REMARKS ON THE PRONILPOTENT COMPLETION OF THE FUNDAMENTAL GROUP

HÉLÈNE ESNAULT AND OLIVIER WITTENBERG

À Pierre Deligne, avec reconnaissance et admiration.

ABSTRACT. Given a smooth and separated variety X over a field k, we associate a "cycle class" in étale cohomology to any continuous section of the natural map from the arithmetic fundamental group of X to the absolute Galois group of k. An étale adaptation of Beilinson's geometrization of the pronilpotent completion of the topological fundamental group allows us to lift this class in suitable cohomology groups.

1. INTRODUCTION

Let X be a geometrically connected variety over a field k of characteristic 0. Denote by $\bar{k}$ an algebraic closure of k. Let $\bar{X} = X \otimes_k \bar{k}$ and $G_k = \mathrm{Gal}(\bar{k}/k)$.

Grothendieck considered in [5] the category $\mathsf{Et}(X)$ of étale covers of X (*i.e.*, of finite étale morphisms of schemes $\pi\colon Y \to X$) and used it to define the fundamental group $\pi_1(X, x)$ of X based at a geometric point x of X as the automorphism group of the fiber functor $\mathsf{Et}(X) \to \mathsf{Sets}$, $\pi \mapsto \pi^{-1}(x)$. The structure morphism $\varepsilon\colon X \to \mathrm{Spec}(k)$ induces a map $\varepsilon_\star\colon \pi_1(X, x) \to G_k$, since $\pi_1(\mathrm{Spec}(k), x)$ is canonically isomorphic to G_k. Grothendieck proved that $\varepsilon_\star$ is onto and that its kernel identifies with $\pi_1(\bar{X}, x)$. Moreover, for any other geometric point x' of X, there exists an isomorphism $\pi_1(X, x) \simeq \pi_1(X, x')$ which is compatible with the projections to G_k. Any rational point $a \in X(k)$ therefore induces a section $a_\star\colon G_k \to \pi_1(X, x)$ of $\varepsilon_\star$, well-defined up to conjugacy by $\pi_1(\bar{X}, x)$ (indeed, a induces a canonical section of $\pi_1(X, a) \to G_k$).

Let $G_{k(X)}$ denote the absolute Galois group of $k(X)$. The inclusion of the generic point of X induces a map $G_{k(X)} \to \pi_1(X, x)$ which is compatible with the projections to G_k and which is well-defined up to conjugacy by $\pi_1(\bar{X}, x)$. In [2, Section 15], Deligne proves that for $a \in X(k)$, the section $a_\star$ admits liftings

---

Date: July 16, 2008.

Partially supported by the DFG Leibniz Preis and the SFB/TR 45.

to $G_{k(X)}$, as pictured below:

$$\begin{tikzcd}
G_{k(X)} \arrow[d] & \\
\pi_1(X,x) \arrow[r, shift left, "\varepsilon_*"] & G_k \arrow[l, shift left, "a_\star"] \arrow[ul, dotted]
\end{tikzcd}$$

In terms of Galois groups, this can be seen as follows. Assume for simplicity that X is a curve (Deligne writes “Par lassitude, nous ne traiterons que du cas où X est de dimension 1”). The choice of a local parameter t of X at a determines an isomorphism $k(X)_a \simeq k((t))$, where $k(X)_a$ denotes the completion of $k(X)$ at a; hence an embedding $k(X) \subset K$, where $K = \bigcup_{n\geq 1} k((t^{1/n}))$. The resulting map between absolute Galois groups $G_K \to G_{k(X)}$ then provides a lifting of $a_\star$. Indeed, the map $G_K \to G_k$ induced by the inclusion $k \subset K$ is an isomorphism.

The construction just described is easily seen to depend on the local parameter t only to order 1, *i.e.*, it only depends on the choice of a tangent vector τ to X at a. Thus Deligne's *tangential base points* (a, τ) induce splittings of $G_{k(X)} \to G_k$. Equivalently, for any dense open $U \subseteq X$, one can say that the datum of a nonzero tangent vector τ on X at a enables one to define the fundamental group $\pi_1(U, (a, \tau))$ of U based at (a, τ) as the automorphism group of the fiber functor $\mathsf{Et}(U) \to \mathsf{Sets}$, $\pi \mapsto \pi^{-1}(\bar\eta)$ where $\bar\eta = \operatorname{Spec}\left(\bigcup_{n\geq 1} \bar k((t^{1/n}))\right)$.

Deligne's tangential base points (a, τ) produce sections $G_k \to G_{k(X)}$ which factor through $G_{k(X)_a}$. For a given $a \in X(k)$, the set of all sections (up to conjugacy) which satisfy this property is naturally a torsor under the group $H^1(k, \hat{\mathbb{Z}}(1)) := \varprojlim_{n\geq 1} H^1(k, \boldsymbol{\mu}_n)$. Those sections which in addition come from a tangential base point form a subtorsor under the image of $H^1(k, \mathbb{Z}(1)) := k^\times$ in $H^1(k, \hat{\mathbb{Z}}(1))$; one could say that they are motivic, as opposed to profinite.

On the other hand, if k is a number field and X is the complement of finitely many rational points in $\mathbb{P}^1_k$, Deligne constructed the motivic fundamental group $\pi_1^{\mathrm{real}}(X, a)$ based at a rational point $a \in X(k)$. It is a pro-group scheme object in a $\mathbb{Q}$-linear rigid tensor category $\mathcal{C}$. In his first construction [2], $\mathcal{C}$ was the category of realizations, the objects of which are sequences of vector spaces $((M_B)_\sigma, M_{DR}, (M_\ell)_\ell, (M_p)_p)$ over $\mathbb{Q}, k, \mathbb{Q}_\ell, \mathbb{Q}_p$ (where p ranges over almost all prime numbers and σ ranges over the embeddings of k into $\mathbb{C}$), together with comparison isomorphisms, weight filtrations, and so on, so as to resemble the various cohomologies of an algebraic variety over a number field and their interrelations. Given an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, the underlying mixed Hodge structure $((M_B)_\sigma, M_{DR})$ is Morgan's mixed Hodge structure on the pronilpotent completion $\varprojlim_{n\geq 1} \mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$, where π_1^{top} denotes the topological fundamental group and I denotes the augmentation ideal of the group algebra.

Deligne was also able to replace a with a tangential base point (a, τ) and define a motivic fundamental group $\pi_1^{\text{real}}(X, (a, \tau))$.

Later on in [3] he gave an abstract definition. Over $X(\mathbb{C})$, Beilinson [3, Section 3] constructed a cosimplicial scheme $\mathcal{P}_a(X)$ with the property that the Hopf algebra $\varinjlim_{n \geq 1} \text{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi^{\text{top}}(X(\mathbb{C}), a)]/I^{n+1}, \mathbb{Q})$ arises from the cohomology of $\mathcal{P}_a(X)$. One may consider $\mathcal{P}_a(X)$ as an object in Voevodsky's triangulated category $DM_{gm}(k)$ of geometric motives. The assumption on X forces $\mathcal{P}_a(X)$ to lie in the triangulated subcategory spanned by Tate objects, while the assumption on k forces this triangulated subcategory to admit a core $\mathsf{MT}(k)$ of mixed Tate objects (see [7]). The latter is a $\mathbb{Q}$-linear rigid tensor category which is canonically neutralized by the graded gr^W of the weight filtration. It then becomes possible to view the cohomology of $\mathcal{P}_a(X)$ as a pro-group scheme object $\pi_1^{\text{mot}}(X, a)$ in $\mathsf{MT}(X)$; its realization is $\pi_1^{\text{real}}(X, a)$. However, there is no construction of a cosimplicial scheme $\mathcal{P}_{(a,\tau)}(X)$ in $DM_{gm}(k)$ which would allow one to define a pro-group scheme object $\pi_1^{\text{mot}}(X, (a, \tau))$ in $\mathsf{MT}(X)$, realized by $\pi_1^{\text{real}}(X, (a, \tau))$. In [3, Section 4], the existence of $\pi_1^{\text{mot}}(X, (a, \tau))$ is proven only indirectly.

One can describe $\pi_1^{\text{mot}}(X, a)$ in the spirit of Grothendieck's construction. The assumptions on X and k allow one to construct the $\mathbb{Q}$-linear abelian rigid category $\mathsf{MT}(X)$ of mixed Tate motives over X, together with a neutral fiber functor gr^W and a pullback functor $\varepsilon^\star : \mathsf{MT}(k) \to \mathsf{MT}(X)$. The corresponding homomorphism of $\mathbb{Q}$-Tannaka group schemes $\varepsilon_\star : G(\mathsf{MT}(X), \text{gr}^W) \to G(\mathsf{MT}(k), \text{gr}^W)$ is surjective. A section s of $\varepsilon_\star$ defines by conjugacy a $\mathbb{Q}$-linear action on the functions $\mathbb{Q}[K]$ on $K = \text{Ker}(\varepsilon_\star)$, hence it defines, by Tannaka duality, a group scheme object K_s in $\mathsf{MT}(k)$. It is shown in [4, Theorem 1] that $\pi_1^{\text{mot}}(X, a)$ is isomorphic to K_s in $\mathsf{MT}(k)$ if the section s is associated to a rational point $a \in X(k)$. On the other hand, Levine's work [8, §12.5] associates a section s of $\varepsilon_\star$ to any tangential base point (a, τ). Due to the lack of a direct construction of $\pi_1^{\text{mot}}(X, (a, \tau))$, there is no direct identification of K_s with $\pi_1^{\text{mot}}(X, (a, \tau))$ if s comes from a tangential base point (a, τ).

In this note we adapt part of Beilinson's description of $\mathbb{Q}[\pi_1^{\text{top}}(X(\mathbb{C}), a)]/I^{n+1}$ to the étale fundamental group. The purpose is to replace the rational point a by an abstract section $s : G_k \to \pi_1(X, x)$ of Grothendieck's arithmetic fundamental group. If X has dimension 1, this section s does not have to come from one of Deligne's tangential base points. The only assumption we need to make is that X is a $K(\pi, 1)$ (see Definition 2.1), for example X could be a smooth proper curve of genus ≥ 1. We closely follow the topological description in [3, §3.3, §3.4]. There, the authors express in cohomological terms not the $\mathbb{Q}$-vector space $\mathbb{Q}[\pi_1^{\text{top}}(X(\mathbb{C}), a)]/I^{n+1}$ but its dual. The latter turns out to coincide with the hypercohomology of the complex ${}_a\mathcal{K}_a$ described in *loc. cit.* It seems difficult to define an analogous complex if one replaces a by an abstract section s. However, under the $K(\pi, 1)$ assumption, one can replace a by the k-form of the universal cover defined by s to obtain a complex, albeit on a larger space, which gives the correct hypercohomology group (see Proposition 3.2).

If one dualizes, that is if one really comes back to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$ (see §3.1), one finds a complex which is more difficult to write down since it is defined only as an object of the derived category (see Definition 3.4). This is perhaps the reason why nowhere in [3, Section 3] it is explicitly described. However, the cohomology of this complex in other degrees carries some extra information which we now consider. Under the assumption that X is a $K(\pi, 1)$, to any continuous section $s\colon G_k \to \pi_1(X, x)$ we associate a proclass in étale cohomology with compact supports $\widetilde{\alpha}(\Lambda) \in \varprojlim H^{2d}_c(Y, \Lambda(d))$ (Theorem 2.6), where $d = \dim(X)$ and Λ is any torsion ring, and where the inverse limit ranges over all étale covers $Y \to X$ through which the k-form of the universal cover defined by s factors (the transition morphisms being the trace maps in étale cohomology). The projection of $\widetilde{\alpha}(\Lambda)$ to $H^{2d}_c(X, \Lambda(d))$ is simply the étale cycle class of $a \in X(k)$ if s comes from a, so we may call it the étale cycle class of s. (Such a class had been considered by Mochizuki [11, Introduction, Structure of the Proof, (1)] in the case where X is proper and has dimension 1.) The complexes constructed in Definition 3.4 then have the property that their cohomology carries a lifting of the étale cycle class of s (see Proposition 3.8).

An immediate consequence of the construction of the étale cycle class of a section $s : G_k \to G_{k(X)}$, when X is a smooth proper curve and k is a p-adic field, is the existence of a pro-0-cycle of degree 1 up to linear equivalence on all Y running through the system of étale covers of X through which the k-form of the universal cover defined by s factors (see Proposition 2.8). This is a weak version of Koenigsmann's theorem according to which in this situation, the section s indeed comes from a rational point of X. However Koenigsmann's proof relies on results from model theory. One might hope that the liftings of the étale cycle class of s defined in §3.4 give a geometric view on Koenigsmann's point.

Acknowledgments: We thank Alexander Beilinson for his interest. We thank Phùng Hô Hai and Marc Levine for discussions based on earlier work related to the topic of this note.

2. Cycle classes of sections

Let k be a field and $\bar{k}$ be a separable closure of k. We denote by X a geometrically connected separated variety over k endowed with a $\bar{k}$-point $x \in X(\bar{k})$, and we let $\bar{X} = X \otimes_k \bar{k}$.

Grothendieck [5] defines a *universal cover of* X *at* x to be a directed inverse system of connected pointed étale covers of (X, x) such that the inverse limit $\widetilde{\pi}_x : \widetilde{X}_x \to X$ factors through every étale cover of X. (A universal cover of X at x exists and is unique up to a unique isomorphism.) The fundamental group

$\pi_1(X,x)$ can be identified with the automorphism group of $\widetilde{\pi}_x : \widetilde{X}_x \to X$. Closed subgroups of $\pi_1(X,x)$ then correspond to "sub-pro-étale covers" of $\widetilde{X}_x \to X$, that is, to factorisations $\widetilde{X}_x \to X' \to X$ of $\widetilde{\pi}_x$ where X' is a connected pro-étale cover of X.

The morphism $\bar{X} \to X$ is a connected pro-étale cover. Moreover $\bar{X}$ is canonically endowed with a $\bar{k}$-point above x. Hence $\widetilde{\pi}_x : \widetilde{X}_x \to X$ factors canonically through $\bar{X}$, so that $\widetilde{X}_x$ can be viewed as the universal cover of $\bar{X}$ at x. In general, there need not exist a pro-cover of X which, after extension of scalars from k to $\bar{k}$, becomes isomorphic to the universal cover of $\bar{X}$ at x. Such a pro-cover of X exists if and only if the natural map $\pi_1(X,x) \to G_k$ admits a (continuous, as will always be implied) section. More precisely, let $s\colon G_k \to \pi_1(X,x)$ be a section. Let us denote by $\pi_s\colon X_s \to X$ the sub-pro-étale cover of $\widetilde{\pi}_x\colon \widetilde{X}_x \to X$ which corresponds to the subgroup $s(G_k) \subset \pi_1(X,x)$. The scheme $X_s \otimes_k \bar{k}$ is then canonically isomorphic to $\widetilde{X}_x$ over $\bar{X}$. In particular X_s is a directed inverse limit of étale covers of X each of which is geometrically connected over k. If s is the section associated (up to conjugacy by $\pi_1(\bar{X},x)$) to a rational point $a \in X(k)$, then $X_s(k) \neq \varnothing$; more precisely, the set $X_s(k)$ then contains a canonical lifting $\widetilde{a} \in X_s(k)$ of a.

Let Λ denote the ring $\mathbb{Z}/N\mathbb{Z}$ for some integer $N \geq 1$ which is invertible in k. For $m \in \mathbb{Z}$, we denote by $\Lambda(m)$ the m-th Tate twist of Λ (so $\Lambda(1) = \boldsymbol{\mu}_N$).

Definition 2.1. We say that the variety X *is a* $K(\pi,1)$ if $\varinjlim H^i(Y,\Lambda) = 0$ for all $i \geq 1$ and all $N \geq 1$ such that N is invertible in k. The direct limit ranges over all étale covers $Y \to X$ such that $\widetilde{X}_x \to X$ factors through $Y \to X$ (for some fixed $x \in X(\bar{k})$). This property is purely geometric: it only depends on the scheme $\bar{X}$.

Examples 2.2. (i) Smooth curves of nonpositive Euler–Poincaré characteristic (*i.e.*, smooth curves of genus g with n punctures such that $2g-2+n \geq 0$, or equivalently smooth curves such that $\deg \omega_{\hat{X}}(\log\infty) \geq 0$ where ∞ denotes the complement of X in a smooth compactification $\hat{X}$ of X), are $K(\pi,1)$'s.

(ii) If X is a $K(\pi,1)$ variety, any étale cover of X is also a $K(\pi,1)$.

(iii) According to Artin [1], on any smooth variety, the $K(\pi,1)$ open subsets form a basis of the Zariski topology.

Under the assumption that X is a $K(\pi,1)$, we shall now associate, to any section $s\colon G_k \to \pi_1(X,x)$, a "cycle class" in the étale cohomology group with compact supports $H^{2d}_c(X,\Lambda(d))$, where $d = \dim(X)$.

Definition 2.3. (i) If $\hat{V}$ is a variety over k and $V \subseteq \hat{V}$ is a dense open subset with complement $Z \subseteq \hat{V}$, we shall denote the étale cohomology group $H^i(\hat{V}, j_!\Lambda(m))$, where j stands for the inclusion $j\colon V \hookrightarrow \hat{V}$, by $H^i(\hat{V}, Z, \Lambda(m))$. When $\hat{V}$ is a proper variety, this group depends only on V and is usually denoted $H^i_c(V, \Lambda(m))$.

(ii) If $s\colon G_k \to \pi_1(X, x)$ is a section, we shall say that an étale cover $\pi\colon Y \to X$ *appears in* $\pi_s\colon X_s \to X$ if the morphism π_s factors through π. Given an étale cover $Y \to X$ appearing in π_s, we shall say that an étale cover $Z \to Y$ *appears in* π_s if the composition $Z \to X$ appears in π_s. A property depending on an étale cover $Y \to X$ will be said to *hold if* $Y \to X$ *appears high enough in* π_s if there exists an étale cover $Y_0 \to X$ appearing in π_s such that all étale covers $Y \to X$ which appear in π_s and which factor through Y_0 satisfy the given property. Finally, given an étale cover $Y \to X$ appearing in π_s, we shall say that a property depending on an étale cover $Z \to Y$ *holds if* $Z \to Y$ *appears high enough in* π_s if there exists an étale cover $Z_0 \to Y$ appearing in π_s such that all étale covers $Z \to Y$ which appear in π_s and which factor through Z_0 satisfy the given property.

Proposition 2.4. *Let X be a geometrically connected separated variety over k. Let $X \subseteq \hat{X}$ be a compactification, with complement $D = \hat{X} \setminus X$. Let $x \in X(\bar{k})$. Let $s\colon G_k \to \pi_1(X, x)$ be a section of the natural map $\pi_1(X, x) \to G_k$, Assume X is a $K(\pi, 1)$. Then for any étale cover $\pi\colon Y \to X$ appearing high enough in π_s, the following conditions hold: for any $m \geq 0$ and any $n \in \mathbb{Z}$, the pullback map*

$$H^m(\hat{X}, D, \Lambda(n)) \longrightarrow H^m(\hat{X} \times Y, D \times Y, \Lambda(n))$$

induced by the first projection $\hat{X} \times Y \to \hat{X}$ is injective and its image is equal to the image of the pullback map

$$(1 \times \pi)^\star\colon H^m(\hat{X} \times X, D \times X, \Lambda(n)) \longrightarrow H^m(\hat{X} \times Y, D \times Y, \Lambda(n)).$$

Proof. Let j denote the inclusion $j\colon X \times Y \hookrightarrow \hat{X} \times Y$. Consider the Leray spectral sequence for the first projection $p\colon \hat{X} \times Y \to \hat{X}$ and the étale sheaf $j_!\Lambda(n)$:

$$E_2^{a,b}(Y) := H^a(\hat{X}, R^b p_\star(j_!\Lambda(n))) \Longrightarrow E^{a+b}(Y) := H^{a+b}(\hat{X} \times Y, j_!\Lambda(n)) \tag{2.1}$$

We are interested in determining the kernel and the image of the natural maps $E_2^{m,0}(Y) \to E^m(Y)$ under the assumption that Y appears high enough in π_s, since $E_2^{m,0}(Y) = H^m(\hat{X}, D, \Lambda(n))$ and $E^m(Y) = H^m(\hat{X} \times Y, D \times Y, \Lambda(n))$. Let us denote by $(F^i E^m(Y))_{i \geq 0}$ the filtration on $E^m(Y)$ determined by (2.1), so that $F^i E^m(Y)/F^{i+1}E^m(Y) = E_\infty^{i,m-i}(Y)$.

Lemma 2.5. *Let $Y \to X$ be an étale cover appearing in π_s. If $Z \to Y$ is an étale cover which appears high enough in π_s, then for every $a \geq 0$ and every $b \geq 1$, the natural map $E_2^{a,b}(Y) \to E_2^{a,b}(Z)$ is zero.*

Proof. The étale sheaf $R^b p_\star(j_!\Lambda(n))$ on $\hat{X}$ coincides with the extension by zero from X to $\hat{X}$ of the pullback, by the structure morphism $X \to \mathrm{Spec}(k)$, of the étale sheaf defined by the G_k-module $H^b(\bar{Y}, \Lambda(n)) = H^b(\bar{Y}, \Lambda) \otimes_\Lambda \Lambda(n)$, where $\bar{Y} = Y \otimes_k \bar{k}$. As a consequence we need only check that there exists an étale cover $Z \to Y$ appearing in π_s such that for every $b \geq 1$, the pullback map $H^b(\bar{Y}, \Lambda) \to H^b(\bar{Z}, \Lambda)$ is zero. Since X (and therefore Y) is a $K(\pi, 1)$, there exists an étale cover $\bar{Z} \to \bar{Y}$ such that for every $b \geq 1$, the pullback map $H^b(\bar{Y}, \Lambda) \to H^b(\bar{Z}, \Lambda)$ is zero. After replacing $\bar{Z}$ with an étale cover of $\bar{Z}$, we may assume that the composition $\bar{Z} \to \bar{Y} \to Y$ is Galois. This implies that there exists an étale cover $Z \to Y$ which appears in π_s and whose base change to $\bar{Y}$ is $\bar{Z} \to \bar{Y}$. □

It follows from Lemma 2.5 that for any $Y \to X$ appearing in π_s and any $i < m$, if $Z \to Y$ appears high enough in π_s then the image of the natural map $F^i E^m(Y) \to F^i E^m(Z)$ is contained in $F^{i+1} E^m(Z)$. By iterating, we see that if $Y \to X$ appears high enough in π_s, then the image of the natural map $E^m(X) \to E^m(Y)$ is contained in $F^m E^m(Y)$. Now $F^m E^m(Y)$ is the image of the map $E_2^{m,0}(Y) \to E^m(Y)$. Hence the second assertion of the proposition is established.

To prove that the map $E_2^{m,0}(Y) \to E^m(Y)$ is injective if Y appears high enough in π_s, it suffices to check (by iteration again) that for every $p \geq 2$, the natural map $E_p^{m,0}(Y) \to E_{p+1}^{m,0}(Y)$ is injective if Y appears high enough in π_s. The kernel of the latter map is a quotient of $E_p^{m-p,p-1}(Y)$. Hence the result follows from Lemma 2.5 (since $p - 1 \geq 1$). □

Construction–Theorem 2.6. *Let X be a smooth geometrically irreducible and separated variety of dimension d over k. We assume that X is a $K(\pi, 1)$. Let $x \in X(\bar{k})$ and let s be a section of the natural map $\pi_1(X, x) \to G_k$. Let $N \geq 1$ be an integer invertible in k. Put $\Lambda = \mathbb{Z}/N\mathbb{Z}$. To s we associate a class $\alpha(\Lambda) \in H_c^{2d}(X, \Lambda(d))$. More generally, to s and to any étale cover $Y \to X$ appearing in π_s, we associate a class $\alpha(\Lambda, Y) \in H_c^{2d}(Y, \Lambda(d))$.*

All these classes are compatible as Y and Λ vary, in the sense that for any prime number ℓ which is invertible in k, they define an element $\widetilde{\alpha}$ of the inverse limit

$$\varprojlim_{m \geq 1} \varprojlim_{Y \to X} H_c^{2d}(Y, \mathbb{Z}/\ell^m\mathbb{Z}(d))$$

where $Y \to X$ runs through all étale covers of X which appear in π_s and where the transition morphism associated to an étale cover $Z \to Y$ appearing in π_s is the trace map $H_c^{2d}(Z, \mathbb{Z}/\ell^m\mathbb{Z}(d)) \to H_c^{2d}(Y, \mathbb{Z}/\ell^m\mathbb{Z}(d))$.

Moreover, if s is the section associated (up to conjugacy) to a rational point $a \in X(k)$, then $\widetilde{\alpha}$ is the cycle class of the rational point $\widetilde{a} \in X_s(k)$.

Proof. Let $X \subseteq \hat{X}$ be a compactification of X and let $D = \hat{X} \setminus X$. Since X is smooth and separated over k, the diagonal embedding $X \hookrightarrow \hat{X} \times X$ defines a class $[\Delta] \in H^{2d}(\hat{X} \times X, D \times X, \Lambda(d))$. According to Proposition 2.4, if $\pi \colon Z \to X$ is an étale cover which appears high enough in π_s, the inverse image $(1 \times \pi)^\star[\Delta]$

of $[\Delta]$ in $H^{2d}(\hat{X} \times Z, D \times Z, \Lambda(d))$ comes, by pullback, from a unique element of $H^{2d}(\hat{X}, D, \Lambda(d)) = H_c^{2d}(X, \Lambda(d))$. It is this element which we denote $\alpha(\Lambda)$. It does not depend on the choice of Z.

For any étale cover $Y \to X$ appearing in π_s, the section s induces a section of the natural map $\pi_1(Y, y) \to G_k$ for some $y \in Y(\bar{k})$ above x. By applying the previous construction to Y and to this new section, we therefore obtain a class $\alpha(\Lambda, Y) \in H_c^{2d}(Y, \Lambda(d))$.

That these classes are compatible if Λ and Y are allowed to vary follows from the fact that the construction of $\alpha(\Lambda)$ can be carried out with an étale cover $\pi \colon Z \to X$ as soon as it appears high enough in π_s.

Suppose now that s is associated to a rational point $a \in X(k)$. To prove that $\widetilde{\alpha}$ is the cycle class of $\widetilde{a}$, it suffices to prove that $\alpha(\Lambda)$ is the cycle class of a (one can then apply the argument to every étale cover $Y \to X$ which appears in π_s). Let $\pi \colon Z \to X$ be as in the construction of $\alpha(\Lambda)$. Let $a_Z \in Z(k)$ denote the image of $\widetilde{a} \in X_s(k)$. Restriction to $\hat{X} \times \{a_Z\} \subseteq \hat{X} \times Z$ defines a map $H^{2d}(\hat{X} \times Z, D \times Z, \Lambda(d)) \to H^{2d}(\hat{X}, D, \Lambda(d))$ which is a retraction of the natural map in the other direction. Therefore $\alpha(\Lambda)$ is the image of $(1 \times \pi)^\star[\Delta]$ by this map. Now $(1 \times \pi)^\star[\Delta]$ is the cycle class of the graph of π; hence its restriction to $\hat{X} \times \{a_Z\}$ is the cycle class of $\pi(a_Z) = a$. □

Remark 2.7. Let us not assume that X is a $K(\pi, 1)$. Let s be a section of the natural map $G_{k(X)} \to G_k$, where $G_{k(X)}$ denotes the absolute Galois group of $k(X)$. Then s determines (up to conjugacy) a section of $\pi_1(U, u) \to G_k$ for any dense open $U \subseteq X$ and any $u \in U(\bar{k})$. In particular, by choosing U to be small enough, one may assume that U is a $K(\pi, 1)$ and apply Theorem 2.6 to produce a class $\alpha(\Lambda, U) \in H_c^{2d}(U, \Lambda(d))$. The image of $\alpha(\Lambda, U)$ in $H_c^{2d}(X, \Lambda(d))$ does not depend on the choice of U. We shall again denote it by $\alpha(\Lambda)$.

Building upon earlier results of Pop, Koenigsmann [6] showed that if X is a smooth and geometrically connected proper curve over a p-adic field k, then every section s of the natural map $G_{k(X)} \to G_k$ determines a unique rational point $a \in X(k)$ (in the sense that the section of $\pi_1(X, x) \to G_k$ induced by s is associated to a). Koenigsmann's proof is model-theoretic. We observe here that Theorem 2.6 gives a geometric understanding of the "abelian" part of the rational point a (*i.e.*, of a considered as a divisor of degree 1 on X):

Proposition 2.8. *Let X be a smooth geometrically connected proper curve over a p-adic field k. Let s be a section of the natural map $G_{k(X)} \to G_k$. Let $\alpha \in H^2(X, \hat{\mathbb{Z}}(1))$ denote the inverse limit of the classes $\alpha(\mathbb{Z}/n\mathbb{Z}) \in H^2(X, \boldsymbol{\mu}_n)$ constructed in Theorem 2.6 and Remark 2.7. Then α is the cycle class of a degree* 1 *divisor on X (uniquely determined up to linear equivalence).*

Proof. Multiplication by n on $\mathbf{G}_{\mathrm{m}}$ induces an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/n\mathrm{Pic}(X) \longrightarrow H^2(X, \boldsymbol{\mu}_n) \longrightarrow \mathrm{Br}(X)$$

where $\mathrm{Br}(X) = H^2(X, \mathbf{G}_{\mathrm{m}})$. According to Remark 2.7, the class $\alpha(\mathbb{Z}/n\mathbb{Z})$ belongs to the image of the map $H^2_c(U, \boldsymbol{\mu}_n) \to H^2(X, \boldsymbol{\mu}_n)$ for every dense open $U \subseteq X$. Hence its image in $\mathrm{Br}(X)$ belongs to the image of $H^2_c(U, \mathbf{G}_{\mathrm{m}}) \to H^2(X, \mathbf{G}_{\mathrm{m}})$ for every dense open $U \subseteq X$. In other words the image of $\alpha(\mathbb{Z}/n\mathbb{Z})$ in $\mathrm{Br}(X)$ belongs to the right kernel of the natural pairing $\mathrm{Pic}(X) \times \mathrm{Br}(X) \to \mathrm{Br}(k)$. By Lichtenbaum–Tate duality [9] this right kernel is zero; hence finally $\alpha(\mathbb{Z}/n\mathbb{Z}) \in \mathrm{Pic}(X)/n\mathrm{Pic}(X)$. On the other hand, the image of α in $H^2(\bar{X}, \hat{\mathbb{Z}}(1)) = \hat{\mathbb{Z}}$ is equal to 1. Therefore α belongs to the inverse image of $1 \in \hat{\mathbb{Z}}$ by the degree map $\varprojlim_{n\geq 1} (\mathrm{Pic}(X)/n\mathrm{Pic}(X)) \to \hat{\mathbb{Z}}$, which means that α is the image of a unique element of $\mathrm{Pic}(X)$ (see [10, I.3.3]). ☐

3. Nilpotent completion and liftings of cycle classes of sections

3.1. **Beilinson's construction.** Let X be a complex manifold and let $a \in X$. Denote by I the augmentation ideal of the group algebra $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]$. For any $n \geq 1$, Beilinson constructed a complex of sheaves of $\mathbb{Q}$-vector spaces on X^n whose n-th hypercohomology group is canonically dual to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$. His construction is described in [3, Section 3]. We recall it briefly. Consider the manifold $X \times X^n \times X$ with coordinates $(t_0, \ldots, t_{n+1})$. For $i \in \{0, \ldots, n\}$, let A_i denote the submanifold defined by $t_i = t_{i+1}$. For $J \subseteq \{0, \ldots, n\}$, let $A_J = \cap_{j\in J} A_j$ and let $\mathbb{Q}_{A_J}$ denote the direct image of the constant sheaf $\mathbb{Q}$ by the inclusion $A_J \hookrightarrow X \times X^n \times X$. Let $\mathcal{B}(n)$ denote the complex of sheaves on $X \times X^n \times X$ defined by $\mathcal{B}(n)^p = \bigoplus_{J\subseteq\{0,\ldots,n\}, \#J=p} \mathbb{Q}_{A_J}$ for $p \in \{0, \ldots, n\}$ and $\mathcal{B}(n)^p = 0$ for all other p; the differential $\mathcal{B}(n)^p \to \mathcal{B}(n)^{p+1}$ is the sum, over all J and all c such that $c \notin J$, of $(-1)^{\#\{m\in J; m<c\}}$ times the restriction map $\mathbb{Q}_{A_J} \to \mathbb{Q}_{A_{J\cup\{c\}}}$. For $(b, a) \in X \times X$, let i_{ba} denote the closed immersion $X^n = \{b\} \times X^n \times \{a\} \hookrightarrow X \times X^n \times X$ and let j_{ba} denote the open immersion $X^n \setminus i_{ba}^{-1}(\bigcup_{i=0}^n A_i) \hookrightarrow X^n$. Set $\mathcal{B}(n)_{ba} = i_{ba}^\star \mathcal{B}(n)$. If $b \neq a$, then $\mathcal{B}(n)_{ba}$ is quasi-isomorphic to $j_{ba!}\mathbb{Q}$. On the other hand, if $b = a$, the defect of exactness of $\mathcal{B}(n)$ at $\mathcal{B}(n)^n$ provides a map $\mathcal{B}(n) \to \mathbb{Q}_{A_{\{0,\ldots,n\}}}[-n]$ which in hypercohomology induces a map $\theta_{aa} \colon \mathbb{H}^n(X^n, \mathcal{B}(n)_{aa}) \to \mathbb{Q}$. Proposition 3.4 of *loc. cit.* then asserts that the $\mathbb{Q}$-vector space $\mathbb{H}^n(X^n, \mathcal{B}(n)_{aa})$, endowed with θ_{aa}, is canonically dual to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$, endowed with the map $\mathbb{Q} \to \mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$ which sends 1 to 1.

3.2. **Replacing the base point by the universal cover.** We first remark that when X is a $K(\pi, 1)$, Beilinson's construction can be reformulated in terms of the (topological) universal cover of X at a instead of the point a itself. Let X be as above. For $a \in X$, let $\widetilde{\pi}_a^{\mathrm{top}}\colon (\widetilde{X}_a^{\mathrm{top}}, \widetilde{a}) \to (X, a)$ denote the (topological) universal pointed cover of the pointed space (X, a). Its fiber above a is $\pi_1^{\mathrm{top}}(X, a)$.

Definition 3.1. (i) We say that X is *topologically a* $K(\pi, 1)$ if $H^i(\widetilde{X}_a^{\mathrm{top}}, \mathbb{Z}) = 0$ for all $i \geq 1$ (or equivalently if $\varinjlim_{Y \to X} H^i(Y, \mathbb{Z}) = 0$, where $Y \to X$ ranges over the topological covers such that $\widetilde{X}_a^{\mathrm{top}} \to X$ factors through $Y \to X$).

(ii) Let $\pi\colon Y \to X$ be a topological cover. Let $p\colon Y \times X^n \times Y \to X \times X^n \times X$ denote the map $\pi \times 1 \times \pi$. We put $\mathcal{B}(n)(Y) = p^\star \mathcal{B}(n)$. The defect of exactness of the complex $\mathcal{B}(n)(Y)$ at $\mathcal{B}(n)(Y)^n$ provides a map $\mathcal{B}(n)(Y) \to \iota_\star \mathbb{Q}[-n]$ where $\iota\colon Y \to Y \times X^n \times Y$ is the closed immersion $1 \times \pi^n \times 1$. We denote by $\theta_\pi\colon \mathbb{H}^n(Y \times X^n \times Y, \mathcal{B}(n)(Y)) \to \mathbb{Q}$ the linear form it induces in hypercohomology.

Proposition 3.2. *Assume X is topologically a $K(\pi, 1)$. For any $a \in X$, the $\mathbb{Q}$-vector space $\mathbb{H}^n(X^n, \mathcal{B}(n)_{aa})$, endowed with the linear form θ_{aa}, is canonically isomorphic to $\mathbb{H}^n(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}}))$ endowed with $\theta_{\widetilde{\pi}_a^{\mathrm{top}}}$.*

Proof. Let $i\colon X^n \hookrightarrow \widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}$ denote the closed immersion $\{\widetilde{a}\} \times 1 \times \{\widetilde{a}\}$. The inverse image of $\mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}})$ by i is equal to $\mathcal{B}(n)_{aa}$. As a consequence, to establish the proposition it suffices to check that the restriction map

$$\mathbb{H}^n(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}})) \longrightarrow \mathbb{H}^n(X^n, i^\star \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}}))$$

is an isomorphism. For this, in view of the definition of $\mathcal{B}(n)$, it suffices to check that the restriction map

$$H^m(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, p^\star \mathbb{Q}_{A_J}) \longrightarrow H^m(X^n, i^\star p^\star \mathbb{Q}_{A_J})$$

is an isomorphism for any m and for any $J \subsetneq \{0, \ldots, n\}$, where p is as in Definition 3.1 (ii). Now this is a direct consequence of the hypothesis that X is topologically a $K(\pi, 1)$ together with the Künneth formula. $\square$

As a consequence, the $\mathbb{Q}$-vector space $\mathbb{H}^n(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}}))$, endowed with $\theta_{\widetilde{\pi}_a^{\mathrm{top}}}$, is canonically dual to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$, endowed with the map $\mathbb{Q} \to \mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$ which sends 1 to 1.

3.3. **In the algebraic setting.** Let X be a smooth geometrically irreducible separated variety of dimension d over a field k with separable closure $\bar{k}$. Let N be an integer invertible in k and let $\Lambda = \mathbb{Z}/N\mathbb{Z}$. Fix $n \geq 1$. For $J \subseteq \{0, \ldots, n\}$, let $A_J \subseteq X \times X^n \times X$ be, as in §3.1, the subvariety defined by the equations $t_j = t_{j+1}$ for $j \in J$, and let Λ_{A_J} denote the direct image of the constant étale sheaf Λ by the inclusion $A_J \hookrightarrow X \times X^n \times X$. Let $\mathcal{B}^e(n)$ (where e stands for "étale") denote the complex of étale sheaves on $X \times X^n \times X$ defined in the same way as $\mathcal{B}(n)$ was defined in §3.1, except that $\mathbb{Q}_{A_J}$ is now replaced by Λ_{A_J}. For $a \in X(\bar{k})$, we denote by $\mathcal{B}^e(n)_{aa}$ the inverse image of $\mathcal{B}^e(n)$ by the natural map $\bar{X}^n = \{a\} \times \bar{X}^n \times \{a\} \to X \times X^n \times X$, where $\bar{X} = X \otimes_k \bar{k}$. The natural map

$\mathcal{B}^e(n) \to \Lambda_{A_{\{0,\dots,n\}}}[-n]$ induces a linear form $\theta_{aa}\colon \mathbb{H}^n(\bar{X}^n, \mathcal{B}^e(n)_{aa}) \to \Lambda$. Finally, if $\pi\colon Y \to X$ is an étale cover, we denote by $p\colon Y \times X^n \times Y \to X \times X^n \times X$ the map $\pi \times 1 \times \pi$. Definition 3.1 (ii) still makes sense. It yields a linear form $\theta_\pi\colon \mathbb{H}^n(Y \times X^n \times Y, \mathcal{B}^e(n)(Y)) \to \Lambda$, where $\mathcal{B}^e(n)(Y) = p^\star\mathcal{B}^e(n)$.

With these definitions in hand, we may formulate a statement analogous to Proposition 3.2. Its proof, which we omit, is also entirely analogous.

Proposition 3.3. *Assume X is a $K(\pi, 1)$. For any $a \in X(\bar{k})$, the Λ-module $\mathbb{H}^n(\bar{X}^n, \mathcal{B}^e(n)_{aa})$, endowed with θ_{aa}, is canonically isomorphic to*

$$\varinjlim \mathbb{H}^n(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{B}^e(n)(\bar{Y})) \tag{3.1}$$

endowed with the linear form $\varinjlim \theta_\pi$. The direct limit is taken over all étale covers $\pi\colon \bar{Y} \to \bar{X}$ such that $\widetilde{\pi}_a\colon \widetilde{X}_a \to \bar{X} \to X$ factors through π.

We now turn to the Λ-module dual to $\mathbb{H}^n(\bar{X}^n, \mathcal{B}^e(n)_{aa})$.

Definition 3.4. Let $n \geq 1$. We define an object $\mathcal{C}(n)$ in the bounded derived category of étale sheaves of Λ-modules on $X \times X^n \times X$ by the formula

$$\mathcal{C}(n) = \tau_{\leq n(2d-1)} Rj_\star \Lambda(nd)$$

where j denotes the open immersion $j\colon (X \times X^n \times X) \setminus (\bigcup_{i=0}^n A_i) \hookrightarrow X \times X^n \times X$. For any étale cover $\pi\colon Y \to X$, we set $\mathcal{C}(n)(Y) = p^\star\mathcal{C}(n)$.

For later use, we note that $R^q j_\star \Lambda(nd) = 0$ if q is not divisible by $2d-1$ and that

$$R^q j_\star \Lambda(nd) = \bigoplus_{J \subseteq \{0,\dots,n\}, \#J = m} \Lambda_{A_J}((n-m)d)$$

if $q = m(2d-1)$ for some integer m. In particular there are natural distinguished triangles

$$\tau_{\leq (n-1)(2d-1)} Rj_\star \Lambda(nd) \longrightarrow \mathcal{C}(n) \longrightarrow \bigoplus_{J \subseteq \{0,\dots,n\}, \#J = n} \Lambda_{A_J}[-n(2d-1)] \xrightarrow{+1} \tag{3.2}$$

and

$$\mathcal{C}(n) \longrightarrow Rj_\star \Lambda(nd) \longrightarrow \Lambda_{A_{\{0,\dots,n\}}}(-d)[-(n+1)(2d-1)] \xrightarrow{+1} \quad . \tag{3.3}$$

The following proposition shows that in the algebraic setting, the n-th hypercohomology group of $\mathcal{C}(n)$ twisted by $2d$ plays a rôle analogous to that of the vector space $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$ in Beilinson's original construction. We shall not pursue this analogy further. Instead, we shall use the complexes $\mathcal{C}(n)$ in §3.4 to define liftings of the class $\alpha(\Lambda)$ associated in Theorem 2.6 to a section of $\pi_1(X, a) \to G_k$.

Proposition 3.5. *Assume X is a $K(\pi,1)$. For any $a \in X(\bar{k})$, the Λ-module*

$$\varprojlim \mathbb{H}_c^{2(n+2)d-n}(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{C}(n)(\bar{Y}) \otimes_\Lambda \Lambda(2d)),$$

where the inverse limit ranges over the étale covers $\pi \colon \bar{Y} \to \bar{X}$ such that $\widetilde{\pi}_a \colon \widetilde{X}_a \to \bar{X} \to X$ factors through π, and where the transition maps are the trace maps, is canonically dual to $\mathbb{H}^n(\bar{X}^n, \mathcal{B}^e(n)_{aa})$.

Proof. There is a canonical quasi-isomorphism $\mathcal{C}(n) = R\mathcal{H}\mathrm{om}(\mathcal{B}^e(n), \Lambda(nd))$. (More generally, applying $R\mathcal{H}\mathrm{om}(-, \Lambda(nd))$ to the distinguished triangle

$$\Lambda_{A_{\{0,\dots,n\}}}[-n-1] \longrightarrow j_!\Lambda \longrightarrow \mathcal{B}^e(n) \xrightarrow{+1}$$

yields (3.3).) Hence, by Poincaré duality, a perfect pairing of Λ-modules

$$\mathbb{H}_c^{2(n+2)d-n}(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{C}(n)(\bar{Y})) \times \mathbb{H}^n(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{B}^e(n)(\bar{Y})) \longrightarrow \Lambda(-2d).$$

for any étale cover $\bar{Y} \to \bar{X}$. We conclude by applying Proposition 3.3. □

3.4. **Liftings of the classes $\alpha(\Lambda)$.** The aim of §3.4 is to use the definitions of §3.3 to construct liftings of the classes $\alpha(\Lambda)$ associated in Theorem 2.6 to sections of $\pi_1(X,x) \to G_k$. We retain the notations and hypotheses of §3.3. In addition, we assume throughout that X is a $K(\pi,1)$ and we fix a section s of the natural map $\pi_1(X,x) \to G_k$ for some $x \in X(\bar{k})$. For simplicity we also assume that X is proper.

First we remark that (3.2) yields, for $n = 1$, a map

$$H^0(A_0, \Lambda) \oplus H^0(A_1, \Lambda) \longrightarrow H^{2d}(X \times X \times X, \Lambda(d))$$

and hence two classes in $H^{2d}(X \times X \times X, \Lambda(d))$. Concretely, these two classes are the classes of the algebraic cycles $A_i \subset X \times X \times X$ for $i \in \{0,1\}$. Let c_0, c_1 denote their images in $\varinjlim H^{2d}(Y \times X \times Y, \Lambda(d))$, where the direct limit ranges over all étale covers $Y \to X$ which appear in $\pi_s \colon X_s \to X$. As in Proposition 2.4, we have $\varinjlim H^{2d}(Y \times X \times Y, \Lambda(d)) = H^{2d}(X, \Lambda(d))$. Now it follows from the construction of $\alpha(\Lambda)$ that c_0 and c_1 coincide, via this equality, with $\alpha(\Lambda) \in H^{2d}(X, \Lambda(d))$. Hence $\alpha(\Lambda)$ can be read off of (3.2) for $n = 1$; which begs us to also consider this triangle for $n > 1$.

Definition 3.6. Let $n \geq 1$. We define $\mathcal{C}'(n)$ by the formula

$$\mathcal{C}'(n) = \tau_{\leq n(2d-1)} Rj'_\star \Lambda(nd)$$

where j' denotes the open immersion $j' \colon (X \times X^n \times X) \backslash (\bigcup_{i=0}^{n-1} A_i) \hookrightarrow X \times X^n \times X$. For any étale cover $\pi \colon Y \to X$, we set $\mathcal{C}'(n)(Y) = p^\star \mathcal{C}'(n)$.

The complexes $\mathcal{C}(n)$ and $\mathcal{C}'(n)$ are related by a distinguished triangle

$$\mathcal{C}'(n) \longrightarrow \mathcal{C}(n) \xrightarrow{\mathrm{res}} i_\star \mathcal{C}(n-1)[-(2d-1)] \xrightarrow{+1} \tag{3.4}$$

where i denotes the closed immersion $i\colon X \times X^{n-1} \times X \simeq A_n \hookrightarrow X \times X^n \times X$ and res is the residue map. (We take the convention that $\mathcal{C}(0) = \Lambda$.)

Definition 3.7. For any $n \geq 1$, we define

$$c_{1,\dots,n} \in \varinjlim H^{n(2d-1)+1}(Y \times X^n \times Y, \tau_{\leq (n-1)(2d-1)} p^\star Rj_\star \Lambda(nd)),$$

where the direct limit ranges over all étale covers $Y \to X$ which appear in $\pi_s\colon X_s \to X$, to be the class of the image of 1 by the map

$$H^0(A_{\{1,\dots,n\}}, \Lambda) \longrightarrow H^{n(2d-1)+1}(X \times X^n \times X, \tau_{\leq (n-1)(2d-1)} Rj_\star \Lambda(nd))$$

stemming from (3.2).

Proposition 3.8. *The class* $c_{1,\dots,n}$ *is a lifting of* $\alpha(\Lambda) \in H^{2d}(X, \Lambda(d))$ *(by an "iterated residue" map).*

Proof. Let $n \geq 2$. For any $Y \to X$ appearing in π_s, the residue map in (3.4), together with (3.2), gives rise to a commutative square

$$\begin{tikzcd}
\bigoplus_{J \subseteq \{0,\dots,n\}, \#J = n} H^0(A_J, \Lambda) \arrow[r] \arrow[d] & H^{n(2d-1)+1}(Y \times X^n \times Y, \tau_{\leq (n-1)(2d-1)} p^\star Rj_\star \Lambda(nd)) \arrow[d] \\
\bigoplus_{J \subseteq \{0,\dots,n-1\}, \#J = n-1} H^0(A_J, \Lambda) \arrow[r] & H^{(n-1)(2d-1)+1}(Y \times X^{n-1} \times Y, \tau_{\leq (n-2)(2d-1)} p^\star Rj_\star \Lambda((n-1)d))
\end{tikzcd}$$

where the vertical map on the left sends $1 \in H^0(A_J, \Lambda)$ to $1 \in H^0(A_{J \setminus \{n\}}, \Lambda)$ if $n \in J$, to 0 otherwise, and the vertical map on the right is the residue along $Y \times X^{n-1} \times Y = p^{-1}(A_n) \subset Y \times X^n \times Y$. As a consequence, for every $n \geq 2$, the class $c_{1,\dots,n}$ maps to $c_{1,\dots,n-1}$ by the residue map. Since $c_1 = \alpha(\Lambda)$, this proves the proposition. $\square$

One might hope to use the liftings of $\alpha(\Lambda)$ defined in Proposition 3.8 in order to "rigidify" the 0-cycle of degree 1 constructed up to linear equivalence in Proposition 2.8 (so as to show that it lies in $X(k) \subset \mathrm{Pic}^1(X)$). For example the first lifting $c_{1,2}$ belongs to $\varinjlim H^3(Y \times X^2 \times Y, \tau_{\leq 1} p^\star Rj_\star \Lambda(2))$. If the section s comes from a rational point $a \in X(k)$, then $c_{1,2}$ is the class of $X \times \{a\}$ in $H^3(X^2, \tau_{\leq 1} R\lambda_\star \Lambda(2))$, where λ denotes the open immersion $\lambda\colon X^2 \setminus ((X \times \{a\}) \cup \Delta \cup (\{a\} \times X)) \hookrightarrow X^2$.

References

[1] Artin, M., *Comparaison avec la cohomologie classique: cas d'un schéma lisse*, Séminaire de Géométrie Algébrique du Bois–Marie 1963–1964 (SGA 4 III), Exp. XI, Lecture Notes in Mathematics, Vol. 305, Springer–Verlag, Berlin–New York, 1973.

[2] Deligne, P., *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, MSRI Publications, Vol. 16, Springer, New York, 1989.

[3] Deligne, P., Goncharov, A, *Groupes fondamentaux motiviques de Tate mixtes*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 1, 1–56.

[4] Esnault, H., Levine, M., *Tate motives and the fundamental group*, preprint 2007, 65 pages.

[5] Grothendieck, A., Revêtements étales et groupe fondamental, Séminaire de Géométrie Algébrique du Bois–Marie 1960–1961 (SGA 1), Lecture Notes in Mathematics, Vol. 224, Springer–Verlag, Berlin–New York, 1970.

[6] Koenigsmann, J., *On the 'section conjecture' in anabelian geometry*, J. reine angew. Math. **588** (2005), 221–235.

[7] Levine, M., *Tate motives and the vanishing conjectures for algebraic K-theory*, Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), 167–188, NATO ASI Series C, Vol. 407, Kluwer Acad. Publ., Dordrecht, 1993.

[8] Levine, M., *Motivic tubular neighborhoods*, Doc. Math. **12** (2007), 71-146.

[9] Lichtenbaum, S., *Curves over discrete valuation rings*, Amer. J. Math. **90** (1968), 380-405.

[10] Milne, J., Arithmetic duality theorems, Perspectives in Mathematics, Vol. 1, Academic Press, Boston, MA, 1986.

[11] Mochizuki, S., *The local pro-p anabelian geometry of curves*, Invent. math. **138** (1999), no. 2, 319-423.

Universität Duisburg–Essen, Mathematik, 45117 Essen, Germany

E-mail address: `esnault@uni-due.de`

Institut de Recherche Mathématique Avancée, CNRS – Université Louis Pasteur,
7 rue René Descartes, 67084 Strasbourg Cedex, France

E-mail address: `olivier.wittenberg@math.u-strasbg.fr`