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Parabolic Raynaud bundles

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Abstract. Let X be an irreducible smooth projective curve defined over the field of complex numbers, $S = \{p_1, p_2, \dots, p_n\} \subset X$ a finite set of closed points and $N \geq 2$ a fixed integer. For any pair $(r, d) \in \mathbb{N} \times \frac{1}{N}\mathbb{Z}$, there exists a parabolic vector bundle $R_{r,d,*}$ on X , with parabolic structure over S and all parabolic weights in $\frac{1}{N}\mathbb{Z}$, that has the following property: Take any parabolic vector bundle E_* of rank r on X whose parabolic points are contained in S , all the parabolic weights are in $\frac{1}{N}\mathbb{Z}$ and the parabolic degree is d . Then E_* is parabolically semistable if and only if there is no nonzero parabolic homomorphism from $R_{r,d,*}$ to E_* .

1. Introduction

Let X be an irreducible smooth projective curve defined over \mathbb{C} . Faltings showed that a vector bundle E over X is semistable if and only if there exists a vector bundle F on X such that $H^i(X, \mathcal{H}om(F, E)) = 0$ for $i = 0, 1$ [4]. This result was developed further by Popa (see [10]) to obtain estimates for the rank of F which depend only on the rank of E . Indeed, the theory of generalized Theta-functions can be build upon this vanishing. Furthermore, as was shown by Faltings in [4] (see also [6] for a generalization to projective surfaces) this allows a new construction of the moduli space of S-equivalence classes of semistable vector bundles over x .

Given r and d , a vector bundle $R_{r,d}$ on X is called a *Raynaud bundle* if the following holds: A vector bundle E on X of rank r and degree d is semistable if and only if $\text{Hom}(R_{r,d}, E) = 0$ [7]. We note that these are called Raynaud bundles because the vector bundles considered by Raynaud in [11] are Raynaud bundles for $r = 2$ and $d = 2(\text{genus}(X) - 1)$ [7, Theorem 1.1].

In [7], building on Popa's result it was shown that Raynaud bundles exist. It is quite interesting that semistability of a sheaf E is determined just by the vanishing of $\text{Hom}(R_{r,d}, E)$. Remember that the original definition of semistability requires the vanishing of $\text{Hom}(F, E)$ for all semistable F of smaller rank and higher slope than E .

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Parabolic vector bundles arose in investigations of flat vector bundles over varieties that are not complete. Over the years they have acquired the status of objects of independent interest. In [3], an analog of Faltings' semistability criterion is given for parabolic vector bundles over a Riemann surface.

Our aim here is to prove the existence of the Raynaud bundles in the parabolic context.

Given two parabolic vector bundles E_* and F_* on X , the global parabolic homomorphisms from F_* to E_* will be denoted by $\text{Hom}_{\text{par}}(F_*, E_*)$ (see [8, p. 212, Definition 1.5(II)] for the definition of a parabolic homomorphism).

We prove the following theorem:

Theorem 1.1. *Let $S = \{p_1, p_2, \dots, p_n\} \subset X$ a finite set of closed points of X , and $N \geq 2$ a fixed integer. For any pair $(r, d) \in \mathbb{N} \times \frac{1}{N}\mathbb{Z}$, there exists a parabolic vector bundle $R_{r,d,*}$ on X , with parabolic structure over S and all parabolic weights in $\frac{1}{N}\mathbb{Z}$, that has the following property: Take any parabolic vector bundle E_* on X such that the following three hold:*

- (1) *The parabolic points of E_* are contained in S ,*
- (2) *All the parabolic weights of E_* are in $\frac{1}{N}\mathbb{Z}$, and*
- (3) *The rank of E_* is r , and the parabolic degree of E_* is d .*

Then E_ is parabolically semistable $\iff \text{Hom}_{\text{par}}(R_{r,d,*}, E_*) = 0$.*

2. Preliminaries

2.1. The equivalence of equivariant bundles and parabolic bundles

We will recall a correspondence between parabolic vector bundles and equivariant vector bundles which will be used in the proof of Theorem 1.1.

We assume that at least one of the following two conditions hold:

- $\text{genus}(X) \geq 1$
- $|S| \neq 1$.

Fix S and d as in Theorem 1.1. Fix a Galois algebraic covering

$$f : Y \longrightarrow X \tag{1}$$

such that

- f is ramified exactly over S , and
- The ramification index of each point in $f^{-1}(S)$ is $N - 1$.

See [9, p. 26, Proposition 1.2.12] for the existence of f satisfying these conditions. We note that the assumption $(\text{genus}(X), |S|) \neq (0, 1)$ is needed for the existence of f . Let

$$\Gamma := \text{Gal}(f)$$

be the Galois group for the covering f . A Γ -linearized vector bundle on Y is an algebraic vector bundle E equipped with a lift of the action of Γ as vector

bundle automorphisms. This means that Γ acts on the total space of E as algebraic automorphisms, and the action of each $\gamma \in \Gamma$ on E is an isomorphism of the vector bundle E with $(\gamma^{-1})^*E$.

In [1], a natural bijective correspondence between the following two classes was established:

- (1) The Γ -linearized vector bundles W on Y , and
- (2) The parabolic vector bundles E_* over X for which the parabolic divisor is contained in S and all the parabolic weights are in $\frac{1}{N}\mathbb{Z}$.

This bijective correspondence takes the usual tensor product (respectively, direct sum) of Γ -linearized vector bundles to the tensor product (respectively, direct sum) of the corresponding parabolic vector bundles. Similarly, for any Γ -linearized vector bundle W , this bijective correspondence takes W^* to the parabolic dual of the parabolic vector bundle corresponding to W . (See [2, 12] for the above mentioned operations on parabolic vector bundles.)

For any parabolic vector bundle E_* of the above type, if \widehat{E} is the corresponding Γ -linearized vector bundle, then

$$\deg(\widehat{E}) = |\Gamma| \cdot \deg_{\text{par}}(E_*), \quad (2)$$

where $\deg_{\text{par}}(E_*)$ is the parabolic degree of E_* [1, p. 318, (3.12)]. Furthermore,

$$E_* \text{ is parabolically semistable} \iff \widehat{E} \text{ is semistable} \quad (3)$$

(see [1, p. 318, Lemma 3.13]).

2.2. Raynaud bundles on smooth algebraic curves

In [7], an irreducible smooth projective curve Y is considered. It is shown that for any pair of integers (r, d) , there exists a vector bundle $R_{r,d}$ on Y (which is called a Raynaud bundle) with the following property:

For a vector bundle E of rank r and degree d on Y ,

$$E \text{ is semistable} \iff \text{Hom}(R_{r,d}, E) = 0 \quad (4)$$

(see (i) \iff (v) in Theorem 2.12 of [7]).

This way we obtain a short cohomological criterion which enables us to check semistability. We can compute the rank and degree of $R_{r,d}$ in terms of the genus g_Y of Y and the integers r and d (see Proposition 2.2 and Corollary 3.4 in [7]).

3. Proof of Theorem 1.1

As in Sect. 2.1, we will assume that at least one of the following two conditions hold:

- $\text{genus}(X) \geq 1$
- $|S| \neq 1$.

The remaining case where $(\text{genus}(X), |S|) = (0, 1)$ will be treated separately.

Let E_* be a parabolic vector bundle on X satisfying conditions (1)–(3) of Theorem 1.1. Denoting by \widehat{E} the associated Γ -linearized vector bundle on Y in (1), we have that \widehat{E} is of rank r and degree $d|\Gamma|$, and the parabolically semistability of E_* is equivalent to the semistability of \widehat{E} (see (3)).

Using the result (4) together with (2) and (3) we obtain that

$$E_* \text{ is parabolically semistable} \iff \text{Hom}(R_{r,d|\Gamma|}, \widehat{E}) = 0. \quad (5)$$

Step 1: Set now

$$\widetilde{R}_{r,d} := \bigoplus_{\gamma \in \Gamma} \gamma^* R_{r,d}. \quad (6)$$

Since \widehat{E} is Γ -linearized,

$$\text{Hom}(\widetilde{R}_{r,d|\Gamma|}, \widehat{E}) \cong \text{Hom}(R_{r,d|\Gamma|}, \widehat{E})^{\oplus |\Gamma|}.$$

Therefore,

$$\text{Hom}(\widetilde{R}_{r,d|\Gamma|}, \widehat{E}) = 0 \iff \text{Hom}(R_{r,d|\Gamma|}, \widehat{E}) = 0.$$

Combining this with (5) we conclude that

$$E_* \text{ is parabolically semistable} \iff \text{Hom}(\widetilde{R}_{r,d|\Gamma|}, \widehat{E}) = 0. \quad (7)$$

Step 2: The vector bundle $\widetilde{R}_{r,d}$ in (6) admits a canonical Γ -linearization. Let $R'_{r,d,*}$ be the parabolic vector bundle over X corresponding to this Γ -linearized vector bundle $\widetilde{R}_{r,d}$.

Consider the trivial vector bundle over Y

$$\widehat{W} := \mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}(\Gamma), \quad (8)$$

where $\mathbb{C}(\Gamma)$ is the group algebra of Γ . The action of Γ on Y lifts to an action of Γ on \mathcal{O}_Y . The natural action of Γ on $\mathbb{C}(\Gamma)$ and the action of Γ on \mathcal{O}_Y together define a Γ -linearization on the vector bundle \widehat{W} in (8).

Let W_* denote the parabolic vector bundle over X corresponding to the Γ -linearized vector bundle \widehat{W} .

Lemma 3.1. *Let F_* be a parabolic vector bundle over X satisfying condition (1) and condition (2) in Theorem 1.1. Let \widehat{F} be the Γ -linearized vector bundle on Y corresponding to F_* . Then*

$$\text{Hom}(\widetilde{R}_{r,d|\Gamma|}, \widehat{F}) = \text{Hom}_{\text{par}}(R'_{r,d,*} \otimes W_*, F_*),$$

where $R'_{r,d,*} \otimes W_*$ is the parabolic tensor product of the parabolic vector bundles $R'_{r,d,*}$ and W_* constructed above.

Proof. The parabolic vector bundle W_* is constructed as follows. Consider the direct image

$$W = f_*\mathcal{O}_Y, \quad (9)$$

where f is the covering map in (1). We have a filtration of subsheaves

$$W_1 \subset \cdots \subset W_i \subset \cdots \subset W_{N-1} \subset W_N = W,$$

where $W_j := f_*\mathcal{O}_Y(-(N-j)f^{-1}(S)_{\text{red}})$. For any point $x \in S$, let

$$0 \subset W_x^1 \subset \cdots \subset W_x^j \subset W_x^{j+1} \subset \cdots \subset W_x^{N-1} \subset W_x^N = W_x \quad (10)$$

be the filtration of subspaces given by the above filtration of subsheaves. The dimension of each successive quotient in (10) is $|\Gamma|/N$.

The vector bundle underlying the parabolic vector bundle W_* is W (defined in (9)), its parabolic divisor is S , its quasiparabolic filtration on each point $x \in S$ is the one in (10), and the parabolic weight of the subspace $W_x^j \subset W_x$ in (10) is $(N-j)/N$.

Let V_* be a parabolic vector bundle over X satisfying condition (1) and condition (2) in Theorem 1.1. The vector bundle underlying V_* will be denoted by V_0 . Let \widehat{V} be the Γ -linearized vector bundle over Y corresponding to V_* . Then

$$H^0(Y, \widehat{V})^\Gamma = H^0(X, V_0) \quad (11)$$

[1, p. 310, (2.9)]. We next note that given any finite dimensional complex left Γ -module M , there is a canonical \mathbb{C} -linear isomorphism

$$M \longrightarrow \text{Hom}_{\mathbb{C}(\Gamma)}(\mathbb{C}(\Gamma), M) = \text{Hom}_{\mathbb{C}}(\mathbb{C}(\Gamma), M)^\Gamma \quad (12)$$

that sends any $v \in M$ to the homomorphism of Γ -modules $\rho_v : \mathbb{C}(\Gamma) \longrightarrow M$ uniquely determined by the condition that $\rho_v(\gamma) = \gamma \cdot v$ for all $\gamma \in \Gamma$. Using the isomorphism in (12) it follows that

$$H^0(Y, \widehat{W}^* \otimes \widehat{V})^\Gamma = H^0(Y, \widehat{V}), \quad (13)$$

where \widehat{W} is the Γ -linearized vector bundle constructed in (8).

The bijective correspondence between the parabolic vector bundles and the Γ -linearized vector bundles is compatible with the operations of taking tensor product and dual. Hence the parabolic vector bundle over X corresponding to the Γ -linearized vector bundle $\widehat{W}^* \otimes \widehat{V}$ is the parabolic tensor product $W_*^* \otimes V_*$, where W_*^* is the parabolic dual of W_* . We also note that

$$\text{Hom}_{\text{par}}(U_*, V_*) = (\text{Hom}(\widehat{U}, \widehat{V}))^\Gamma,$$

where \widehat{U} is any Γ -linearized vector bundle on Y and U_* is the corresponding parabolic vector bundle. Therefore, substituting the Γ -linearized vector bundle $\widehat{U}^* \otimes \widehat{V}$ in place of \widehat{V} in (11) it follows that

$$H^0(X, (U_*^* \otimes V_*)_0) = \text{Hom}_{\text{par}}(U_*, V_*),$$

where $(U_*^* \otimes V_*)_0$ is the vector bundle underlying the parabolic tensor product $U_*^* \otimes V_*$ with U_*^* being the parabolic dual of U_* . In particular, we have

$$H^0(X, (W_*^* \otimes V_*)_0) = \text{Hom}_{\text{par}}(W_*, V_*), \quad (14)$$

where $(W_*^* \otimes V_*)_0$ is the vector bundle underlying the parabolic bundle $W_*^* \otimes V_*$.

In (11), substitute the Γ -linearized vector bundle $\widehat{W}^* \otimes \widehat{V}$ in place of \widehat{V} . Combining this with (13) and (14) we have

$$H^0(Y, \widehat{V}) = \text{Hom}_{\text{par}}(W_*, V_*). \quad (15)$$

Since the bijective correspondence between the parabolic vector bundles and the Γ -linearized vector bundles is compatible with the operations of taking tensor product and dual, the parabolic vector bundle corresponding to the Γ -linearized vector bundle $\widetilde{R}_{r,d|\Gamma}^* \otimes \widehat{F}$ is the parabolic tensor product $(R'_{r,d,*})^* \otimes F_*$, where $(R'_{r,d,*})^*$ is the parabolic dual of $R'_{r,d,*}$.

Now the lemma follows by substituting the Γ -linearized vector bundle $\widetilde{R}_{r,d|\Gamma}^* \otimes \widehat{F}$ in place of \widehat{V} in (15). \square

Let

$$R_{r,d,*} := W_* \otimes R'_{r,d,*} \quad (16)$$

be the parabolic tensor product. In view of (7) and Lemma 3.1, we conclude that the parabolic vector bundle $R_{r,d,*}$ constructed in (16) satisfies the condition in Theorem 1.1.

Step 3: To complete the proof of Theorem 1.1 we now consider the remaining case where $(\text{genus}(X), |S|) = (0, 1)$. So $X = \mathbb{C}\mathbb{P}^1$ and $S = \{x\}$ is a singleton set.

Any vector bundle over $\mathbb{C}\mathbb{P}^1$ decomposes into a direct sum of line bundles [5, p. 126, Théorème 2.1]. Using this it follows immediately that a parabolic vector bundle E_* of rank r on $\mathbb{C}\mathbb{P}^1$ with parabolic structure over x is parabolically semistable if and only if the vector bundle E_0 underlying E_* is a direct sum

$$E_0 = L^{\oplus r},$$

and the quasiparabolic filtration is $0 \subset (E_0)_x$.

Given r and d as in Theorem 1.1, let

$$d_0 = \lfloor d/r \rfloor$$

be the integral part of d/r . Set

$$\alpha := d/r - d_0 \in [0, 1). \quad (17)$$

If $\alpha < (N - 1)/N$, then set $R_{r,d,*}$ to be the line bundle $\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(d_0)$ equipped with parabolic weight $\alpha + 1/N$ at x .

If $\alpha = (N - 1)/N$, then set $R_{r,d,*}$ to be the line bundle $\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(d_0 + 1)$ with the trivial parabolic structure.

Take any parabolic vector bundle E_* of rank r and parabolic degree d on $\mathbb{C}\mathbb{P}^1$ with parabolic structure on x . If E_* is parabolically semistable, then from the above observation that the underlying vector bundle E_0 is a direct sum of r copies of a

line bundle, and the quasiparabolic flag of E_* is trivial, it follows immediately that $\text{Hom}_{\text{par}}(R_{r,d,*}, E_*) = 0$.

Now assume that

$$\text{Hom}_{\text{par}}(R_{r,d,*}, E_*) = 0. \quad (18)$$

Let

$$E_0 = \bigoplus_{i=1}^r L_i$$

be a decomposition into a direct sum of line bundles of the underlying vector bundle E_0 . From (18) it follows immediately that $\text{degree}(L_i)$ is independent of i . Therefore, E_0 is a direct sum of r copies of a line bundle. Again from (18) it follows that all the parabolic weights are at most α defined in (17). Therefore, the parabolic weight must be α with multiplicity r . Consequently, E_* is parabolically semistable. This completes the proof of Theorem 1.1.

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