1^{a(1)}x_2^{a(2)}t^k$ with $k \leq a(1)$ and $k \leq a(2)$. Hence the S -module $A_k(\Sigma_2^{(0)})$ is generated by the monomial $x_1^kx_2^k$. Hence the S -algebra $A(\Sigma_2^{(0)})$ is generated by x_1x_2t .

Let a monomial $x^at^k = x_1^{a(1)} \cdots x_n^{a(n)}t^k$ belong to $A_k(\Sigma_n^{(j)})$. Write $\text{supp}(x^a)$ for the support of x^a . Thus, $\text{supp}(x^a) = \{i \in [n] : a(i) \neq 0\}$.

First, let $\text{supp}(x^a) = [n]$ and, say, $0 < a(1) \leq a(2) \leq \cdots \leq a(n)$. Then $k \leq a(1) + \cdots + a(j + 1)$. If $a(1) < a(n)$, then one has $k < a(i_1) + \cdots + a(i_j) + a(n)$, where $1 \leq i_1 < \cdots < i_j < n$. Thus, the monomial $(x^a/x_n)t^k$ belongs to $A_k(\Sigma_n^{(j)})$. Let $a(1) = a(n)$. Then $x^at^k = (x_1x_2 \cdots x_n)^{a(1)}t^k$ and $k \leq (j + 1)a(1)$. If $k < (j + 1)a(1)$, then the monomial $(x^a/x_n)t^k$ belongs to $A_k(\Sigma_n^{(j)})$. If $k = (j + 1)a(1)$, then $x^at^k = ((x_1x_2 \cdots x_n)^{t^{j+1}})^{a(1)}$.

Second, let $\text{supp}(x^a) \neq [n]$ and, say, $\text{supp}(x^a) = \{1, 2, \dots, \ell\}$. Since $\Sigma_n^{(j)} \neq \Sigma_n$, one has $\ell \geq 2$. Since $\{n - j, \dots, n\}$ is a facet of $\Sigma_n^{(j)}$, one has $n - j \leq \ell$. Thus, $0 \leq j - (n - \ell) \leq \ell - 2$. If $X \subset [\ell]$ with $|X| = j + 1 - (n - \ell)$, then $X \cup ([n] \setminus [\ell])$ is a facet of $\Sigma_n^{(j)}$. Thus, the monomial x^at^k must belong to $A_k(\Sigma_\ell^{(j-(n-\ell))})$. Since $\ell < n$, one has $A_{\ell, j-(n-\ell)} = A(\Sigma_\ell^{(j-(n-\ell))})$. Since $\ell - (j - (n - \ell)) = n - j$, the S -subalgebra $A_{\ell, j-(n-\ell)}$ is generated by those monomials $x_{i_1}x_{i_2} \cdots x_{i_{n-j+q-1}}t^q$ with $1 \leq q \leq j - (n - \ell) + 1$, where $1 \leq i_1 < i_2 < \cdots < i_{n-j+q-1} \leq \ell$. Hence $A_{\ell, j-(n-\ell)} \subset A_{n,j}$. Thus, the monomial x^at^k must belong to $A_{n,j}$, as desired. \square

Example 4.7. The graph of a triangle is just the skeleton $\Sigma_3^{(1)}$. Its vertex cover algebra with respect to the canonical weight function is the subalgebra of $K[x_1, x_2, x_3, t]$ generated by the elements $x_1x_2t, x_1x_3t, x_2x_3t, x_1x_2x_3t^2$ over $K[x_1, x_2, x_3]$.

5. MAXIMAL GENERATING DEGREE

In this section we are interested in the maximal degree of the minimal generators of the vertex cover algebras.

It turns out that this degree can be estimated for finite graphs. Every finite graph considered here is assumed to have no loops and no multiple edges.

Let G be a finite graph on the vertex set $[n]$ and $E(G)$ its edge set. Let $w : E(G) \rightarrow \mathbb{N} \setminus \{0\}$ be a weight function on G . Recall that a vector $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$ is a vertex cover of a weighted graph (G, w) of order k if $a(i) + a(j) \geq kw(e)$ for all $e = \{i, j\} \in E(G)$.

Let, as before, $S = K[x_1, \dots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. The vertex cover algebra $A(G, w)$ of (G, w) is the graded S -algebra $A(G, w) = \bigoplus_{k \geq 0} A_k(G, w) \subset S[t]$, where $A_k(G, w)$ is spanned by those monomials $x^a t^k = x_1^{a(1)} \dots x_n^{a(n)} t^k$ such that $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$ is a vertex cover of (G, w) of order k , and where $A_0(G, w) = S$.

First of all, the vertex cover algebra of a finite graph G together with the canonical weight function is discussed.

Theorem 5.1. *Let G be a finite graph on $[n]$. Then*

- (a) *The graded S -algebra $A(G)$ is generated in degree at most 2. In particular, the second Veronese subalgebra $A(G)^{(2)}$ is a standard graded S -algebra.*
- (b) *The graded S -algebra $A(G)$ is a standard graded S -algebra if and only if G is a bipartite graph.*

Proof. (a) Let $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$ be a vertex cover of G of order k with $k \geq 3$. What we must prove is the existence of a vertex cover $\varepsilon = (\varepsilon(1), \dots, \varepsilon(n))$ of G of order 2 such that $a - \varepsilon$ is a vertex cover of G of order $k - 2$.

Let $A \subset [n]$ denote the set of $i \in [n]$ with $a(i) = 0$, $B \subset [n]$ the set of $j \in [n]$ such that there is $i \in A$ with $\{i, j\} \in E(G)$, and $C = [n] \setminus (A \cup B)$. We then define $\varepsilon \in \mathbb{N}^n$ by setting $\varepsilon(i) = 0$ if $i \in A$, $\varepsilon(i) = 2$ if $i \in B$, and $\varepsilon(i) = 1$ if $i \in C$.

It is clear that ε is a vertex cover of G of order 2. We claim that $a - \varepsilon$ is a vertex cover of G of order $k - 2$. Note that $a(i) \geq k$ if $i \in B$. Let $\{i, j\}$ be an edge of G . Then $a(i) + a(j) \geq k + 1$ if $i \in B$ and $j \in C$; $a(i) + a(j) \geq 2k$ if both i and j belong to B . Thus, $(a(i) + a(j)) - (\varepsilon(i) + \varepsilon(j)) \geq k - 2$. Hence $a - \varepsilon$ is a vertex cover of G of order $k - 2$, as desired.

(b) (“if”) Let G be a bipartite graph on $[n] = U \cup V$ with $U \cap V = \emptyset$, where each edge of G is of the form $\{i, j\}$ with $i \in U$ and $j \in V$. Let $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$ be a vertex cover of G of order k with $k \geq 2$. What we must prove is the existence of a vertex cover $\varepsilon = (\varepsilon(1), \dots, \varepsilon(n))$ of G of order 1 such that $a - \varepsilon$ is a vertex cover of G of order $k - 1$. Let $A \subset U$ denote the set of $i \in U$ with $a(i) = 0$, and $B \subset V$ the set of those $j \in V$ such that there is $i \in A$ with $\{i, j\} \in E(G)$. We then define $\varepsilon = (\varepsilon(1), \dots, \varepsilon(n)) \in \mathbb{N}^n$ by setting $\varepsilon(i) = 0$ if $i \in A \cup (V \setminus B)$, and $\varepsilon(i) = 1$ if $i \in (U \setminus A) \cup B$. It is clear that ε is a vertex cover of G of order 1. As was done in (a) it follows that $a - \varepsilon$ is a vertex cover of G of order $k - 1$, as required.

(“only if”) Let G be a finite graph on $[n]$ which is not bipartite. Then there is a cycle C of odd length, say, of length $2\ell - 1$. Let each of $a = (a(1), \dots, a(n))$ and

$b(b(1), \dots, b(n))$ be a vertex cover of G of order 1. Let, say, $1, 2, \dots, 2\ell - 1$ be the vertices of C . Then the number of $1 \leq i \leq 2\ell - 1$ with $a(i) > 0$ is at least ℓ and the number of $1 \leq i \leq 2\ell - 1$ with $b(i) > 0$ is at least ℓ . Hence $(a + b)(i) > 1$ for some $1 \leq i \leq 2\ell - 1$. Thus, the vertex cover $(1, 1, \dots, 1) \in \mathbb{N}^n$ of G of order 2 cannot be the sum of the form $a + b$, where each of a and b is a vertex cover of G of order 1. In other words, the monomial $x_1 x_2 \cdots x_n t^2 \in A_2(G)$ must belong to the minimal system of monomial generators of the S -algebra $A(G)$. \square

Example 5.2. The graph of a triangle is not bipartite. That is why its vertex cover algebra is not standard graded, as we have seen in Example 4.7. On the other hand, the graph of a square is bipartite. Hence its vertex cover algebra is standard graded. In fact, it is the subalgebra of $K[x_1, x_2, x_3, x_4, t]$ generated by the elements $x_1 x_3 t, x_2 x_4 t$ over $K[x_1, x_2, x_3, x_4]$.

Recall that, in combinatorics on finite graphs, a *minimal vertex cover* of G is a vertex cover $(a(1), \dots, a(n))$ of G of order 1 with the property that if $b = (b(1), \dots, b(n))$ is a vertex cover of G of order 1 with $b(i) \leq a(i)$ for all i , then one has $a(i) = b(i)$ for all i . The S -module $A_1(G)$ is generated by those monomials $x_1^{a(1)} \cdots x_n^{a(n)} t$ such that $(a(1), \dots, a(n))$ is a minimal vertex cover of G .

Proposition 5.3. *Let G be a finite graph on $[n]$ with $E(G)$ its edge set. Then the following conditions are equivalent:*

- (i) *The graded S -algebra $A(G)$ is generated by the monomial $x_1 x_2 \cdots x_n t^2$ together with those monomials $x_1^{a(1)} \cdots x_n^{a(n)} t$ such that $(a(1), \dots, a(n))$ is a minimal vertex cover of G ;*
- (ii) *For every cycle C of G of odd length and for every vertex $i \in [n]$ there exists a vertex j of C with $\{i, j\} \in E(G)$.*

Proof. (i) \implies (ii): Let C be a cycle of G of odd length and i_0 a vertex of G . Suppose that $\{i_0, j\} \notin E(G)$ for every vertex j of C . Write $V(C)$ for the vertex set of C and define $a = (a(1), \dots, a(n)) \in \mathbb{N}^n$ by setting $a(j) = 1$ if $j \in V(C)$, $a(i_0) = 0$, and $a(k) = 2$ if $k \in [n] \setminus (V(C) \cup \{i_0\})$. Since $\{i_0, j\} \notin E(G)$ for every $j \in V(C)$, it follows that a is a vertex cover of G of order 2. Since $a(i_0) = 0$, the monomial $x^a t^2$ cannot be divided by $x_1 \cdots x_n t^2$. Since $a(j) = 1$ for all $j \in V(C)$, as was seen in the proof of “only if” part of Theorem 5.1, it is impossible to write $a = b + b'$, where each of b and b' is a vertex cover of G of order 1. Hence the graded S -algebra $A(G)$ cannot be generated by the monomial $x_1 x_2 \cdots x_n t^2$ together with those monomials $x_1^{a(1)} \cdots x_n^{a(n)} t$ such that $(a(1), \dots, a(n))$ is a vertex cover of G of order 1.

(ii) \implies (i): Let G be a finite graph on $[n]$ with the property that, for every cycle C of G of odd length and for every vertex $i \in [n]$, there exists a vertex j of C with $\{i, j\} \in E(G)$. If $a = (a(1), \dots, a(n))$ is a vertex cover of G of order 2 with each $a(i) > 0$, then the monomial $x_1 x_2 \cdots x_n t^2$ divides $x^a t^2$.

Let $a = (a(1), \dots, a(n))$ be a vertex cover of G of order 2 with $a(i_0) = 0$ for some $i_0 \in [n]$. What we must prove is the existence of b and b' with $a = b + b'$, where each of b and b' is a vertex cover of G of order 1. Let C be a cycle of G of odd length. Then there is a vertex j of C with $\{i_0, j\} \in E(G)$. Since $a(i_0) = 0$, one has

$a(j) \geq 2$. In other words, if $V \subset [n]$ is the set of those $i \in [n]$ with $a(i) = 1$ and if G' is the induced subgraph of G on V , then G' possesses no odd cycle. Hence G' is a bipartite graph. Let $V = V_1 \cup V_2$ be a decomposition of V such that each edge of G' is of the form $\{p, q\}$ with $p \in V_1$ and $q \in V_2$. Now, define $b = (b(1), \dots, b(n)) \in \mathbb{N}^n$ by setting $b(i) = 1$ if either $a(i) \geq 2$ or $i \in V_1$, and $b(i) = 0$ if either $a(i) = 0$ or $i \in V_2$. In addition, define $b' = (b'(1), \dots, b'(n)) \in \mathbb{N}^n$ by setting $b'(i) = a(i) - 1$ if $a(i) \geq 2$, $b'(i) = 1$ if $i \in V_2$, and $b'(i) = 0$ if either $a(i) = 0$ or $i \in V_1$. Then each of b and b' is a vertex cover of G of order 1 with $a = b + b'$, as desired. \square

We now improve the result of Theorem 5.1(b) by the following

Theorem 5.4. *Let G be a finite bipartite graph on $[n]$ and w an arbitrary weight function on G . Then the vertex cover algebra $A(G, w)$ is a standard graded S -algebra.*

Proof. Given a real number r we write $\lceil r \rceil$ for the least integer q for which $r \leq q$ and write $\lfloor r \rfloor$ for the greatest integer q for which $q \leq r$. Let a and b be nonnegative integers, and let k and N be positive integers. Suppose that $a + b \geq kN$. We claim

$$(1) \quad \lceil a/k \rceil + \lfloor b/k \rfloor \geq N$$

and

$$(2) \quad (a - \lceil a/k \rceil) + (b - \lfloor b/k \rfloor) \geq (k - 1)N.$$

To see why these inequalities are true, let $a/k = \ell + \varepsilon$ and $b/k = m + \delta$, where ℓ and m are integers and where $0 \leq \varepsilon < 1$ and $0 \leq \delta < 1$. Since $a + b \geq kN$, one has $a/k + b/k \geq N$. Thus $(\ell + m) + (\varepsilon + \delta) \geq N$. Since $0 \leq \varepsilon + \delta < 2$, one has $\ell + m \geq N - 1$ if $\varepsilon > 0$ and $\ell + m \geq N$ if $\varepsilon = 0$. If $\varepsilon > 0$, then $\lceil a/k \rceil = \ell + 1$. Thus

$$\lceil a/k \rceil + \lfloor b/k \rfloor = (\ell + 1) + m \geq N.$$

If $\varepsilon = 0$, then $\lceil a/k \rceil = \ell$. Thus $\lceil a/k \rceil + \lfloor b/k \rfloor = \ell + m \geq N$.

On the other hand, one has $(k - 1)(a/k + b/k) \geq (k - 1)N$. Thus

$$(a - a/k) + (b - b/k) \geq (k - 1)N.$$

Hence

$$(a - \ell) + (b - m) - (\varepsilon + \delta) \geq (k - 1)N.$$

Let $\varepsilon > 0$. Then $(a - \ell) + (b - m) \geq (k - 1)N + 1$. Hence

$$(a - \lceil a/k \rceil) + (b - \lfloor b/k \rfloor) = (a - \ell) + (b - m) - 1 \geq (k - 1)N.$$

Let $\varepsilon = 0$. Then $(a - \ell) + (b - m) \geq (k - 1)N$. Hence

$$(a - \lceil a/k \rceil) + (b - \lfloor b/k \rfloor) = (a - \ell) + (b - m) \geq (k - 1)N.$$

Let G be a finite bipartite graph on $[n]$ and w an arbitrary weight function on G . Let $[n] = U \cup V$ be the decomposition of $[n]$, where each edge of G is of the form $\{i, j\}$ with $i \in U$ and $j \in V$. Let $x^a t^k = x_1^{a(1)} \dots x_n^{a(n)} t^k$ be a monomial belonging to $A_k(G, w)$. Thus $a(i) + a(j) \geq kw(\{i, j\})$ for each edge $\{i, j\}$ of G . We then define $b = (b(1), \dots, b(n)) \in \mathbb{N}^n$ by setting $b(i) = \lceil a(i)/k \rceil$ if $i \in U$ and $b(j) = \lfloor a(j)/k \rfloor$ if $j \in V$. Let $c = a - b \in \mathbb{N}^n$. Then the above inequalities (1) and (2) guarantee that $b(i) + b(j) \geq w(\{i, j\})$ and $c(i) + c(j) \geq (k - 1)w(\{i, j\})$ for each edge $\{i, j\}$ of

G . Thus $x^{bt} \in A_1(G, w)$ and $x^{ct^{k-1}} \in A_{k-1}(G, w)$. Moreover, $x^{at^k} = (x^{bt})(x^{ct^{k-1}})$. Hence the graded S -algebra $A(G, w)$ is generated by $A_1(G, w)$, as desired. \square

We have seen in Lemma 4.1 that the vertex cover algebra $A(\Delta, w)$ of a weighted simplicial complex Δ is the symbolic Rees algebra of the ideal $I^*(\Delta, w)$. Therefore, $A(\Delta, w)$ is standard graded over S if and only if $I^*(\Delta, w)^{(n)} = I^*(\Delta, w)^n$ for all $n > 0$.

Viewing in this way, the sufficient part of Theorem 5.1(b) was already proved by Gitler, Reyes and Villarreal [14, Corollary 2.6]. In fact, it follows from their result that $I^*(G)^{(n)} = I^*(G)^n$ for all $n > 0$ if G is a bipartite graphs. On the other hand, Simis, Vasconcelos and Villarreal [24, Theorem 5.9] showed that if I is the edge ideal of a graph G , then $I^{(n)} = I^n$ for all $n \geq 0$ if and only if G is bipartite.

Due to a result of Bahiano [2, Theorem 2.14], the symbolic Rees algebra of the edge ideal I of a graph is generated by elements of degree at most $(n-1)(n-h)$, where n is the number of vertices and $h = \text{height}(I)$. Since the bound is attained by a complete graph, where $h = n-1$, Vasconcelos has asked whether this bound can be improved to $n-1$. In the following we give a family of counter-examples to this question. These examples also show that the maximal degree of the generators of vertex cover algebras of simplicial complexes on n vertices can not be bounded by any linear function of n .

Example 5.5. Let $n = m + 2k + 1$, where $m, k \geq 2$. Let G be the graph on the vertex set $V = \{1, \dots, n\}$ with the edges (i, j) for $i = 1, \dots, m, j = 1, \dots, n, j \neq i$, and $(i, i+k), (i, i+k+1)$ for $i = m+1, \dots, n$. Here we replace any occurring index h with $n < h \leq n + 2k + 1$ by the index $h - n + m \leq n$.

Let Σ be the simplicial complex of G , then the minimal non-faces of Σ are the vertices $1, \dots, m$ and the sets $\{i, \dots, i+k-1\}, i = m+1, \dots, n$. Let Δ be the simplicial complex with the facets $V \setminus \{1\}, \dots, V \setminus \{m\}$ and $V \setminus \{i, \dots, i+k-1\}, i = m+1, \dots, n$. Then $A(\Delta)$ is the symbolic Rees algebra of the edge ideal of G .

Let $a \in \mathbb{N}^{2k+1}$ be the vertex cover of Δ with $a(i) = k$ for $i = 1, \dots, m$, and $a(j) = 1$ for $j = m+1, \dots, n$. Obviously, a is a vertex cover of order $mk + k + 1$ with respect to the canonical weight. In fact, we have $\sum_{i \in F} a(i) = mk + k + 1$ for all facets $F \in \Delta$. We claim that a is an indecomposable vertex cover. In other words, the vertex cover algebra $A(\Delta)$ has a generator of degree $mk + k + 1$. For any linear function en , where $e > 0$ is an integer, we choose $m = 4e$ and $k = 2e$. Then

$$mk + k + 1 = 8e^2 + 2e + 1 > 8e^2 + e = en.$$

In order to show that a is indeed an indecomposable vertex cover of Δ suppose to the contrary that a is the sum of two vertex covers b and c of order $< mk + k + 1$. Let ℓ be the order of b . Then c is of order $mk + k + 1 - \ell$. We first notice that

$$\sum_{i \in F} b(i) + \sum_{i \in F} c(i) = mk + k + 1$$

for all facets $F \in \Delta$. Since $\sum_{i \in F} b(i) \geq \ell$ and $\sum_{i \in F} c(i) \geq mk + k + 1 - \ell$, this implies that $\sum_{i \in F} b(i) = \ell$. From this it follows that $\sum_{i \notin F} b(i)$ is the same for all facets F

of Δ . Comparing this sum for the facets $V \setminus \{1\}$ and $V \setminus \{m+k+1, \dots, m+2k+1\}$, we get $b(1) = \sum_{i=m+1}^{m+k} b(i)$. Similarly,

$$b(j) = \sum_{i=m+1}^{m+k} b(i)$$

for all $j = 2, \dots, m$. If we consider the facets $V \setminus \{i, \dots, i+k-1\}$ and $V \setminus \{i+1, \dots, k+1\}$, we get $b(i) = b(i+k)$ for $i = m+1, \dots, m+2k+1$. From this it follows that

$$b(m+1) = \dots = b(m+2k+1)$$

and hence $b(j) = kb(m+1)$ for $j = 2, \dots, m$. Thus,

$$\ell = \sum_{i=2}^{m+2k+1} b(i) = (mk+k+1)b(m+1).$$

Now, if $b(m+1) = 0$, we would get $\ell = 0$, which implies that c is of order $mk+k+1$, a contradiction. If $b(m+1) \geq 1$, we would get $\ell \geq mk+k+1$, a contradiction, too.

Using the program `normaliz` by Bruns and Koch [7] we found that if $m = k = 2$, $A(\Delta)$ has 52 generators over K and that the generator of maximal degree corresponds to the vertex cover described above.

On the other hand, we have the following general upper bound for the degrees of the generators of a vertex cover algebra. Recall that $d(A)$ denotes the maximal degree of the generators of a graded S -algebra A .

Theorem 5.6. *Let (Δ, w) be a weighted simplicial complex on the vertex set $[n]$. Then*

$$d(A(\Delta, w)) < \frac{(n+1)^{\frac{n+3}{2}}}{2^n}.$$

Proof. We may view $A(\Delta, w)$ as the semigroup ring of the normal semigroup $H \subset \mathbb{Z}^{n+1}$ which is the set of integral points of the rational cone C defined by the set of integral inequalities

$$\sum_{i \in F} z_i - w_F y \geq 0 \quad \text{for } F \in \mathcal{F}(\Delta) \quad \text{and} \quad z_1 \geq 0, \dots, z_n \geq 0, \quad y \geq 0.$$

In other words, $x_1^{a_1} \dots x_n^{a_n} t^k \in A(\Delta, w)$ if and only if $(a_1, \dots, a_n, k) \in C$. For each element $q \in H$ we denote by $\deg q$ the last component of q , which is equal to the degree of the corresponding element in $A(\Delta, w)$.

Let E be a set of integral vectors spanning the extremal arrays of C . Then every element $q \in H$ must lie in a rational cone spanned by $n+1$ elements of E . Let q_1, \dots, q_{n+1} be an arbitrary subset of $n+1$ elements of E . Let D be the rational cone spanned by these elements. Then the affine semigroup of integral points of D is generated by the elements of the form $a_1 q_1 + \dots + a_{n+1} q_{n+1}$ with $a_j \in [0, 1]$, $a_1 + \dots + a_{n+1} < n+1$. This implies that the maximal degree of a generator of this affine semigroup is less than $\deg q_1 + \dots + \deg q_{n+1}$. Since the union of all generators of such affine semigroups forms a set of generators for H , we get

$$d(A(\Delta, w)) < (n+1) \max\{\deg q \mid q \in E\}.$$

On the other hand, every element $q \in E$ is a solution of n inequalities of the form $\sum_{i \in F} z_i - w_F y \geq 0$. Let A denote the square $(0,1)$ -matrix of the coefficients of the variables z_i . Then we may assume that $\deg q = |\det A|$. In 1893 Hadamard proved that the determinant of any $n \times n$ complex matrix A with entries in the closed unit disk satisfies the inequality $|\det A| \leq n^{n/2}$. As an improvement of this bound it was shown by Faddeev and Sominskii [13] in 1965 that for $(0,1)$ -square matrix of size n one has the inequality

$$|\det A| \leq \frac{(n+1)^{\frac{n+1}{2}}}{2^n}.$$

This then yields the desired conclusion. \square

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