ALGEBRAIC CYCLE COMPLEXES

Basic Properties

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Abstract. We collect several basic properties of algebraic cycle complexes defined by Bloch, Friedlander, Suslin and Voevodsky, like moving lemmas, localization, homotopy invariance and Mayer-Vietoris exact sequences. We also explain a generalization of the theorem of Nesterenko/Suslin/Totaro from fields to smooth, semilocal algebras of geometric type over an infinite base field. After this survey we give a new cubical proof of Bloch/Nart's elementary vanishing theorem in codimension one. Then we show how these results give rise to a framework in which we can study the relationship between motivic cohomology (higher Chow groups) of smooth varieties and Zariski cohomology with respect to Quillen or Milnor K-sheaves. Finally we indicate how moving lemmas can be used to derive properties of algebraic cycle complexes over fields and give several examples.

1. Introduction

During the conference at Banff, I felt that there was a need to describe some fairly elementary arguments that are frequently used in the study of various complexes (built out of algebraic cycles) which compute motivic cohomology or some of its variants. This paper is an extended version of my lecture. It contains also some other well known material from other sources. I want to emphasize that, except from some elementary proofs, this paper is mostly of expository nature.

In the next section we will briefly explain the definitions of various motivic (co)homology theories with or without compact supports. The article (Friedlander/Voevodsky, 1999) contains a comparison of all 4 existing theories (together with a bivariant theory) with all of their standard properties. This is in analogy with algebraic topology where we have homology and cohomology with or without compact supports. In the following chapters however, we will mostly describe the theory of higher Chow groups, which is the Borel-Moore homology theory in this picture. Since our presentation of the general theory is quite short, we refer to the original papers (Bloch, 1986), (Suslin/Voevodsky, 1996), (Suslin/Voevodsky, 1999),(Suslin/Voevodsky, 1995), (Friedlander/Voevodsky, 1999), (Voevodsky, 1999), (Voevodsky, 1999) and (Voevodsky, 1996). Furthermore we recommend the following excellent survey articles: (Bloch, 1998), (Friedlander, 1997), (Levine, 1997) and (Weibel, 1997).

In the third section we explain various moving lemmas and their consequences, in particular a generalization of the theorem of Nesterenko, Suslin and Totaro from fields to smooth, semilocal algebras of geometric type over an infinite base field k.

In section 4, we discuss the vanishing theorem in codimension one, a special and easy case of the Beilinson-Soulé conjecture, (Soulé, 1985). We give an elementary proof, following (Nart, 1989).

Section 5 is devoted to applications. We study the relationship between motivic cohomology of smooth varieties and K-cohomology with respect to the Quillen- or Milnor K-sheaves.

In the last section we indicate how the previous results may be applied to derive properties of algebraic cycle complexes over spectra of fields and give several examples.

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2. Algebraic Cycle Complexes

All schemes occuring here are assumed to be Noetherian, separated and equidimensional over a field k. First let us give two different but equivalent definitions for Bloch's higher Chow groups:

Simplicial Higher Chow groups (Bloch, 1986):

Here $\Delta^n = \mathrm{Spec}(k[t_0,\dots,t_n]/\sum t_i - 1) \cong \mathbb{A}^n_k$ = affine n-space over an arbitrary field k with coordinates $t_0,...,t_n$ satisfying $\sum t_i = 1$. We denote by $\Sigma := \cup\{t_i = 0\} \subseteq \mathbb{A}^n$ the union of all codimension one faces; in general, a face is a subsimplex $\Delta^m \subseteq \Delta^n$ obtained by setting $n-m$ coordinates equal to zero. Let $Z^p(X,n)$ be the free abelian group generated by all closed, integral subvarieties of codimension p in $X \times \Delta^n$, meeting all faces of all codimensions properly. $CH^p(X,n)$ is defined as the n-th homology group of the complex

$$\dots \to Z^p(X,n+1)\xrightarrow{\partial} Z^p(X,n)\xrightarrow{\partial} Z^p(X,n-1)\xrightarrow{\partial}\dots\xrightarrow{\partial} Z^p(X,0),$$

where $\partial = \sum(-1)^i\partial_i$ is given by the alternating sum of restrictions to faces.

Cubical Higher Chow groups (Levine, 1994):

Here we use instead the algebraic cubes $\square^n := (\mathbb{P}^1_k \setminus \{1\})^n \cong \mathbb{A}^n_k$ and the faces are defined by setting $x_i = 0$, resp. $x_i = \infty$, while the boundary is given by $\partial = \sum(-1)^i(\partial_i^0 - \partial_i^\infty)$. $C^p(X,n)$ is defined as the free abelian group generated by all closed integral subvarieties of $X \times \square^n$ meeting all cubical faces properly modulo the subgroup of degenerate cycles, i.e. those which arise from pulling back along projection maps $\square^n \to \square^{n-1}$. In (Levine, 1994) it is shown that the resulting complex is quasiisomorphic to the simplicial version.

Equidimensional Cycles (Suslin, 1999):

Let X be a quasiprojective scheme of pure dimension d over k and $r \geq 0$. There are contravariant functors (presheaves) from the category of smooth schemes over k to abelian groups:

$$z_{equi}(X,r) : \mathfrak{Sm}/k \mapsto \mathfrak{Ab},$$

where $z_{equi}(X,r)(Y)$ is the free abelian group generated by all closed integral subvarieties $W \subseteq X \times Y$ such that W is equidimensional of relative dimension r over Y. Note that equidimensional of dimension zero is equivalent to being quasifinite, of finite type and dominant over a component of Y. In particular, we may choose $Y = \Delta^n$ and note that every equidimensional cycle meets all faces properly by definition.
Therefore, for $i \leq d$, we obtain a subcomplex

$$Z^i_{equi}(X,*) \subseteq Z^i(X,*)$$

of the complex of Bloch, by setting $Z^i_{equi}(X,n) := z_{equi}(X,d-i)(\Delta^n)$. It consists out of cycles of codimension i which are equidimensional of relative dimension $d-i$, where d is the dimension of X. We will see later, that the inclusion gives rise to an isomorphism on homology, see also (Suslin, 1999).

Algebraic singular (Suslin) homology (Suslin/Voevodsky, 1996):

Let S be a smooth, connected scheme over k and X an arbitrary scheme of finite type over k. Define $C_0(S;X)$ as the free abelian group generated by all closed, integral subvarieties $W \subseteq X \times S$, such that the projection $pr_2 : W \to S$ is finite (in particular proper) and surjective. Now consider the abelian groups

$$C_n(S;X) := C_0(S \times \Delta^n_k; X)$$

together with restriction maps to faces

$$\partial_n : C_n(S;X) \to C_{n-1}(S;X)$$

It is very instructive to view elements in $C_n(S;X)$ as zero cycles on X parametrised by $S \times \Delta^n_k$. $C_n(S;X)$ is covariant in X and contravariant in S for arbitrary morphisms. Suslin homology is defined as the $n-$th homology group of the complex $C_*(S;X)$, and denoted by $H_n^{\mathrm{sing}}(S;X)$. It defines a functor $\mathfrak{Sch}/k \to \mathfrak{Ab}$. There is one important special case when $S = Spec(k)$. Then we just write $C_n(X)$ for $C_n(Spec(k);X)$ and $H_n^{\mathrm{sing}}(X)$ for $H_n^{\mathrm{sing}}(Spec(k);X)$.

Motivic Complexes of Suslin/Voevodsky (Suslin/Voevodsky, 1995):

We first define the complexes $\mathbb{Z}(i)$ from (Suslin/Voevodsky, 1995): these are chain complexes of Zariski sheaves. Let $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\} = Spec(k[t,t^{-1}])$. Then

$$\mathbb{Z}(i) := C_*((\mathbb{G}_m)^i/\partial\mathbb{G}_m^{i-1})[-i]$$

where $\partial(\mathbb{G}_m)^{i-1} \subseteq (\mathbb{G}_m)^i$ is the union of all embeddings which are obtained by setting one coordinate equal to 1. Here, when we write $\mathbb{G}_m$, we really mean the presheaf representing it and then take the associated chain complex C_* afterwards. $\mathbb{Z}(i)$ is bounded above and, for example, $\mathbb{Z}(0) \cong \mathbb{Z}$ and $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$.

Motivic Cohomology (Suslin/Voevodsky, 1995) is defined as the hypercohomology group

$$H^*(X, \mathbb{Z}(i)) := H^*_{cdh}(X, \mathbb{Z}(i))$$

Here the $cdh-$topology is the minimal Grothendieck topology, such that Nisnevich coverings (étale coverings, such that above each scheme point there is a point with isomorphic residue field) and morphisms of the form

$$(p,i) : W \coprod U_1 \to U$$

are coverings, where $i : U_1 \subseteq U$ is a closed embedding and $p : W \to U$ a proper surjective morphism with the property that $p^{-1}(U - U_1) \to U - U_1$ is an isomorphism. Examples for cdh-coverings are resolutions of singularities and inclusions of components of reducible schemes.

Bivariant Cycle Cohomology (Friedlander/Voevodsky, 1999)

Let X and U be schemes of finite type over k. Bivariant cycle cohomology is defined as the hypercohomology group

$$A_{r,n}(U,X) := H^{-n}_{cdh}(U, C_* z_{equi}(X,r))$$

Bivariant cycle cohomology is related to higher Chow groups in the following way: if X is an affine, equidimensional scheme of dimension n over k, then $A_{r,i}(Spec(k),X) \cong CH^{n-r}(X,i)$. Furthermore it is the main device used to prove duality between motivic cohomology and higher Chow groups for smooth schemes (see below).

3. Moving Lemmas and Corollaries

First we sketch the proof of homotopy invariance for higher Chow groups. The same argument (without the moving lemma part) works also for Suslin homology, motivic cohomology and bivariant cycle cohomology in the first argument!

Theorem 3.1 (Bloch, 1986)
Higher Chow groups are homotopy invariant:

$$pr_1^* : CH^p(X,n) \xrightarrow{\cong} CH^p(X \times_k \mathbb{A}^1_k, n).$$

Proof: The projection map

$$pr_1 : X \times_k \mathbb{A}^1_k \to X$$

is flat, i.e. pullbacks are well defined on the level of algebraic cycles.
Step 1: Homotopy (formal sum of maps):

$$T^n : \Delta^{n+1} \to \Delta^n \times \Delta^1$$

defined (on vertices) by $T^n = \sum (-1)^i f_i^n$, where

$$f_i^n(j) = \begin{cases} (j,0) & j \leq i \\ (j-1,1) & i < j \leq n+1 \end{cases}.$$

This defines homotopies

$$T^* : Z^p(X \times_k \mathbb{A}^1_k \times_k \Delta^n) \to Z^p(X \times_k \Delta^{n+1})$$

and one obtains:

$$T^* \circ d + d \circ T^* = i_1^* - i_0^*$$

wherever the maps are defined. The next step guarantees that we may apply this homotopy.
Step 2: It is necessary to show that the subcomplex of cycles in $Z^p(X,\cdot)$ that intersect properly all additional faces occuring in the triangulation above, has the same homology. The proof of that quasiisomorphism is a consequence of the following type of moving lemma (applied to $X \times \mathbb{A}^1$):

Theorem 3.2 Moving Lemma I, (Bloch, 1986, chap. 2)
Let X be an integral $k-$variety endowed with an action of a connected algebraic group G on it. Let K be an extension field of k and $\psi : \mathbb{A}^1_K \to G_K$ be a morphism with $\psi(0) = id$ such that $\psi(x)$ is k-generic for each non-zero $x \in \mathbb{A}^1(\overline{k})$. Assume that there is a finite collection $\mathcal{Y}$ of closed subsets of X such that $G \cdot Y = X$ for all $Y \in \mathcal{Y}$ and denote by $Z^p(X,\cdot)_{\mathcal{Y}}$ the subspace of all cycles which additionally also meet the subsets $Y \times \Delta^m$ properly for all faces Δ^m. Then the following pullback map from k to K is homotopic to zero:

$$\pi^* : \frac{Z^p(X_k,\cdot)}{Z^p(X_k,\cdot)_{\mathcal{Y}}} \longrightarrow \frac{Z^p(X_K,\cdot)}{Z^p(X_K,\cdot)_{\mathcal{Y}}}$$

Back to the proof of homotopy invariance: by specialization, π^* (applied to $X \times \mathbb{A}^1$) is injective on homology and this finishes step 2. □

Corollary 3.3 *The complexes $Z^p(X,\cdot)$ (simplicial) and $C^p(X,\cdot)$ (cubical) are quasisomorphic.*

Proof: (Levine, 1994), using homotopy invariance for both simplicial and cubical complexes. □

Theorem 3.4 Moving Lemma II, (Bloch, 1994)
Let X be a quasiprojective variety over a field of arbitrary characteristic and $A \subseteq X$ a closed subset of pure codimension r with open complement $U = X - A$. Then the restriction map

$$Z^p(X,\cdot)/Z^{p-r}(A,\cdot) \longrightarrow Z^p(U,\cdot)$$

is a quasiisomorphism inducing the long exact localization sequence.

A refinement of moving lemma I was written up by Bloch (unpublished) and Levine (Levine, 1998):

Theorem 3.5 Moving Lemma III, (Levine, 1998)
Let Y be a scheme of finite type over k and X a smooth, affine variety over k together with a morphism $f : Y \to X$ defined over k. Denote by $Z^p_f(X, \cdot) \subseteq Z^p(X, \cdot)$ the subcomplex generated by those integral cycles W such that $f^{-1}(W)$ is defined (i.e. meets properly). Then the inclusion of complexes is a quasiisomorphism.

Thus higher Chow groups admit pullback maps in the category of smooth affine varieties. As far as I know it remains an open problem, whether higher Chow groups admit pullback maps between arbitrary smooth varieties. However it is easy to check that the groups $CH^p(X, n)$ are covariant for proper morphisms and contravariant for flat morphisms between arbitrary schemes. Furthermore, by (Levine, 1994), $CH^p(X, n) \otimes \mathbb{Q}$ is contravariant for arbitrary morphisms between smooth schemes and the complexes $Z^p(-, n) \otimes \mathbb{Q}$ satisfy the Mayer-Vietoris property, i.e. for open subsets $U, V \subseteq X$, there is a distinguished triangle

$$Z^p(U \cup V, \cdot)_{\mathbb{Q}} \to Z^p(U, \cdot)_{\mathbb{Q}} \oplus Z^p(V, \cdot)_Q \to Z^p(U \cap V, \cdot)_{\mathbb{Q}} \to Z^p(U \cup V, \cdot - 1)_{\mathbb{Q}}$$

inducing the Mayer-Vietoris long exact sequences

$$\to CH^p(U \cup V, n)_{\mathbb{Q}} \to CH^p(U, n)_{\mathbb{Q}} \oplus CH^p(V, n)_{\mathbb{Q}} \to CH^p(U \cap V, n)_{\mathbb{Q}} \to$$

Higher Chow groups also admit products and regulator maps to Deligne cohomology and étale cohomology. Their most important property is given by the theorem of Bloch (Bloch, 1986) (refined by Levine in (Levine, 1998)) which puts them into relation with (the weight-graded pieces of) Quillen K-theory in the case of a smooth, quasiprojective variety X of dimension d over a field k:

$$gr^p_\gamma K_n(X) \otimes \mathbb{Z}[\frac{1}{(n+d-1)!}] \cong CH^p(X, n) \otimes \mathbb{Z}[\frac{1}{(n+d-1)!}]$$

Let us mention a few further results:

Theorem 3.6 (Nesterenko/Suslin, 1990), (Totaro, 1992)
If k is an arbitrary field,

$$K^M_n(k) \cong CH^n(k, n).$$

The methods of Nesterenko/Suslin/Totaro together with moving lemma III and other techniques imply the following generalization:

Theorem 3.7 (Elbaz-Vincent/Müller-Stach, 1998)
Let R be a smooth, semilocal $k-$algebra of geometric type over an infinite field k. Then there is a natural surjective map

$$\varphi_n : K_n^M(R) \to CH^n(Spec(R), n)$$

which takes a Milnor symbol $\{r_1, ..., r_n\}$ to the graph cycle, which is defined by the n rational functions $r_1, ..., r_n$ on $Spec(R)$.

Gabber, cf. (Gabber, 1992), has proved an equivalent statement about Gersten resolutions for Milnor K-theory, see the corollary below. In (Elbaz-Vincent/Müller-Stach, 1998) we show furthermore that the maps φ_n are injective, if R is a one-dimensional, smooth, semilocal $k-$algebra with split maps to its residue fields.
Open Problem: Is φ_n always injective for R a smooth, semilocal k-algebra of geometric type over an infinite field?
The following should be compared with (Colliot-Thélène/Hoobler/Kahn, 1998):

Corollary 3.8 ("Gersten resolution for Milnor K-theory")
If R is a smooth, semilocal k-algebra of geometric type over an infinite field k, then the following sequence is (universally) exact:

$$K_n^M(R) \to \bigoplus_{x \in X^{(0)}} K_n^M(k(x)) \to \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} \mathbb{Z} \to 0$$

Furthermore, if X is a smooth quasiprojective variety over an infinite field k, then we have the following exact sequence of sheaves of Milnor K-groups:

$$\mathcal{K}_n^M \to \bigoplus_{x \in X^{(0)}} i_* K_n^M(k(x)) \to \bigoplus_{x \in X^{(1)}} i_* K_{n-1}^M(k(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} i_* \mathbb{Z} \to 0$$

The following results underline the good relationship between K-theory and higher Chow groups for arbitrary fields k:

Theorem 3.9 (Suslin, 1986), (Bloch/Lichtenbaum, 1995, chap.7)

$$CH^2(k, 3) \cong K_3^{ind}(k) := K_3(k)/K_3^M(k).$$

Theorem 3.10 (Bloch/Lichtenbaum, 1995)
If k is any field, there is the Bloch-Lichtenbaum spectral sequence,

$$E_2^{p,q} = CH^{-q}(k, -p-q) \Longrightarrow K_{-p-q}(k).$$

Meanwhile Friedlander, Levine and Suslin have announced a version of this spectral sequence for arbitrary quasiprojective varieties over a field (converging to G-theory). Finally we mention two more moving lemmas with important consequences:

Theorem 3.11 Moving Lemma IV, (Suslin, 1999)
Let X be affine over k. Then the inclusion $Z^i_{equi}(X,*) \subseteq Z^i(X,*)$ is a quasiisomorphism, if $i \leq d$.

Theorem 3.12 Moving Lemma V, (Suslin, 1999), (Friedlander/Voevodsky, 1999)
Localization also holds for equidimensional cycles: if Y is a closed subvariety of X with complement U, then the sequence

$$0 \to Z_0(Y) \to Z_0(X) \to Z_0(U)$$

is exact in the category of abelian groups (presheaves) and even right exact for the associated sheaves in the cdh-topology (moving by blowing up).

Theorem 3.13 Moving Lemma VI, (Friedlander/Voevodsky, 1999)
Let X be a scheme of finite type over k, assume that k admits resolution of singularities and let U be a smooth, quasiprojective scheme of pure dimension n. Then the embedding

$$Z_{equi}(X,r)(U \times -) \to Z_{equi}(X \times U, r+n)(-)$$

induces a quasiisomorphism of associated sheaves with transfers. If both X, U are smooth and projective, then the restriction on the characteristic is unnecessary.

Moving Lemma VI means that one may move cycles $W \subseteq X \times U \times S$, which are equidimensional of relative dimension r over S, such that they are equidimensional of relative dimension $r+n$ over $U \times S$.

Corollary 3.14 (Friedlander/Voevodsky, 1999)
Assume that k admits resolution of singularities and let X be a smooth, quasiprojective scheme over k. Then:

$$CH^p(X,i) \cong H^{2p-i}(X, \mathbb{Z}(p))$$

Furthermore, if X, Y are arbitrary schemes, U smooth of dimension d:

$$(Duality)\ A_{r,n}(X \times U, Y) \cong A_{r+d,n}(X, U \times Y).$$

4. A Vanishing theorem in codimension one

In this section let X be a smooth, quasiprojective $k-$variety. The vanishing part of the following result will be valid for certain singular varieties too (cf. (Nart, 1989), (Friedlander, 1994, thm.3.3.)).

Theorem 4.1 (Bloch, 1986), (Nart, 1989)
$CH^1(X,1) = \Gamma(X, \mathcal{O}_X^*)$ and $CH^1(X,n) = 0$ for $n \geq 2$.

Proof: We will work in cubical coordinates. As mentioned in section 1, we have to consider the complex $C^1(X,*)$, which is the quotient of all cubical cycles modulo degenerate ones, i.e. of those which arise from pullbacks from projection maps $\square^i \to \square^{i-1}$.
An integral, effective divisor W in $X \times \square^n$ is either horizontal (i.e. the projection onto X is dominant) or vertical (i.e. $W = W' \times \square^n$). We will denote the subgroups of those types of cycles by V_*, H_* respectively. V_* defines a subcomplex, since vertical cycles restrict to vertical cycles. All differentials in V_* are trivial, since the restrictions to $t_i = 0$ and $t_i = \infty$ are isomorphic and therefore cancel. In fact vertical cycles in V_n are degenerate for $n \geq 1$ and therefore by definition zero in $C^1(X,*)$. Hence there is an exact sequence of complexes

$$0 \to V_0 \to C^1(X,*) \to H_* \to 0$$

and $C^1(X,*)$ reduces to

$$\ldots \to H_2 \to H_1 \to V_0 \to 0.$$

For $n = 1$, the assertion is clear, since elements of H_1 correspond to non-zero rational functions on finite coverings of X and therefore via norms to rational functions on X. To vanish in V_0 means that they have no zeroes or poles, hence are invertible.
So assume $n \geq 2$. Let $k(X)$ be the function field of X. For every open subset $U \subseteq X$, the map $CH^1(X,n) \to CH^1(U,n)$ is injective for $n \geq 1$ and an isomorphism for $n \geq 2$ by the localization sequence and the trivial vanishing $CH^0(W,n)$ for all W and $n \geq 1$. Going to the limit, this easily implies the following

Lemma 4.2 *The restriction map $H_* \to C^1(k(X),*)$ is a quasiisomorphism in degrees $n \geq 2$.*

Therefore we may assume that $X = Spec(k)$ for a field k.
For $n \geq 2$, take any horizontal divisor W in $\square^n$ given by a rational function $F(X_1,...,X_n) \in k(X_1,....,X_n)$ and assume that $\partial(W) = 0 \in C^1(k,n-1)$.

Considering the normalized chain complexes instead, we may assume that for all $i = 1, ..., n$

$$F(X_1, ..., X_{i-1}, 0, X_{i+1}, ..., X_n) = F(X_1, ..., X_{i-1}, 1, X_{i+1}, ..., X_n) = 1, \quad (\sharp)$$

since this rational function has to be constant on each face of $\square^n$ and the union of all codimension one faces of $\square^n$ is connected for $n \geq 2$. Therefore we may assume that the common value is equal to 1. The following argument is a cubical translation of the argument in (Nart, 1989): $F(X_1, ..., X_n) - 1$ vanishes on the union $\partial\square^n$ of codimension one faces whose ideal is the principal ideal $(\prod X_i(1-X_i))$. In particular there exists a rational function $H(X_1, ..., X_n)$ such that

$$F(X_1, ..., X_n) = 1 + H(X_1, ..., X_n) \cdot \prod_i X_i(1 - X_i)$$

and the denominator of H being coprime to $\prod_i X_i(1 - X_i)$.
Now set

$$G(X_1, ..., X_{n+1}) := 1 + (1 - X_{n+1}) \cdot H(X_1, ..., X_n) \cdot \prod_i X_i(1 - X_i).$$

It is easy to compute that $\partial(div(G)) = (-1)^{n+1} div(F)$, since all boundaries vanish by $(\sharp)$, except $\partial^0_{n+1}(div(G)) = (-1)^{n+1} div(F)$. $\square$

Conjecture 4.3 (Soulé, 1985)
For $n \geq 2p \geq 2$, one expects the vanishing $CH^p(Spec(k), n)_{\mathbb{Q}} = 0$. This is an open problem for $p \geq 2$.

5. Zariski cohomology for Milnor and Quillen K-Sheaves

In this section let X be a smooth, quasiprojective $k-$variety. Let $\mathcal{CH}^p(r)$ be the sheafification of higher Chow groups in the Zariski topology. Recall that, by (Elbaz-Vincent/Müller-Stach, 1998) and (Gabber, 1992), we have an isomorphism of sheaves: $\mathcal{CH}^r(r) \cong \overline{\mathcal{K}}_r^M$, where $\overline{\mathcal{K}}_r^M$ is the image of the natural map

$$\mathcal{K}_r^M \longrightarrow \bigoplus_{x \in X^{(0)}} i_* K_r^M(k(x))$$

in the generic points. There is a flasque resolution of the sheaf $\mathcal{CH}^p(r)$:

$$0 \to \mathcal{CH}^p(r) \to \bigoplus_{x \in X^{(0)}} i_* CH^p(k(x), r) \to \bigoplus_{x \in X^{(1)}} i_* CH^{p-1}(k(x), r-1) \to$$

$$\to \bigoplus_{x \in X^{(p-1)}} i_* CH^1(k(x), r-p+1) \to \bigoplus_{x \in X^{(p)}} i_* CH^0(k(x), r-p) \to 0.$$

Clearly, by the theorem of Nesterenko/Suslin/Totaro, this resolution coincides for $p = r$ with the Gersten resolution for Milnor K-theory mentioned before, see (Elbaz-Vincent/Müller-Stach, 1998), and yields a resolution of the sheaf $\overline{\mathcal{K}}_r^M$.

Proposition 5.1 (Bloch, 1986),(Bloch, 1994)
There is a local-to-global spectral sequence

$$E_2^{m,n}(p) := H^m(X, \mathcal{CH}^p(2p-n)) \Longrightarrow CH^p(X, 2p-m-n).$$

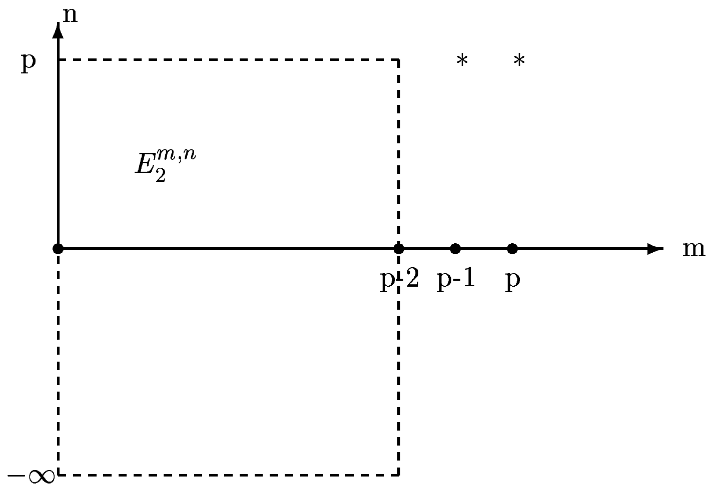
Proof: This is theorem 3.2. in (Bloch, 1986) together with the correction of the localization property in (Bloch, 1994). Another possibility (in char 0) is to use motivic cohomology and the local-to-global spectral sequence there. □

From the partial degeneration of this spectral sequence one can deduce the following well known result, see also (Rost, 1996), (Kato, 1986):

Corollary 5.2 *There are natural isomorphisms:*
(a) $CH^p(X) \cong H^p(X, \mathcal{CH}^p(p)) \cong H^p(X, \overline{\mathcal{K}}_p^M)$.
(b) $CH^p(X,1) \cong H^{p-1}(X, \mathcal{CH}^p(p)) \cong H^{p-1}(X, \overline{\mathcal{K}}_p^M)$.
(c) $CH^p(X,2) \cong H^{p-2}(X, \mathcal{CH}^p(p)) \cong H^{p-2}(X, \overline{\mathcal{K}}_p^M)$.
(d) We always have an exact sequence:

$$H^{p-2}(X, \mathcal{CH}^p(p+1)) \to CH^p(X,3) \to H^{p-3}(X, \mathcal{CH}^p(p)) \to 0.$$

Proof: I learnt the following argument from Uwe Jannsen ((Jannsen, 1987)). The non-zero entries in the E_2-table of the spectral sequence are diplayed inside the dashed box and on the two asterisks in the diagram below. The vanishing of the other entries follows from the fact that the Gersten resolution for $\mathcal{CH}^p(r)$ has length $\leq \min(p,r)$ and from the vanishing of higher Chow groups of fields for $p > r$ and $p = 0, r \geq 1$ as well as $p = 1, r \geq 2$. The corollary now follows from the partial degeneration of the spectral sequence near the upper right corner of the dashed box.



□

Now recall *Bloch's formula*: $CH^p(X) \cong H^p(X, \mathcal{K}_p)$. We will see, how far we are able generalize that. Recall that, for a field k, one has $K_n^M(k) \cong K_n(k)$ for $n \leq 2$. Comparing the Gersten resolutions for Milnor and Quillen K-theory, we get immediately :

Corollary 5.3 (Jannsen, 1987), (Landsburg, 1991), (Müller-Stach, 1997) *There is a natural isomorphism*

$$CH^p(X,1) \cong H^{p-1}(X, \mathcal{K}_p).$$

Remark: The following proof of part (b) is slightly more elementary than the proof given above, or the one in (Landsburg, 1991). Many people have told me that they have obtained similar proofs. Our proof grew out of discussions with M. Levine. Consider the diagram

$$\begin{array}{ccccc} C^p(X,2) & \to & C^p(X,1) & \to & C^p(X,0) \\ \downarrow N & & \downarrow N & & || \\ \oplus_{x \in X^{(p-2)}} K_2(k(x)) & \to & \oplus_{x \in X^{(p-1)}} k(x)^* & \to & Z^p(X) \end{array}$$

The middle vertical arrow N is the norm map, defined as follows: if W is a closed, integral subvariety of $X \times \square^1$, then take the closure of $W' = pr_X(W)$ together with the $Norm_{k(W)/k(W')}(f)$ where f is given by the projection $X \times \square^1 \to \square^1$. On the left, N is defined in a similar way using the norm in Milnor K-theory. This gives a well defined map $\varphi : CH^p(X,1) \to H^{p-1}(X, \mathcal{K}_p)$, since the tame symbol in the lower sequence coincides with the boundary map for higher Chow groups. We show that φ is bijective.
φ is **surjective**: N has a partial inverse given by the graph map; given

(W, f), $f \in k(W)^*$, the graph of f defines a closed, integral subvariety of $W \times \mathbb{P}^1$. We denote by $\Gamma(W, f)$ its restriction to $W \times \square^1$. $\Gamma(W, f)$ clearly meets all faces properly, since the restriction to the faces $t = 0, \infty$ is given by the divisor of f, which is empty or of codimension one in W and therefore of codimension p in X. Obviously always $N(\Gamma(W, f)) = (W, f)$ since the graph of a map is an isomorphism onto its image.
φ is **injective**: here it is important to notice that N has only an inverse defined for the middle vertical arrow. The left vertical arrow has no well-defined inverse by taking graphs since the graph of a symbol in $K_2(k(x))$ is not necessarily in good position. So we use the following trick: assume we are given a cycle $W = \sum W_i \in C^p(X, 1)$ with $\partial(W) = 0$ and $N(W) = 0$ in $H^{p-1}(X, \mathcal{K}_p)$, i.e. $N(W) = Tame(\sum\{f_k, g_k\})$ with$f_k, g_k \in k(V_k)^*$. Using the well known equation $\{a, a\} = \{-1, a\}$ in K_2 of any field together with bilinearity, cf. (Bass/Tate, 1973), we may assume that each pair of rational functions f_k, g_k has different divisors. Also we may assume that all V_k are normal after passing to the normalization. In this situation we take the sum of the graph maps defined by (f_k, g_k) on V_k (resp. on the normalization) and take their image in $C^p(X, 2)$, denoted by $V = \sum \Gamma(V_k, f_k, g_k)$. Now $W - \partial(V)$ has the property that $N(W - \partial(V))$ is zero as an element of $\oplus_{x \in X^{(p-1)}} k(x)^*$ already, i.e. has trivial norm.
So far we have proved, that the inverse of φ is in fact well defined and it remains to show that an element in $C^p(X, 1)$ with trivial norm is equivalent modulo $\partial C^p(X, 2)$ to the trivial element. In order to prove this, it is enough to show that every closed, integral element $W \in C^p(X, 1)$ is equivalent to $\Gamma(N(W))$.
Case 1: W is not generically finite over X. Then W is necessarily degenerate and we are done.
Case 2: W is generically finite over X with closure $V = \overline{pr_X(W)} \in X^{(p-1)}$. Let $f \in k(W)^*$ be the projection to $\square^1$. f is integral over $k(V)$, therefore defines a polynomial $F(T) = T^d + \ldots + (-1)^d c_0 \in k(V)[T]$ such that $c_0 = N(f)$. Consider the closure in $C^p(X, 2)$ of the subvariety C with the equation

$$P(T, Y) = F(T) - (T - 1)^{d-1}(T - c_0)Y = 0$$

as a subvariety of $Spec(k(V)) \times \square^2$. C meets all faces properly (check) and (modulo degenerate cycles) the boundaries of C are: $C \cap \{Y = 0\} = W$, $C \cap \{Y = \infty\} = \Gamma(N(W))$ and $C \cap \{T = 0\} = C \cap \{T = \infty\} = \emptyset$. This proves that $\partial(C) = W - \Gamma(N(W))$ and finishes the proof of injectivity. □

Note that in the previous proof it was essential that a subvariety in $X \times \square^1$ is either degenerate (case 1) or generically finite over X (case 2). This fails in other groups, but essentially the same method shows the surjectivity part of the following result for X smooth and quasiprojective over k (see also (Rost, 1996), (Jannsen, 1987) and (Müller-Stach, 1997)):

Theorem 5.4 *There is a natural isomorphism* $CH^p(X,2) \to H^{p-2}(X,\mathcal{K}_p)$ *for all* $p \geq 0$.

Proof: Almost elementary proof for $p = 2$: the fact that the norm map $CH^p(X,2) \to H^{p-2}(X,\mathcal{K}_p)$ is well-defined and surjective follows as in the previous proof, see also (Landsburg, 1991). For $p = 2$, we want to show that it is an isomorphism. Let $U \subseteq X$ be any open subset with complement W, a closed subset of codimension one. Part of the localization sequence is

$$\ldots \to CH^1(W,2) \to CH^2(X,2) \to CH^2(U,2) \to \ldots$$

Now $CH^1(W,n) = 0$ for $n \geq 2$, which follows from the previous vanishing theorem and localization (take out a divisor containing the singular set in W) or by using (Friedlander, 1994, thm. 3.3.). Therefore $CH^2(X,2) \to CH^2(U,2)$ is injective. Taking the limit over all open subsets U, we get an injection $CH^2(X,2) \to CH^2(k(X),2)$. It is known that $H^0(X,\mathcal{K}_2) \to K_2(k(X))$ is injective, by the exactness of the Gersten resolution. Combining both statements, we obtain that our map $H^0(X,\mathcal{K}_2) \to CH^2(X,2)$ is injective, since by Nesterenko/Suslin/Totaro we have that $CH^2(k(X),2) \cong K_2(k(X))$ and therefore the composed map $H^0(X,\mathcal{K}_2) \to CH^2(X,2) \to K_2(k(X))$ is injective.
For general p, we know that $CH^p(X,2) \cong H^{p-2}(X,\overline{\mathcal{K}}_p^M)$. Comparing the Gersten resolutions for both $\overline{\mathcal{K}}_p^M$ and $\mathcal{K}_p$, the assertion follows for $p \leq 2$ immediately. However Jannsen remarks in (Jannsen, 1987) that this still holds for $p = 3$, since the image of

$$\bigoplus_{x \in X^{(0)}} K_3(k(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} K_2(k(x))$$

is the same as the image of

$$\bigoplus_{x \in X^{(0)}} K_3^M(k(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} K_2(k(x))$$

by a result of Merkurjev and Suslin, see (Merkurjev/Suslin, 1991, prop. 11.11.). Finally, in (Rost, 1996) it is shown that $CH^p(X,2) \cong H^{p-2}(X,\mathcal{K}_p)$ for all $p \geq 3$, by deriving it in a similar way again from (Merkurjev/Suslin, 1991), prop. 11.11. □

Remark: Soulé had shown earlier that $CH^p(X,2)$ is always rationally (i.e. after tensoring with $\mathbb{Q}$) isomorphic to $H^{p-2}(X,\mathcal{K}_p)$ for all $p \geq 0$, see (Soulé, 1985).

6. Cycles Complexes over Fields

In this section we describe universal relations in $CH^m(k,2m-1)_{\mathbb{Q}}$ for arbitrary fields k. The search for such relations is inspired by the program of (Zagier, 1991) which studies the relation between higher Bloch groups $B_m(k)$ and the algebraic K-theory of k. For $m = 2$ this project has its roots in the close relation between $B_2(k)$ (defined by Bloch), $K_3(k)^{\mathrm{ind}}$ and $CH^2(k,3)$ (the last two are integrally isomorphic by (Bloch/Lichtenbaum, 1995) and the first two are rationally isomorphic by (Suslin, 1986)). The groups $B_m(k)$ are defined from a free group via relations connected with higher polylogarithms. Consequently, in order to have a well defined homomorphism

$$\overline{\rho}_m : B_m(k) \longrightarrow CH^m(k,2m-1)_{\mathbb{Q}},$$

it is necessary to verify those relations inside K-theory or motivic cohomology. Especially for $m = 3$, there has been substantial progress in (Goncharov, 1995), using Grassmannian homology, a theory closely related to the simplicial version of higher Chow groups. For example the five term relation in $CH^2(k,3)$ (stated below) can be proved in simplicial coordinates in a very simple way by taking the boundary of a linear 2-plane in $\mathbb{A}^4$, cf. (Goncharov, 1995). For $m \geq 3$ not all possible relations are known, but I think that a translation into the language of motivic cohomology will lead to further progress.

We work in cubical coordinates. In order to make certain manipulations, it is convenient to consider alternating cubical higher Chow groups, cf. (Levine, 1994). Let $G = G_n$ be the semidirect product of the symmetric group S_n with $(\mathbb{Z}/2\mathbb{Z})^n$, where S_n acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the factors. G acts on $Z^p(k,n) \otimes \mathbb{Q}$ via permutation and inversion of coordinates. Define $\mathrm{sgn} : S_n \to \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{sgn}_j : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ to be the non-trivial characters and let

$$\chi = \mathrm{sgn} \cdot \prod_{j=1}^{n} \mathrm{sgn}_j \, .$$

Then there is a natural choice of a projection of vector spaces

$$Alt_n : Z^p(k,n) \otimes \mathbb{Q} \to Z^p(k,n) \otimes \mathbb{Q}, \qquad Z \mapsto \frac{1}{|G|} \sum_{g \in G} \chi(g)\, g(Z).$$

Define $C^p(k,n) \subseteq Z^p(k,n) \otimes \mathbb{Q}$ to be the image of Alt_n together with the differential induced by $Alt_n \circ \partial$ (denoted again by ∂). The resulting homological complex

$$C^m(k,\cdot): \qquad \ldots \to C^m(k,2m) \to C^m(k,2m-1) \to \ldots \to C^m(k,m) \to 0$$

still computes $CH^m(k,n)_{\mathbb{Q}}$:

Theorem 6.1 (Levine, 1994)
The inclusion $C^m(k,\cdot) \hookrightarrow Z^m(k,\cdot)_{\mathbb{Q}}$ is a quasiisomorphism.

For $m=2$, the complex $C^2(k,\cdot)$ has the acyclic subcomplex

$$\ldots \to C^1(k,1) \wedge C^1(k,3) \to C^1(k,1) \wedge C^1(k,2) \to C^1(k,1) \wedge \partial C^1(k,2) \to 0\,,$$

i.e. the truncation of the subcomplex consisting of subvarieties where one coordinate entry is constant. The proof of acyclicity is the same as in (Nart, 1989). The quotient complex will be denoted by $A^2(k,\cdot)$. It is quasiisomorphic to $C^2(k,\cdot)$ and has certain advantages; for example cycles in $C^2(k,3)$ with one coordinate entry being constant have zero image in $A^2(k,3)$. We call such cycles negligible.

Definition 6.2 *Let C_a be the cycle in $A^2(k,3)$ which is given by Alt_3 of the parametrized curve $x \mapsto [x, 1-x, 1-a/x]$, cf.* (Totaro, 1992).

Theorem 6.3 (Gangl/Müller-Stach, 1999)
(a) If k contains a primitive n-th root of unity, then every $a \in k^*$ gives rise to a distribution relation:

$$C_{a^n} - n \sum_{\zeta^n=1} C_{\zeta a} = 0 \in CH^2(k,3)_{\mathbb{Q}}.$$

(b) For $a \neq b, 1-b$, and $a, b \neq 0, 1$, one obtains the five term relation

$$\mathcal{V}_{a,b} := C_{\frac{a(1-b)}{b(1-a)}} - C_{\frac{1-b}{1-a}} + C_{1-b} - C_{\frac{a}{b}} + C_a = 0 \in CH^2(k,3)_{\mathbb{Q}}.$$

(c) The inversion relation holds for $a \in k^*$:

$$C_a + C_{\frac{1}{a}} = 0 \in CH^2(k,3)_{\mathbb{Q}}.$$

(d) $C_1 = C_{-1} = 0 \in CH^2(k,3)_{\mathbb{Q}}$.
(e) Let $f: \mathbb{P}^1 \to \mathbb{P}^1$ be any rational function of degree n. Then the following relation holds:

$$nC_{cr(f(x),a,b,c)} - \sum_{\gamma \in f^{-1}(c)} \sum_{\beta \in f^{-1}(b)} \sum_{\alpha \in f^{-1}(a)} C_{cr(x,\alpha,\beta,\gamma)} = 0 \in CH^2(k,3)_{\mathbb{Q}},$$

assuming that $x, a, b, c \in k$ and all α, β, γ are mutually distinct and lie in k. Here $cr(a,b,c,d)$ denotes the cross ratio $\frac{(a-c)(b-d)}{(a-d)(b-c)}$.

Remark: As we showed in (Gangl/Müller-Stach, 1999), this gives rise to a well defined homomorphism

$$\overline{\rho}_2 : B_2(k) \longrightarrow CH^2(k,3)_{\mathbb{Q}}, \quad a \mapsto C_a,$$

which can be used to give an alternative presentation of K-groups (i.e. higher Chow groups) via generators and relations. This should be seen in relation with the work of (Goncharov, 1995), (Suslin, 1986) and (Zagier, 1991). See (Gangl/Müller-Stach, 1999) for more details.

Now let $m \geq 3$. The cycles of Totaro were generalized by Bloch and Kriz, cf. (Bloch, 1998), and we again denote them by $C_a \in C^m(k, 2m-1)$:

$$(x_1, ..., x_{m-1}) \mapsto Alt_{2m-1}\Big[x_1, ..., x_{m-1}, 1-x_1, 1-\frac{x_2}{x_1}, ..., 1-\frac{x_{m-1}}{x_{m-2}}, 1-\frac{a}{x_{m-1}}\Big].$$

As in the case $m=2$, we modify the complex $C^3(k,\cdot)$ in such a way that we can manipulate parametrized cycles modulo negligible ones. To simplify our computations, we mod out by a subcomplex $S^3(k,\cdot)$.

Definition 6.4 *For $n \geq 5$, let $S^3(k,n) :=$*

$$= C^1(k,1) \wedge C^2(k,n-1) + C^2(k,3) \wedge C^1(k,n-3) + C^2(k,2) \wedge C^1(k,n-2).$$

$$S^3(k,4) := \partial C^1(k,2) \wedge C^2(k,3) + C^1(k,1) \wedge \partial C^2(k,4) + C^2(k,2) \wedge C^1(k,2)$$

$$S^3(k,3) := C^2(k,2) \wedge \partial C^1(k,2)$$

The quotient complex $C^3(k,\cdot)/S^3(k,\cdot)$ will be denoted by $A^3(k,\cdot)$.

In (Gangl/Müller-Stach, 1999), we show that if $CH^2(k,n)_{\mathbb{Q}} = 0$ for $n \geq 4$ (Beilinson-Soulé conjecture for $m=2$), then $S^3(k,\cdot)$ is acyclic. This vanishing holds in particular for number fields by Borel's theorem, cf. (Borel, 1974) together with the main result of (Levine, 1998).

Theorem 6.5 (Gangl/Müller-Stach, 1999)
Assume that $CH^2(k,n)_{\mathbb{Q}} = 0$ for $n \geq 4$.
(a) The Kummer-Spence relation for the trilogarithm holds in the form

$$\mathcal{S}(a,b) = 0 \in CH^3(k,5)_{\mathbb{Q}},$$

whenever $a \neq b, 1-b$, $a,b \neq 0,1$ and $\mathcal{S}(a,b) := C_{\frac{b(1-b)}{a(1-a)}} + C_{\frac{a(1-b)}{b(1-a)}} + C_{\frac{(1-a)(1-b)}{ab}} - 2(C_{\frac{b}{a}} - C_{\frac{1-b}{a}} - C_{\frac{b}{1-a}} - C_{\frac{1-b}{1-a}} + C_{\frac{1}{1-a}} + C_{\frac{1}{a}} - C_{1-\frac{1}{b}})$.
(b) The inversion relation and the "3-term relation" hold:

$$C_a - C_{\frac{1}{a}} = \mathcal{T}_a - \mathcal{T}_b = 0 \in CH^3(k,5)_{\mathbb{Q}} \quad \text{where} \quad \mathcal{T}_a = C_a + C_{\frac{1}{1-a}} + C_{1-\frac{1}{a}}.$$

For every $m \geq 3$, we get distribution relations:

Theorem 6.6 (Gangl/Müller-Stach, 1999)
If k contains a primitive n-th root of unity, then every $a \in k$ gives rise to a distribution relation for each $m \geq 2$:

$$C_{a^n} - n^{m-1} \sum_{\zeta^n=1} C_{\zeta a} = 0 \in CH^m(k, 2m-1)_{\mathbb{Q}}.$$

Remarks: (1) Again, one can ask whether one obtains a well defined homomorphism $\overline{\rho}_3 : B_3(k) \to CH^3(k,5)_{\mathbb{Q}}$, but we are not yet able to prove the so-called Goncharov 22-term relation in $CH^3(k,5)_{\mathbb{Q}}$ which holds in $B_3(k)$.
(2) One can study higher Chow groups $CH^p(X,q)$ of "Milnor type" (i.e. the case $p=q$) also for higher dimensional projective varieties. For example let X be a smooth, projective surface and $Z = \cup Z_i$ a strict normal crossing divisor on X. Denote by $W = \coprod_{i<j} Z_i \cap Z_j$ the singularity set of Z. Then, for $q \geq 1$, there is an exact sequence

$$\bigoplus_i CH^q(Z_i,q) \to CH^q(Z,q) \to \mathrm{Ker}[\bigoplus_W K^M_{q-1}(\mathbb{C}) \stackrel{\varepsilon}{\to} \oplus CH^q(Z_i,q-1)] \to 0,$$

deduced from the Mayer-Vietoris sequence for Z. In (Müller-Stach, 1997) and (Müller-Stach/Saito, 1998) this computation is used to construct interesting non-decomposable cycles in $CH^{q+1}(X,q)$ on certain smooth projective surfaces over $\mathbb{C}$.

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