LATTICE INVARIANTS FROM THE HEAT KERNEL

JUAN MARCOS CERVIÑO AND GEORG HEIN

Abstract. We derive lattice invariants from the heat flux of a lattice. Using spherical theta functions we obtain for integer lattices invariant modular forms. The invariants include the theta function. However, in dimension four we show that our invariants can distinguish two isospectral lattices.

1. Introduction

For a lattice $\Lambda \subset \mathbb{E}^n$ in the n-dimensional euclidean space $\mathbb{E}^n$ one of the most important invariants is its theta function Θ_Λ. However, the theta function alone does not determine the isomorphism class of the lattice. In this article we explain how one obtains further lattice invariants using the heat flux of the lattice Λ. Considering Λ as a distribution on functions on $\mathbb{E}^n$, its heat flux yields a function

$$f_\Lambda : \mathbb{R}^+ \times \mathbb{E}^n \to \mathbb{R}\,.$$

This function itself determines $\Lambda \subset \mathbb{E}^n$, and is hence not a lattice invariant. However, one can use f_Λ to construct invariants of the lattice. For example, the restriction of f_Λ to $\mathbb{R}^+ \times \{0\}$ yields, up to some scaling, the theta function Θ_Λ of the lattice. Another source of invariants is obtained by integrating products of homogeneous parts of f_Λ over the unit sphere $S^{n-1} \subset \mathbb{E}^n$. In order to obtain modular forms, we evaluate differentials associated to homogeneous harmonic polynomials of f_Λ along $\mathbb{R}^+ \times \{0\}$. Whenever we find an invariant sum of such products, then we obtain an invariant modular form. In Section 3 we construct such invariants for lattices $\Lambda \subset \mathbb{E}^2$. If Λ is integral of level N, we obtain as lattice invariants the holomorphic functions on the upper half plane

$$\Theta_{n,n,\Lambda}(\tau) = \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(\cos(2n\measuredangle(\gamma,\delta))\|\gamma\|^{2n}\|\delta\|^{2n}\right) q^{\|\gamma\|^2+\|\delta\|^2} \quad \text{with } q = \exp(2\pi i\tau)$$

which are modular forms with integer coefficients of weight $4n+2$ and of level N. In the final Section 4 we introduce the modular form $\Theta_{1,1,\Lambda}$ for an integral lattice $\Lambda \subset \mathbb{E}^n$. This modular form is of weight $n+4$, has integer coefficients, and is given by

$$\Theta_{1,1,\Lambda}(\tau) = \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(n\cos^2(\measuredangle(\gamma,\delta)) - 1\right) \|\gamma\|^{2n}\|\delta\|^{2n} q^{\|\gamma\|^2+\|\delta\|^2} \quad \text{with } q = \exp(2\pi i\tau)\,.$$

We compute these modular forms for two isospectral lattices in $\mathbb{E}^4$ found by A. Schiemann in [4] and find that they are different. Thus, the modular lattice invariant $\Theta_{1,1,\Lambda}$ can distinguish at least two isospectral lattices.

Notation: By a lattice we mean a free $\mathbb{Z}$ module $\Lambda \cong \mathbb{Z}^n$ with a positive definite quadratic form $Q : \Lambda \to \mathbb{R}$. A lattice possesses isometric embeddings $\Lambda \to \mathbb{E}^n$. Two such embeddings differ by an element of the orthogonal group $\mathrm{O}(n)$. By abuse of notation we call an embedded lattice $\Lambda \subset \mathbb{E}^n$ also a lattice. The length or norm of $\gamma \in \Lambda$ is defined to be $\|\gamma\| := \sqrt{Q(\gamma)}$. We call the number $Q(\gamma)$ the square length of $\gamma \in \Lambda$.

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2. The function f_Λ

2.1. **The heat flux of a lattice.** We consider a cocompact lattice $\Lambda \subset \mathbb{E}^n$ in the n dimensional euclidean space. We consider two lattices $\Lambda, \Lambda' \subset \mathbb{E}^n$ to be isomorphic when there exists an element $\varphi \in \mathrm{O}(n)$ which transforms Λ into Λ', in short $\Lambda' = \varphi(\Lambda)$.
We regard $\Lambda \in D^n(\mathbb{E}^n)$ as a distribution on the Schwartz space $S(\mathbb{E}^n)$ by

$$\Lambda(f) = \sum_{\gamma\in\Lambda} f(\gamma).$$

Smooth n forms ω on $\mathbb{E}^n$ can be considered as currents $[\omega] \in D^n(\mathbb{E}^n)$ by $[\omega]f := \int_{\mathbb{E}^n} \omega \cdot f$. A current $[\omega]$ associated to a smooth form ω is called a smooth current. We want to apply the heat flux to the singular distribution Λ to obtain a smooth current $(\Lambda)_t$ for any positive $t \in \mathbb{R}$. We follow the approach in [2, Section 2]:

$$\text{The Laplace operator is given by:} \qquad \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,.$$

For any smooth k-form $f \in A^k(\mathbb{E}^n)$ we have the heat flux of f:

$$P_t(f) \in A^k(X) \text{ for all } t \in \mathbb{R}_+ \text{ such that } \left(\frac{\partial}{\partial t} + \Delta\right) P_t(f) = 0\,, \text{ and } \lim_{t\to 0} P_t(f) = f.$$

The heat flux of a distribution $T \in D^n(\mathbb{E}^n)$ is defined by

$$P_t(T)(f) := T(P_t(f)).$$

If the distribution is smooth, then we have the equality:

$$P_t([\omega]) = [P_t(\omega)].$$

That is the extension to distributions is compatible with the embedding $A^n(\mathbb{E}^n) \hookrightarrow D^n(\mathbb{E}^n)$ and the heat flux on smooth forms. Furthermore, for $T = \Lambda$ the heat flux smooths the distribution, that is

$$P_t(\Lambda) = [\omega_t] \quad \text{for some smooth } \omega_t \in A^n(\mathbb{E}^n).$$

Using the explicit form of the heat kernel for a point (cf. [1, Chapter 2]) we can write down the heat flux of Λ directly as $\omega_t = f_\Lambda d\mu$, with $d\mu = dx_1 \wedge \ldots dx_n$ the volume form on $\mathbb{E}^n$, and

$$f_\Lambda(t,x) = (4\pi t)^{\frac{-n}{2}} \sum_{\gamma\in\Lambda} \exp\left(\frac{-\|x-\gamma\|^2}{4t}\right)\,.$$

2.2. f_Λ **determines the lattice** $\Lambda \subset \mathbb{E}^n$**.** Indeed we have on the level of distributions $\lim_{t\to+0}[f_\Lambda d\mu] = \Lambda$. However, the function f_Λ depends on the embedding of Λ. We try to deduce more information by considering the values of derivatives of f_Λ along $\mathbb{R}^+ \times \{0\}$.

2.3. **Polynomial derivations.** We need the subring $A = \mathbb{R}[x_1, \ldots, x_n] \subset A^0(\mathbb{E}^n)$ of polynomials in the smooth functions. We define a pairing on

$$A \times A^0(\mathbb{E}^n) \to \mathbb{R}\,, \quad \langle P, f\rangle = P\left(\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n}\right) f|_0\,.$$

Note that the pairing is also well defined when P is smooth and f is a polynomial. We have the following properties of that pairing:

(i) The pairing is bilinear.
(ii) The pairing is symmetric, in the sense that $\langle P, f\rangle = \langle f, P\rangle$ whenever one side is defined.
(iii) the restriction of $\langle\,,\,\rangle$ to $A\times A$ is positive definite.
(iv) The monomials form a orthogonal (not orthonormal!) basis.
(v) For two polynomials $P, Q \in A$ we have $\langle P\cdot Q, f\rangle = \langle P, Q\left(\frac{\partial}{\partial x_1},\dots \frac{\partial}{\partial x_n}\right) f\rangle$.

For any $P\in A$ we obtain with $\langle P, f_\Lambda\rangle$ a smooth function $\langle P, f_\Lambda\rangle : \mathbb{R}^+ \to \mathbb{R}$.

Lemma 2.4. *For the* $\mathrm{O}(n)$*-invariant polynomial* $r^2 := \sum_{i=1}^n x_1^2$ *we have*

$$\langle r^{2k}, f_\Lambda\rangle = \frac{\partial^k}{\partial t^k} f_\Lambda(t,0)\,.$$

Proof. Keeping in mind, that f_Λ is the heat flux of Λ, that is $(\Delta + \frac{\partial}{\partial t})f_\Lambda = 0$, we get

$$\langle P\cdot r^2, f_\Lambda\rangle = \langle P, r^2\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) f_\Lambda\rangle = \langle P, -\Delta(f_\Lambda)\rangle = \langle P, \frac{\partial}{\partial t} f_\Lambda\rangle.$$

The first equality is 2.3 (v). We inductively conclude the assertion of the lemma. □

So evaluation of $\mathrm{O}(n)$-invariant polynomials $P = \sum_i a_i r^{2i}$ gives only expressions of type $\sum_i a_i \frac{\partial^i}{\partial t^i} f_\Lambda(t,0)$. After having seen what $\langle r^{2k}, f_\Lambda\rangle$ is, we study the pairing $\langle h, f_\Lambda\rangle$ for homogeneous harmonic polynomials h, i.e. $\Delta(h) = 0$. We have the following result:

Proposition 2.5. *If* $h\in A$ *is a homogeneous harmonic polynomial of degree* d*, then*

$$(2t)^d\langle h, f_\Lambda\rangle = (4\pi t)^{\frac{-n}{2}} \sum_{\gamma\in\Lambda} h(\gamma)\exp\left(\frac{-\|\gamma\|^2}{4t}\right).$$

In order to prove this proposition we need an auxiliary result:

Lemma 2.6. *Let* h *be a homogeneous polynomial of degree* d*,* $a \in \mathbb{C}^*$*, and* $f = f(x_1,\dots,x_n) = \exp\left(\frac{a}{2}\cdot\sum_{i=1}^n x_i^2\right)$*. Then we have an equality*

$$h\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) f = a^d\left(\sum_{k\geq 0}\left(\frac{-1}{2a}\right)^k \frac{1}{k!}\,\Delta^k(h)\right)\cdot f.$$

Proof. We have to prove the lemma only for h a monomial. We proceed by induction on $d = \deg(h)$. For $d = 0$, the statement is obvious. Suppose the assertion holds for h' of degree $d-1$. We set $h = x_i h'$. A direct calculation shows that

$$\Delta(h) = x_i\Delta(h') - 2\frac{\partial}{\partial x_i}h'.$$

Since Δ and $\frac{\partial}{\partial x_i}$ commute we deduce from that formula inductively that

$$\Delta^k(h) = x_i\Delta^k(h') - 2k\frac{\partial}{\partial x_i}\Delta^{k-1}(h') \tag{1}$$

holds for all integers $k\geq 0$. Now we compute using the induction hypothesis, the Leibniz rule, and equation (1):

$$\begin{aligned} h\left(\tfrac{\partial}{\partial x_1},\dots,\tfrac{\partial}{\partial x_n}\right) f &= \tfrac{\partial}{\partial x_i}\left(a^{d-1}\left(\textstyle\sum_{k\geq 0}\left(\tfrac{-1}{2a}\right)^k \tfrac{1}{k!}\,\Delta^k(h')\right)\cdot f\right)\\ &= a^d\left(\textstyle\sum_{k\geq 0}\left(\tfrac{-1}{2a}\right)^k \tfrac{1}{k!}(x_i\Delta^k(h') - 2k\tfrac{\partial}{\partial x_i}\Delta^{k-1}(h'))\right)\cdot f\\ &= a^d\left(\textstyle\sum_{k\geq 0}\left(\tfrac{-1}{2a}\right)^k \tfrac{1}{k!}\,\Delta^k(h)\right)\cdot f\,. \end{aligned}$$

□

Proof of Proposition 2.5. Setting $a = \frac{-1}{2t}$ in Lemma 2.6 we obtain that

$$h\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \exp\left(\frac{-\sum_{i=1}^n x_i^2}{4t}\right) = \left(\frac{-1}{2t}\right)^d h(x_1, \ldots, x_n) \exp\left(\frac{-\sum_{i=1}^n x_i^2}{4t}\right).$$

Substituting $x_i = \tilde{x}_i - \gamma_i$ for some $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Lambda$ yields

$$h\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \exp\left(\frac{-\|\gamma - x\|^2}{4t}\right) = \left(\frac{-1}{2t}\right)^d h(x_1 - \gamma_1, \ldots, x_n - \gamma_n) \exp\left(\frac{-\|\gamma - x\|^2}{4t}\right)$$

Multiplying this equality with $(4\pi t)^{\frac{-n}{2}}$ and summing up over all $\gamma \in \Lambda$ yields the statement of the proposition when specializing to $x = 0$. □

2.7. **Definition of the $c_{k_1,k_2,\ldots,k_m}$.** In order to find an invariant we consider the Taylor expansion of f_Λ around $\mathbb{R}^+ \times \{0\}$, $f_\Lambda = f_{\Lambda,0} + f_{\Lambda,1} + \ldots$ with

$$f_{\Lambda,k} = \sum_{I \subset \mathbb{N}^n, |I| = 2k} a_I \frac{x^I}{I!} \text{ where for } I = (i_1, i_2, \ldots, i_n) \;\; I! := \prod_{m=1}^n i_m! \,,\; x^I := \prod_{m=1}^n x_m^{i_m} \;, \text{ and}$$

$$a_I := \langle x^I, f_\Lambda \rangle = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} f_\Lambda|_{\mathbb{R}^+ \times \{0\}}.$$

These polynomials are not $\mathrm{O}(n)$-invariant but the integrals of their products

$$c_{k_1,k_2,\ldots,k_m} := \int_{S^{n-1}} f_{\Lambda,k_1} \cdot f_{\Lambda,k_2} \cdot \ldots \cdot f_{\Lambda,k_m} d\bar{\mu}$$

are. Here $d\bar{\mu}$ denotes the normalized $\mathrm{O}(n)$-invariant measure on S^{n-1} such that $\int_{S^{n-1}} d\bar{\mu} = 1$. We restrict to the parts of even degree because the parts of odd degree vanish for f_Λ being an even function. We will see that the $c_{k_1,k_2,\ldots,k_m}$ can be expressed as sums of products of type $\langle g, f_\Lambda \rangle$. The decomposition of a polynomial $g \in A$ into a sum $g = h_n + r^2 h_{n-2} + r^4 h_{n-4} + \ldots$ with h_i harmonic (see Appendix) will allow applications of the following:

2.8. **Spherical theta function principle.** We want to compute these invariants and relate them to modular forms for integral lattices. Here we call a lattice $\Lambda \subset \mathbb{E}^n$ integral if $\|\gamma\|^2 \in \mathbb{N}$ for all $\gamma \in \Lambda$. Such a lattice has a level $N \in \mathbb{N}$ and a discriminant D (cf. [5, Chapter 3.1]). For a homogeneous harmonic function h on $\mathbb{E}^n$ of degree d we have the spherical theta function $\Theta_{h,\Lambda}$ defined by

$$\Theta_{h,\Lambda}(\tau) = \sum_{\gamma \in \Lambda} h(\gamma) q^{\|\gamma\|^2}, \qquad q = \exp(2\pi i \tau).$$

These spherical theta functions are modular forms of weight $\frac{n}{2} + d$, level N, and character $\left(\frac{D}{\cdot}\right)$ (see again [5, Chapter 3.2]).

Definition. A harmonic invariant datum $p = ((h_{ij})_{i=1,..,m\; j=1,\ldots,k})$ for lattices in $\mathbb{E}^n$ consists of harmonic homogeneous polynomials h_{ij} of degree d_i on $\mathbb{E}^n$, such that the map

$$p : \{\text{lattices } \Lambda \subset \mathbb{E}^n\} \to \{\text{smooth functions } \mathbb{R}^+ \to \mathbb{C}\} \qquad \Lambda \mapsto p(\Lambda)$$

$$\text{by } p(\Lambda) := \sum_{j=1}^k \prod_{i=1}^m \langle h_{ij}, f_\Lambda \rangle \qquad \text{where} \qquad f_\Lambda(t,x) = (4\pi t)^{\frac{-n}{2}} \sum_{\gamma \in \Lambda} \exp\left(\frac{-\|x - \gamma\|^2}{4t}\right),$$

which satisfies $p(\Lambda) = p(\varphi(\Lambda))$ for all $\varphi \in \mathrm{O}(n)$. The sum $d = \sum_{i=1}^m d_i$ is called the degree of the harmonic invariant datum. The harmonic invariant datum is called even (resp. odd) when m is even (resp. odd).

The connection between harmonic invariant data and invariant modular forms is given the by the following proposition.

Proposition 2.9. *Suppose* $p = ((h_{ij})_{i=1,..,m\ j=1,...,k})$ *is a harmonic invariant datum of degree* d *for lattices in* $\mathbb{E}^n$. *Then for any integral lattice* $\Lambda \subset \mathbb{E}^n$ *of level* N *the modular form*

$$\Theta_{p,\Lambda}(\tau) := \sum_{j=1}^{k} \prod_{i=1}^{m} \Theta_{h_{ik},\Lambda}$$

is invariant under $\mathrm{O}(n)$. $\Theta_{p,\Lambda}(\tau)$ *is a modular form of weight* $\frac{nm}{2}+d$, *and of level* N. *If the harmonic invariant datum is odd, then* $\Theta_{p,\Lambda}(\tau)$ *has character* $\left(\frac{D}{\cdot}\right)$ *with* D *the discriminant of the lattice. If it is even, then* $\Theta_{p,\Lambda}(\tau)$ *is a modular form for the trivial character.*

Proof. First we remark that $\Theta_{p,\Lambda}$ being a sum of products of modular forms of level N is a modular form of level N and the stated weight. Likewise we see that it is a modular form of character $\left(\frac{D}{\cdot}\right)^m$. Thus, it remains to show that $\Theta_{p,\Lambda}$ is invariant under $\mathrm{O}(n)$. We start with calculating $p(\Lambda)(t)$ using Proposition 2.5:

$$p(\Lambda)(t) = \sum_{j=1}^{k}\prod_{i=1}^{m}\langle h_{ij}, f_\Lambda\rangle = \sum_{j=1}^{k}\prod_{i=1}^{m}(2t)^{-d_i}(4\pi t)^{\frac{-n}{2}}\sum_{\gamma\in\Lambda} h_{ij}(\gamma)\exp\left(\frac{-\|\gamma\|^2}{4t}\right).$$

Since the d_i sum up to d we get that

$$(2t)^d(4\pi t)^{\frac{mn}{2}}p(\Lambda)(t) = \sum_{j=1}^{k}\prod_{i=1}^{m}\sum_{\gamma\in\Lambda} h_{ij}(\gamma)\exp\left(\frac{-\|\gamma\|^2}{4t}\right).$$

By the definition of a harmonic invariant datum, the left hand side is $\mathrm{O}(n)$-invariant. Moreover, it is the value of $\Theta_{p,\Lambda}\left(\frac{i}{8\pi t}\right)$. By the identity theorem for holomorphic functions the values of the modular form $\Theta_{p,\Lambda}$ along the imaginary axis determine this function, and hence it is $\mathrm{O}(n)$-invariant. ☐

2.10. **Example: The theta series.** The function $f_\Lambda(t,0) = (4\pi t)^{\frac{-n}{2}}\sum_{\gamma\in\Lambda}\exp(-\|\gamma\|^2/4t)$ is a lattice invariant. The theta function of the lattice Λ is

$$\Theta_\Lambda(\tau) := \sum_{\gamma\in\Lambda}\exp(2\pi i\tau\|\gamma\|^2).$$

So up to the scaling factor $(4\pi t)^{\frac{-n}{2}}$ the theta function $\Theta_\Lambda\left(\frac{i}{8\pi t}\right)$ gives $f_\Lambda(t,0)$:

$$f_\Lambda(t,0) = (4\pi t)^{\frac{-n}{2}}\Theta_\Lambda\left(\frac{i}{8\pi t}\right).$$

Vice versa, from $f_\Lambda(t,0)$ we can extract the values of Θ_Λ along the imaginary line, which determines Θ_Λ by the identity theorem for holomorphic functions.
Setting $h_{11}\equiv 1$ we obtain a harmonic invariant datum $f_\Lambda(t,0) = (1)$ of degree zero with $m = k = 1$. This harmonic invariant datum yields the theta function of the lattice up to the factor $(4\pi t)^{\frac{n}{2}}$. We show in the sequel that we can build more harmonic invariant data from our lattice invariants $c_{k_1,\ldots,k_r}$.

3. Lattices in $\mathbb{E}^2$

3.1. **Preparations.** Let $\Lambda \subset \mathbb{E}^2$ be a fixed lattice. From Corollary A.2 we deduce the following table of integrals which will be useful for the following calculations:

$$\begin{array}{rclrclrcl} \int_{S^1} x^2 d\bar\mu &=& \frac{1}{2} & \int_{S^1} x^2y^2 d\bar\mu &=& \frac{1}{8} & \int_{S^1} x^4 d\bar\mu &=& \frac{3}{8} \\ \int_{S^1} x^4y^2 d\bar\mu &=& \frac{1}{16} & \int_{S^1} x^6 d\bar\mu &=& \frac{5}{16} & \int_{S^1} x^4y^4 d\bar\mu &=& \frac{3}{128} \\ \int_{S^1} x^6y^2 d\bar\mu &=& \frac{5}{128} & \int_{S^1} x^8 d\bar\mu &=& \frac{35}{128} \end{array}$$

By definition $c_0 = f_\Lambda(t,0) = \frac{\Theta_\Lambda(\frac{i}{8\pi t})}{4\pi t}$, so we see that c_0 is essentially the theta function of the lattice. Next we compute

$$4c_1 = 4\int_{S_1}\left(a_{20}\frac{x^2}{2} + a_{11}xy + a_{02}\frac{y^2}{2}\right) d\bar\mu = a_{20} + a_{02}\,.$$

Thus, we obtain $c_1 = \frac{1}{4}\frac{\partial}{\partial t}c_0$. This follows a general pattern: $c_2 = \frac{1}{64}\frac{\partial^2}{\partial t^2}c_0$, $c_3 = \frac{1}{2304}\frac{\partial^3}{\partial t^3}c_0$, and in general $c_n = \frac{1}{4^n(n!)^2}\frac{\partial^n}{\partial t^n}c_0$.

3.2. **The harmonic invariant datum $p_{1,1}$ for lattices in $\mathbb{E}^2$.** We start with the computation of the lattice invariant $c_{1,1}$:

$$\begin{aligned} 32c_{1,1} &= 32\int_{S^1}\left(a_{20}\frac{x^2}{2} + a_{11}xy + a_{02}\frac{y^2}{2}\right)^2 d\bar\mu \\ &= 3a_{20}^2 + 3a_{02}^2 + 4a_{11}^2 + 2a_{20}a_{02} \\ &= 2(a_{20}+a_{02})^2 + 4a_{11}^2 + (a_{20}-a_{02})^2\,. \end{aligned}$$

The function $p_{1,1} = 32c_{1,1} - 32c_1^2 = 4a_{11}^2 + (a_{20}-a_{02})^2$ is therefore a harmonic invariant datum:

$$p_{1,1} = 4\langle xy, f_\Lambda\rangle^2 + \langle x^2-y^2, f_\Lambda\rangle^2$$

This harmonic invariant datum yields by Proposition 2.9:

Corollary 3.3. *For any integral lattice $\Lambda \subset \mathbb{E}^2$ of level N the modular form*

$$\Theta_{1,1,\Lambda} = 4\Theta^2_{xy,\Lambda} + \Theta^2_{x^2-y^2,\Lambda}$$

of weight 6 and level N is invariant under the $\mathrm{O}(2)$-action on the embeddings $\Lambda \to \mathbb{E}^2$. If we write $\Theta_{1,1,\Lambda}(\tau) = \sum_{n\in\mathbb{N}} a_n q^n$ with $q = \exp(2\pi i\tau)$, then the a_n are given by

$$a_n = \sum_{\substack{(\gamma,\delta)\in\Lambda\times\Lambda \\ \|\gamma\|^2+\|\delta\|^2=n}} \cos(2\measuredangle(\gamma,\delta))\|\gamma\|^2\|\delta\|^2.$$

Proof. The only thing which remains to be shown is the formula for the a_n. This is the result of a straightforward calculation, where we expand $4\Theta^2_{xy,\Lambda}$ and $\Theta^2_{x^2-y^2,\Lambda}$ as sums over $\Lambda\times\Lambda$, and check that the summand of $q^{\|\gamma\|^2+\|\delta\|^2}$ is $\cos(2\measuredangle(\gamma,\delta))\|\gamma\|^2\|\delta\|^2$ for each pair $(\gamma,\delta)\in\Lambda\times\Lambda$. See the proof of Theorem 3.6 for this calculation. □

3.4. **The harmonic invariant datum $p_{2,2}$ for lattices in $\mathbb{E}^2$.** We start with a decomposition of $c_{2,2}$ into three summands

$$\begin{aligned} 73728c_{2,2} &= 6a_{40}a_{04} + 96a_{31}a_{13} + 60a_{22}a_{04} + 60a_{40}a_{22} + 108{a_{22}}^2 + \\ &\qquad + 80{a_{13}}^2 + 35{a_{40}}^2 + 35{a_{04}}^2 + 80{a_{31}}^2 \\ &= p_{2,2} + s_1 + s_2 \\ \text{with}\quad p_{2,2} &= (a_{40} - 6a_{22} + a_{04})^2 + 16(a_{31}-a_{13})^2 \\ s_1 &= 18(a_{40} + 2a_{22} + a_{04})^2 \\ s_2 &= 16((a_{20}+a_{02})\circ(a_{20}-a_{02}))^2 + 64((a_{20}+a_{02})\circ a_{11})^2 \end{aligned}$$

If we can show that s_1 and s_2 are invariant under the O(2)-action, then it follows that

$p_{2,2}$ is also invariant under that action. However, we see that $s_1 = 73728c_2^2$, which is an invariant. Next we see that

$$s_2 = 16\left(4\langle xy, \frac{\partial}{\partial t} f_\Lambda\rangle^2 + \langle x^2-y^2, \frac{\partial}{\partial t} f_\Lambda\rangle^2\right).$$

As we have see before Corollary 3.3 this is also invariant under the O(2)-action. Eventually we come up with a further harmonic invariant datum:

$$p_{2,2} = \langle x^4 - 6x^2y^2 + y^4, f_\Lambda\rangle^2 + 16\langle xy(x^2-y^2), f_\Lambda\rangle^2 .$$

Using this harmonic invariant datum gives by Proposition 2.9:

Corollary 3.5. *For an integral lattice $\Lambda \subset \mathbb{E}^2$ the modular form*

$$\Theta_{2,2,\Lambda} = \Theta^2_{x^4-6x^2y^2+y^4,\Lambda} + 16\Theta^2_{xy(x^2-y^2),\Lambda}$$

of weight 10 is invariant under the O(2)*-action on the embeddings $\Lambda \to \mathbb{E}^2$. The q-expansion of $\Theta_{2,2,\Lambda} = \sum_{n\geq 1} a_n q^n$ is given by*

$$a_n = \sum_{\substack{(\gamma,\delta)\,\in\,\Lambda\times\Lambda\\ \|\gamma\|^2+\|\delta\|^2=n}} \cos(4\measuredangle(\gamma,\delta))\|\gamma\|^4\|\delta\|^4.$$

Proof. It is enough to derive from the expression $\Theta_{2,2,\Lambda} = \Theta^2_{x^4-6x^2y^2+y^4,\Lambda} + 16\Theta^2_{xy(x^2-y^2),\Lambda}$ the stated q-expansion which is an elementary calculation. However, since this is a special case of Theorem 3.6, we omit the proof. □

Theorem 3.6. *For a positive integer $n \in \mathbb{Z}$ we take the two harmonic polynomials $h_1(x,y) = \mathrm{Re}((x+iy)^{2n})$, and $h_2(x,y) = \mathrm{Im}((x+iy)^{2n})$ of degree $2n$. For any integral lattice $\Lambda \subset \mathbb{E}^2$ of level N there is a* O(2)*-invariant modular form*

$$\Theta_{n,n,\Lambda}(\tau) := \Theta^2_{h_1,\Lambda}(\tau) + \Theta^2_{h_2,\Lambda}(\tau)$$

of level N, and weight $2+4n$. Its q-expansion $\Theta_{n,n,\Lambda}(\tau) = \sum_{n\geq 0} a_n q^n$ is given by

$$a_n = \sum_{\substack{(\gamma,\delta)\,\in\,\Lambda\times\Lambda\\ \|\gamma\|^2+\|\delta\|^2=n}} \cos(2n\measuredangle(\gamma,\delta))\|\gamma\|^{2n}\|\delta\|^{2n}.$$

Furthermore, $\Theta_{n,n,\Lambda} \in (4q^{2k}) \subset \mathbb{Z}[[q]]$ where k is the first minimal square length of the lattice.

Proof. We identify $\mathbb{E}^2$ with $\mathbb{C}$, and compute the q-expansion of $\Theta_{n,n,\Lambda}$.

$$\begin{aligned}
\Theta_{n,n,\Lambda}(\tau) &= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(h_1(\gamma)h_1(\delta) + h_2(\gamma)h_2(\delta)\right) q^{\|\gamma\|^2+\|\delta\|^2}\\
&= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(\mathrm{Re}(\gamma^{2n})\,\mathrm{Re}(\delta^{2n}) + \mathrm{Im}(\gamma^{2n})\,\mathrm{Im}(\delta^{2n})\right) q^{\|\gamma\|^2+\|\delta\|^2}\\
&= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(\mathrm{Re}(\gamma^{2n}\bar\delta^{2n})\right) q^{\|\gamma\|^2+\|\delta\|^2}\\
&= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(\mathrm{Re}\left(\tfrac{\gamma^n\bar\delta^n}{\bar\gamma^n\delta^n}\right)\gamma^n\bar\gamma^n\delta^n\bar\delta^n\right) q^{\|\gamma\|^2+\|\delta\|^2}\\
&= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(\cos(2n\measuredangle(\gamma,\delta))\|\gamma\|^{2n}\|\delta\|^{2n}\right) q^{\|\gamma\|^2+\|\delta\|^2}.
\end{aligned}$$

We deduce from this form of the q-expansion that $a_0 = a_1 = \cdots = a_{2k-1} = 0$, and that $\Theta_{n,n,\Lambda}$ does not depend on the embedding $\Lambda \subset \mathbb{E}^2$. It remains to prove that $a_n \in 4\,\mathbb{Z}$. To show this, we use from the above deduction that $a_n = \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \mathrm{Re}(\gamma^{2n}\bar\delta^{2n})$ where we sum over all pairs (γ,δ) with $\|\gamma\|^2+\|\delta\|^2 = n$. The symmetric bilinear form $\psi : \Lambda\times\Lambda \to \mathbb{R}$ with $\psi(\gamma,\delta) := \mathrm{Re}(\gamma\bar\delta)$ satisfies $\psi(\gamma,\gamma) \in \mathbb{Z}$ because Λ is integral. Thus, $\psi(\gamma,\delta) \in \frac{1}{2}\mathbb{Z}$. We deduce that for $\gamma, \delta \in \Lambda$ the element $x := \gamma\bar{\delta}$ satisfies a quadratic equation $x^2 - b_1 x + b_2 = 0$ with $b_1 = 2\operatorname{Re}(\gamma\bar{\delta}) \in \mathbb{Z}$, and $b_2 = \|\gamma\|^2\|\delta\|^2 \in \mathbb{Z}$. Thus, x is an integer in a (at most) quadratic field extension $K/\mathbb{Q}$. Consequently, x^{2n} is an integer in this field K which implies $2\operatorname{Re}(\gamma^{2n}\bar{\delta}^{2n}) \in \mathbb{Z}$. If $\gamma \neq \pm\delta$, then the summand $\operatorname{Re}(\gamma^{2n}\bar{\delta}^{2n})$ appears eight times in a_n, namely from the pairs $(\pm\gamma, \pm\delta)$ and $(\pm\delta, \pm\gamma)$. If $\gamma = \pm\delta$, then $\operatorname{Re}(\gamma^{2n}\bar{\delta}^{2n})$ is an integer and appears four times. □

Example 1. Some invariants for the lattice Λ associated to $q(x, y) = x^2 + y^2$.

$$\begin{aligned}
\Theta_\Lambda(\tau) &= 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13} + 4q^{16} + 8q^{17} + \dots \\
\Theta_{2,2,\Lambda}(\tau) &= 16(q^2 - 8q^3 + 16q^4 + 32q^5 - 156q^6 + 112q^7 + 256q^8 - 576q^9 + \dots) \\
\Theta_{4,4,\Lambda}(\tau) &= 16(q^2 + 32q^3 + 256q^4 + 512q^5 + 6084q^6 - 33728q^7 + 65536q^8 + \dots) \\
\Theta_{1,1,\Lambda}(\tau) &= 0 = \Theta_{3,3,\Lambda}(\tau)\,.
\end{aligned}$$

Example 2. Some invariants for the lattice Λ associated to $q(x, y) = x^2 + xy + y^2$.

$$\begin{aligned}
\Theta_\Lambda(\tau) &= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + 12q^{19} + \dots \\
\Theta_{3,3,\Lambda}(\tau) &= 36(q^2 - 54q^4 + 128q^5 + 729q^6 - 3456q^7 + 3524q^8 + 16902q^{10} + \dots) \\
\Theta_{1,1,\Lambda}(\tau) &= 0 = \Theta_{2,2,\Lambda}(\tau) = \Theta_{4,4,\Lambda}(\tau)\,.
\end{aligned}$$

4. The harmonic invariant datum $p_{1,1}$ for lattices in $\mathbb{E}^n$

4.1. **An explicit formula for $c_{1,1}$ and definition of $p_{1,1}$.** Let $\Lambda \subset \mathbb{E}^n$ be a lattice. As explained in 2.1 the heat flux of the distribution Λ is given by $\omega_{\Lambda,t} = f_\Lambda dx_1 \wedge \dots dx_n$ with

$$f_\Lambda(t, x) = (4\pi t)^{\frac{-n}{2}} \sum_{\gamma \in \Lambda} \exp\left(\frac{-\|x - \gamma\|^2}{4t}\right).$$

The value of $f_\Lambda(t, 0)$ is the first lattice invariant. To derive the lattice invariant $c_{1,1}$ we need the second Taylor coefficient of f_Λ in zero. We write it in the following manner:

$$f_{\Lambda,1} = \sum_{i=1}^{n} \frac{a_i}{2} x_i^2 + \sum_{1 \le i < j \le n} b_{ij} x_i x_j$$

where $a_i = \langle x_i^2, f_\Lambda \rangle = \frac{\partial^2}{\partial x_i^2} f_\Lambda|_{x=0}$ and $b_{ij} = \langle x_i x_j, f_\Lambda \rangle = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f_\Lambda|_{x=0}$. The invariant $c_{1,1}$ is defined to be the integral $c_{1,1} := \int_{S^{n-1}} f_{\Lambda,1}^2 d\bar{\mu}$.

As a preparation we compute $c_1 := \int_{S^{n-1}} f_{\Lambda,1} d\bar{\mu}$. Using the integral formulas $\int_{S^{n-1}} x_i^2 d\bar{\mu} = \frac{1}{n}$ and $\int_{S^{n-1}} x_i x_j d\bar{\mu} = 0$ (cf. Corollary A.2) we see that

$$c_1 = \frac{1}{2n}(\sum_{i=1}^{n} a_i) = \frac{-1}{2n} \Delta f_\Lambda|_{x=0} = \frac{1}{2n}\frac{\partial f_\Lambda}{\partial t}|_{x=0}.$$

We need the following integrals from Corollary A.2:

$$\int_{S^{n-1}} x_i^4 d\bar{\mu} = \frac{3}{n(n+2)}\,, \qquad \int_{S^{n-1}} x_i^2 x_j^2 d\bar{\mu} = \frac{1}{n(n+2)}\,.$$

Having in mind (Corollary A.2) that the integrals of homogeneous monomials over S^{n-1} vanish if an odd exponent occurs we find the following:

$$\begin{aligned}
4n(n+2)c_{1,1} &= 4 \sum_{1 \le i < j \le n} b_{ij}^2 + 3\sum_{i=1}^{n} a_i^2 + 2 \sum_{1 \le i < j \le n} a_i a_j \\
&= 4 \sum_{1 \le i < j \le n} b_{ij}^2 + 2\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j.
\end{aligned}$$

We want to express $c_{1,1}$ in terms of differential operators which correspond either to harmonic polynomials or to $r := a_1 + a_2 + \cdots + a_n$. This r corresponds to $-\Delta$, as $r = -\Delta(f)|_{x=0}$. So we introduce the harmonic polynomials $h_i = na_i - r$. Using the obvious equality $\sum_{i=1}^n h_i = 0$ we obtain:

$$\begin{aligned}
4n^3(n+2)c_{1,1} &= 4n^2 \sum_{1\le i<j\le n} b_{ij}^2 + 2\sum_{i=1}^n (na_i)^2 + \sum_{i=1}^n\sum_{j=1}^n (na_i)(na_j)\\
&= 4n^2 \sum_{1\le i<j\le n} b_{ij}^2 + 2\sum_{i=1}^n (h_i+r)^2 + \sum_{i=1}^n\sum_{j=1}^n (h_i+r)(h_j+r)\\
&= \left(4n^2 \sum_{1\le i<j\le n} b_{ij}^2 + 2\sum_{i=1}^n h_i^2\right) + \left((n^2+2n)r^2\right).
\end{aligned}$$

So the invariant $c_{1,1}$ decomposes as $c_{1,1} = \frac{1}{2n^3(n+2)}p_{1,1} + \frac{r^2}{4n^2}$. The second summand corresponds to $c_1^2 = \left(\frac{1}{2n}\frac{\partial}{\partial r} f_\Lambda|_{x=0}\right)^2$. Hence, the first summand is also invariant. It yields a harmonic invariant datum:

$$p_{1,1} = 2n^2 \sum_{1\le i<j\le n} b_{ij}^2 + \sum_{i=1}^n h_i^2 = 2n^2 \sum_{1\le i<j\le n} \langle x_i x_j, f_\Lambda\rangle^2 + \sum_{i=1}^n \langle h_i, f_\Lambda\rangle^2.$$

Remark. Note that for $n = 2$ we have $h_1 = -h_2$, so $h_1^2 = h_2^2$. Thus, in this case we have $p_{1,1} = 2\left(4\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f_\Lambda|_{x=0}\right)^2 + \left((\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2^2}) f_\Lambda|_{x=0}\right)^2\right)$. Up to the factor 2 this is our harmonic system from 3.2.

Applying Proposition 2.9 we obtain the next result:

Theorem 4.2. *Let $\Lambda \subset \mathbb{E}^n$ be an integral lattice, of level N. The modular form*

$$\Theta_{1,1,\Lambda}(\tau) = 2n^2\left(\sum_{1\le i<j\le n} \Theta^2_{x_ix_j,\Lambda}(\tau)\right) + \sum_{i=1}^n \Theta^2_{nx_i^2 - \sum_{j=1}^n x_j^2,\Lambda}(\tau)$$

is independent from the embedding $\Lambda \to \mathbb{E}^n$. $\Theta_{1,1,\Lambda}$ is a cusp form of weight $n+4$ of level N. Its q-expansion is given by

$$\Theta_{1,1,\Lambda}(\tau) = \sum_{m\ge 0} a_m q^m \quad \textit{with} \quad a_m = n^2 \sum_{\substack{(\gamma,\delta)\,\in\,\Lambda\times\Lambda\\ \|\gamma\|^2+\|\delta\|^2=m}} \left(\cos^2(\measuredangle(\gamma,\delta)) - \frac{1}{n}\right)\|\gamma\|^2\|\delta\|^2.$$

We have $\Theta_{1,1,\Lambda} \in (2nq^{2k}) \subset \mathbb{Z}[[q]]$ with k the first minimal square length of the lattice Λ. If n is even, then $\Theta_{1,1,\Lambda} \in (4nq^{2k}) \subset \mathbb{Z}[[q]]$.

Proof. On the one hand, when considering the q-expansion it is obvious that $\Theta_{1,1,\Lambda}$ is invariant under the $\mathrm{O}(n)$-action. On the other hand, being a sum of squares of modular forms of weight $\frac{n}{2}+2$ makes $\Theta_{1,1,\Lambda}$ into a modular form of weight $n+4$. Even though $\Theta_{1,1,\Lambda}$ comes from a harmonic invariant datum, we only have to show that the modular form has the stated q-expansion.

$$\begin{aligned}
\Theta_{1,1,\Lambda}(\tau) &= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda}\left(2n^2\sum_{1\le i<j\le n}\gamma_i\gamma_j\delta_i\delta_j + \sum_{i=1}^n (n\gamma_i^2 - \|\gamma\|^2)(n\delta_i^2 - \|\delta\|^2)\right) q^{\|\gamma\|^2+\|\delta\|^2}\\
&= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda}\left(n^2\sum_{i=1}^n\sum_{j=1}^n \gamma_i\gamma_j\delta_i\delta_j - n\|\gamma\|^2\|\delta\|^2\right) q^{\|\gamma\|^2+\|\delta\|^2}\\
&= \sum_{(\gamma,\delta)\in\Lambda\times\Lambda} \left(n^2\langle\gamma,\delta\rangle^2 - n\|\gamma\|^2\|\delta\|^2\right) q^{\|\gamma\|^2+\|\delta|^2}
\end{aligned}$$

Now the definition of the cosine gives the formula for the q-expansion. From this formula we conclude that $a_0 = a_1 = \cdots = a_{2k-1} = 0$. In order to prove that $a_m \in 2n\mathbb{Z}$ we consider the sum

$$\frac{1}{n}a_m = \sum_{\substack{(\gamma,\delta)\in\Lambda\times\Lambda\\ \|\gamma\|^2+\|\delta\|^2=m}} \left(n\langle\gamma,\delta\rangle^2 - \|\gamma\|^2\|\delta\|^2\right) .$$

Since Λ is integral, we have $n\langle\gamma,\delta\rangle^2 \in \frac{1}{4}\mathbb{Z}$ (respectively in $\frac{1}{2}\mathbb{Z}$ when n is even). If $\gamma \neq \pm\delta$, then the eight pairs $(\pm\gamma,\pm\delta)$ and $(\pm\delta,\pm\gamma)$ in $\Lambda\times\Lambda$ give the same contribution $(n\langle\gamma,\delta\rangle^2 - \|\gamma\|^2\|\delta\|^2)$ to a_m. If $\gamma \neq \pm\delta$, then $\langle\gamma,\delta\rangle$ is an integer and the integer summand $(n\langle\gamma,\delta\rangle^2 - \|\gamma\|^2\|\delta\|^2)$ appears four times. □

4.3. **Example: Computing $\Theta_{1,1,\Lambda}$ for two isospectral lattices in dimension four.** We consider the two integral lattices Λ_1 and Λ_2 in $\mathbb{E}^4$ which were investigated by A. Schiemann in [4]. The Gram matrices of these lattices are given by

$$A_1 = \frac{1}{2}\begin{pmatrix} 4 & 2 & 0 & 1\\ 2 & 8 & 3 & 1\\ 0 & 3 & 10 & 5\\ 1 & 1 & 5 & 10 \end{pmatrix} \qquad A_2 = \frac{1}{2}\begin{pmatrix} 4 & 0 & 1 & 1\\ 0 & 8 & 1 & -4\\ 1 & 1 & 8 & 2\\ 1 & -4 & 2 & 10 \end{pmatrix}$$

Schiemann showed in his article that these two lattices are not isometric even though they have the same theta function $\Theta(\tau) = \Theta_{\Lambda_1}(\tau) = \Theta_{\Lambda_2}(\tau)$, which he determined to be

$$\Theta(\tau) = 1+2q^2+4q^4+6q^5+10q^6+6q^7+12q^8+6q^9+6q^{10}+8q^{11}+10q^{12}+8q^{13}+10q^{14}+22q^{15}+\dots .$$

Proposition 4.4. *We have an inequality $\Theta_{1,1,\Lambda_1} \neq \Theta_{1,1,\Lambda_2}$. Thus, the modular forms $\Theta_{1,1,\Lambda_1}$ and $\Theta_{1,1,\Lambda_2}$ of level 1729 and weight eight distinguish the two isospectral lattices Λ_1 and Λ_2.*

Proof. We need embeddings of $\Lambda_i \to \mathbb{E}^4$. We choose a decomposition $A_i = S_i^t \cdot S_i$ with S_i upper triangular. We obtain

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 & 0 & \frac{1}{2}\\ 0 & \sqrt{7} & \frac{3\sqrt{7}}{7} & \frac{\sqrt{7}}{14}\\ 0 & 0 & \frac{\sqrt{427}}{7} & \frac{67\sqrt{427}}{854}\\ 0 & 0 & 0 & \frac{\sqrt{105469}}{122} \end{pmatrix}.$$

The column vectors $\{\gamma_i\}_{i=1,2,3,4}$ of the matrix S_1 generate the lattice Λ_1. From the theta function we see that there are two vectors of norm $\sqrt{2}$ $(\pm\gamma_1)$, and four vectors of norm two $(\pm\gamma_2$, and $\pm(\gamma_1-\gamma_2))$. These six lattice vectors are enough to prove the claim. We compute now in the ring $A = \mathbb{R}[[q]]$. The six spherical theta functions $\Theta_{\Lambda_1,x_ix_j}$ for $i<j$ have zero coefficient at q^2. Since there are no lattice vectors of norm $\sqrt{3}$ we find $\Theta_{\Lambda_1,x_ix_j} \in (q^4)$. Thus, we deduce $\Theta^2_{\Lambda_1,x_ix_j} \in (q^8)$.
Considering the lattice vectors of norm at most 2, we see

$$\begin{aligned}
\Theta_{\Lambda_1,4x_1^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= 12q^2 - 8q^4 + \dots\\
\Theta_{\Lambda_1,4x_2^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= -4q^2 + 40q^4 + \dots\\
\Theta_{\Lambda_1,4x_3^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= -4q^2 - 16q^4 + \dots\\
\Theta_{\Lambda_1,4x_4^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= -4q^2 - 16q^4 + \dots
\end{aligned}$$

Taking the squares of these equations we obtain

$$\begin{aligned}
\Theta^2_{\Lambda_1,4x_1^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= 144q^4 - 192q^6 + \dots \\
\Theta^2_{\Lambda_1,4x_2^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= 16q^4 - 320q^6 + \dots \\
\Theta^2_{\Lambda_1,4x_3^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= 16q^4 + 128q^6 + \dots \\
\Theta^2_{\Lambda_1,4x_4^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= 16q^4 + 128q^6 + \dots .
\end{aligned}$$

Summing up these four squares we obtain that

$$\Theta_{1,1,\Lambda_1}(\tau) = 192q^4 - 256q^6 + \dots .$$

We repeat this construction now with the lattice Λ_2. In this case we find

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2\sqrt{2} & \frac{\sqrt{2}}{4} & -\sqrt{2} \\ 0 & 0 & \frac{\sqrt{122}}{4} & \frac{9\sqrt{122}}{122} \\ 0 & 0 & 0 & \frac{\sqrt{105469}}{122} \end{pmatrix} .$$

The vectors of norm $\sqrt{2}$ are $\pm\gamma_1$, whereas the vectors of norm two are the four vectors $\pm\gamma_2$, and $\pm\gamma_3$. Again, we have $\Theta^2_{\Lambda_2,x_ix_j} \in (q^8)$ for $i < j$. As before we compute

$$\begin{aligned}
\Theta_{\Lambda_2,4x_1^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= 12q^2 - 15q^4 + \dots \\
\Theta_{\Lambda_2,4x_2^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= -4q^2 + \tfrac{33}{2}q^4 + \dots \\
\Theta_{\Lambda_2,4x_3^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= -4q^2 + \tfrac{29}{2}q^4 + \dots \\
\Theta_{\Lambda_2,4x_4^2-(x_1^2+x_2^2+x_3^2+x_4^2)} &= -4q^2 - 16q^4 + \dots .
\end{aligned}$$

Eventually, we obtain $\Theta_{1,1,\Lambda_2}(\tau) = 192q^4 - 480q^6 + \dots \neq \Theta_{1,1,\Lambda_1}(\tau)$. □

Remark. To see that the two modular forms are different, it is enough to compute up to the first Fourier coefficient which is different. Anyway, using a computer we can give more coefficients:

$$\begin{aligned}
\Theta_{1,1,\Lambda_1}(\tau) &= 192q^4 - 256q^6 - 896q^7 + 1120q^8 - 2848q^9 + 3024q^{10} - 2112q^{11} + \\
&\quad +13536q^{12} - 4064q^{13} - 16272q^{14} - 4544q^{15} + \dots \\
\Theta_{1,1,\Lambda_2}(\tau) &= 192q^4 - 480q^6 - 608q^7 + 736q^8 - 1312q^9 + 3216q^{10} + 1056q^{11} - \\
&\quad -2048q^{12} - 2624q^{13} + 2896q^{14} - 12288q^{15} + \dots .
\end{aligned}$$

Appendix A. Integrating polynomials on spheres

Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial. We need the integral $\int_{S^{n-1}} f d\mu$ for further computations. Here $d\mu$ denotes the standard $\mathrm{O}(n)$-invariant measure on S^{n-1}. Since we can decompose f into its homogeneous components, it is enough to consider homogeneous polynomials f. Here we have the following result:

Proposition A.1. *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d. If d is odd, we have that $\int_{S^{n-1}} f d\mu = 0$. If $d = 2k$ is even, $\Delta^k f$ is a real number, and we have*

$$\int_{S^{n-1}} f d\mu = c_d \Delta^k f \quad \textit{with} \quad c_d := \alpha_d \cdot \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})} \text{ , and } \alpha_{2k} := \frac{1}{(-2)^k k! \prod_{m=1}^{k}(n+2m-2)} .$$

Proof. We start with the following observation (see Exercise 33 on page 550 in [3]): f can be uniquely decomposed as

$$f = \sum_{l=0}^{\lfloor \frac{d}{2} \rfloor} r^{2l} h_{d-2l} ,$$

where $r^2 = \sum_{i=1}^n x_i^2$ and the h_i are homogeneous harmonic polynomials of degree i. The mean value principle and the fact that r^2 is constantly one on S^{n-1} together with the fact that the volume of S^{n-1} is $\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})}$ yield

$$\int_{S^{n-1}} f = \int_{S^{n-1}} \sum_{l=0}^{d/2} h_{d-2l} = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})} \sum_{l=0}^{d/2} h_{d-2l}(0)\,.$$

Now, the value of a homogeneous polynomial at zero is zero, unless the degree of the polynomial is zero. Thus, we deduce that $\int_{S^{n-1}} f d\mu = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})} h_0$. This shows, that integration of homogeneous polynomials of odd degree gives zero. Next, we show (by simple and straightforward differentiation) the following equality for any homogeneous harmonic polynomial h:

$$\Delta(r^{2m}h) = (-2m)(n + 2m + 2\deg(h) - 2)r^{2m-2}h\,.$$

In consequence we have $\Delta^k r^{2k} = (-2)^k k! \prod_{m=1}^k (n + 2m - 2) = \frac{1}{\alpha_{2k}}$, and $\Delta^k r^{2m} h = 0$ for $m < k$ and h harmonic. We deduce that $\Delta^k f$ yields $\frac{h_0}{\alpha_{2k}}$, which finishes the proof. □

We need concrete formulas for the normalized measure $d\bar{\mu} = \frac{\Gamma(\frac{n+2}{2})}{n\pi^{\frac{n}{2}}} d\mu$. This measure is O($n$)-invariant and has the property that $\int_{S^{n-1}} d\bar{\mu} = 1$. As an application of the above Proposition A.1 we obtain:

Corollary A.2. *For the monomial $f = x_1^{i_1} \cdots x_n^{i_n}$ of degree $d = i_1 + \cdots + i_n$ we have the equality*

$$\int_{S^{n-1}} f d\bar{\mu} = \begin{cases} 0 & \text{at least one of the } i_k \text{ is odd,} \\ \left(\prod_{k=1}^n \frac{i_k!}{\frac{i_k}{2}!}\right)\left(\prod_{m=1}^{d/2} \frac{1}{2(n+2m-2)}\right) & \text{all the } i_k \text{ are even.} \end{cases}$$

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FB Mathematik, University Duisburg-Essen, 45117 Essen, Germany
E-mail address: juan.cervino@uni-due.de

FB Mathematik, University Duisburg-Essen, 45117 Essen, Germany
E-mail address: georg.hein@uni-due.de