

LATTICE INVARIANTS FROM THE HEAT KERNEL (II)

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ABSTRACT. Given an integral lattice Λ of rank n and a finite sequence $m_1 \leq m_2 \leq \dots \leq m_k$ of natural numbers we construct a modular form $\Theta_{m_1, m_2, \dots, m_k, \Lambda}$ of level $N = N(\Lambda)$. The weight of this modular form is $nk/2 + \sum_{i=1}^k m_i$. This construction generalizes the theta series Θ_Λ of integral lattices, because $\Theta_\Lambda = \Theta_{0, \Lambda}$. We give the q -expansions of the modular forms $\Theta_{m, m, \Lambda}$, and $\Theta_{1, 1, 1, \Lambda}$ and show that (up to some scaling) they are given by power series with integer coefficients.

1. INTRODUCTION

For an integral lattice Λ the theta series $\Theta_\Lambda(\tau) = \sum_{\lambda \in \Lambda} \exp(2\pi i \|\lambda\|^2 \tau)$ is a first invariant. Moreover, Θ_Λ is a modular form of weight $\text{rk}(\Lambda)/2$ and level $N(\Lambda)$. Since the vector space of modular forms of given weight and level is of finite dimension, Θ_Λ can be read off from the first coefficients in its q -expansion. Unfortunately, there are pairs (Λ, Λ') of lattices which possess the same theta series $\Theta_\Lambda = \Theta_{\Lambda'}$ and are not isometric. A first example are the two unimodular lattices $E_8 \oplus E_8$ and E_{16} of rank 16 (see page 1243 in [2] for details). Schiemann constructed in [6] an example of two four dimensional lattices (Λ, Λ') which are isospectral (i.e. $\Theta_\Lambda = \Theta_{\Lambda'}$) but not isometric.

Spherical theta functions $\Theta_{h, \Lambda} := \sum_{\lambda \in \Lambda} h(\lambda) \exp(2\pi i \|\lambda\|^2 \tau)$ define for homogeneous harmonic polynomials h also modular forms of level $N(\Lambda)$. These modular forms depend on h and are of weight $\deg(h) + \text{rk}(\Lambda)/2$. (The term *spherical theta functions* appears in [7] whereas Elkies uses *weighted theta function* in [3].) The authors managed in [1] to find sums of products of these spherical theta functions which give new lattice invariants. These modular forms can be used to distinguish the two isospectral lattices in Schiemann's example (see Proposition 4.4 in [1]). In our article [1] we construct an invariant $c_{m_1, m_2, \dots, m_k, \Lambda}$ which turns out to be a sum of products of modular forms and their derivatives. Out of these invariants we can sometimes construct invariant harmonic data $p_{m_1, m_2, \dots, m_k, \Lambda}$ which give invariant modular forms. However, we computed only $p_{1, 1, \Lambda}$ explicitly for lattices Λ of arbitrary rank in our article [1].

The aim of this article is a direct construction of the invariant harmonic datum $p_{m_1, m_2, \dots, m_k, \Lambda}$. So take an integral lattice Λ of rank n , let N be the level of Λ , and fix a finite sequence $0 \leq m_1 \leq m_2 \leq \dots \leq m_k$ of integers. We start with an isometric embedding $\Lambda \rightarrow \mathbb{E}^n$ of Λ into the Euclidean space. Here we consider Λ as a distribution on the Schwartz functions on \mathbb{E}^n . The heat flux of this distribution is given by a function $f_\Lambda : \mathbb{R}^+ \times \mathbb{E}^n \rightarrow \mathbb{R}$. Using the *harmonic Taylor coefficients* of f_Λ we obtain the harmonic invariant system $p_{m_1, m_2, \dots, m_k, \Lambda}$ which provides a modular form $\Theta_{m_1, m_2, \dots, m_k, \Lambda}$ of level N and weight $nk/2 + \sum_{i=1}^k m_k$, independent from the chosen embedding $\Lambda \rightarrow \mathbb{E}^n$ (see Theorem 2.6).

Next we give for all integers $m \geq 0$ the q -expansion of the invariant modular forms

Date: August 31, 2009.

2000 Mathematics Subject Classification. 11F11, 11F27, 11E45.

Key words and phrases. spherical theta functions, lattice invariants, modular form.

$\Theta_{m,m,\Lambda} = \sum_{k \geq 0} a_{m,m,k} q^k$. It turns out that the coefficients $a_{m,m,k}$ are given by

$$a_{m,m,k} = \sum_{(v,w) \in \Lambda^2, \|v\|^2 + \|w\|^2 = k} p_m(\cos(\angle(v,w))) \|v\|^{2m} \|w\|^{2m},$$

where p_m is an even polynomial of degree $2m$ (see Theorem 3.3). We compute these polynomials in Lemma 5.4. The first ones being

$$p_0(c) = 1, \quad p_1(c) = \frac{c^2}{2} - \frac{1}{2n}, \quad \text{and} \quad p_2(c) = \frac{c^4}{24} - \frac{c^2}{4(n+4)} + \frac{1}{8(n+4)(n+2)}.$$

Knowing these polynomials we can give the modular forms Θ_{m,m,E_8} for the E_8 lattice for $m \leq 9$ in 3.4. We conclude with computing the *triple modular invariant* $\Theta_{1,1,1,\Lambda}$ in Theorem 4.5.

Acknowledgment. This work has been supported by the SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties”.

2. THE MODULAR FORMS $\Theta_{k_1, \dots, k_m, \Lambda}$

2.1. Notation. We consider a lattice $\Lambda \subset \mathbb{E}^n$ embedded in the n -dimensional euclidean space. In [1] we defined the function $f_\Lambda : \mathbb{R}^+ \times \mathbb{E}^n \rightarrow \mathbb{R}$ by

$$f_\Lambda(t, x) = (4\pi t)^{-\frac{n}{2}} \sum_{\gamma \in \Lambda} \exp\left(\frac{-\|x - \gamma\|^2}{4t}\right).$$

As explained in Section 2.1 of [1] this function describes the heat flux of the lattice Λ . We call a function $c_\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}$, which we obtain from f_Λ , a lattice invariant if the action of the orthogonal group $O(n)$ on the isometric embeddings of $\Lambda \rightarrow \mathbb{E}^n$ does not change c_Λ . The first example for such a lattice invariant is $c_{0,\Lambda}(t) = f_\Lambda(t, 0)$. It was shown in [1, Section 2.10] that this function $c_{0,\Lambda}$ determines the theta series of the lattice Λ . We call a lattice Λ integral if the square lengths $\|\gamma\|^2$ are integers for all $\gamma \in \Lambda$. An integral lattice Λ has two integer invariants: its discriminant $D = D(\Lambda)$, and the level $N = N(\Lambda)$. We recall their definitions. If $\frac{1}{2}A \in \text{Mat}_{n \times n}(\mathbb{R})$ is a symmetric Gram matrix for Λ , then $A \in \text{Mat}_{n \times n}(\mathbb{Z})$ and has even integers on its diagonal. We set $D(L) := \det(A)$. The smallest positive integer N such that $N \cdot A^{-1} \in \text{Mat}_{n \times n}(\mathbb{Z})$ and $N \cdot A^{-1}$ has even entries on the principal diagonal is called the level of Λ .

2.2. Harmonic polynomials. We list some well known properties of harmonic polynomials. For proofs see [4, Theorem 3.1], and [5, Chapter XIII, Exercises 33–35]. The ring of polynomial functions on \mathbb{E}^n we denote by $A = \mathbb{R}[x_1, \dots, x_n]$. The ring A has a natural grading $A = \bigoplus_{k \geq 0} A_k$ with A_k the homogeneous polynomials of degree k . We define a pairing on

$$A \times A \rightarrow \mathbb{R}, \quad \langle g, f \rangle = g\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) f|_0.$$

The orthogonal group $O(n)$ acts on \mathbb{E}^n , defining an action on the polynomials $\sigma(f)(x) := f(\sigma^{-1}(x))$. It is convenient, to recall some well known basic properties of this pairing.

- (1) The pairing is a bilinear, symmetric, and positive definite.
- (2) For $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ we set $x^I = \prod_{k=1}^n x_k^{i_k}$, and $I! = \prod_{k=1}^n i_k!$. The normed monomials $\left\{ \frac{x^I}{\sqrt{I!}} \right\}_{I \in \mathbb{N}^n}$ form an orthonormal basis.
- (3) For two polynomials $P, Q \in A$ we have $\langle P \cdot Q, f \rangle = \langle P, Q\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) f \rangle$. In particular, the Laplace operator $\Delta = -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} : A_m \rightarrow A_{m-2}$ has up to sign the multiplication with $r^2 = \sum_{k=1}^n x_k^2$ as adjoint.

- (4) The infinite dimensional representation $O(n) \times A \rightarrow A$ is compatible with the grading of A and with the inner product on A . Therefore it defines finite dimensional representations $O(n) \rightarrow O(A_m, \langle, \rangle)$ for all $m \geq 0$.
- (5) The kernel Harm_m of $\Delta : A_m \rightarrow A_{m-2}$ is an irreducible representation. The vector space Harm_m of harmonic functions is of dimension $\binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$. Its orthogonal complement is $r^2 \cdot A_{m-2}$ which is an $O(n)$ invariant subspace. In consequence, we obtain the decomposition of A_m into irreducible subspaces

$$A_m = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} r^{2k} \text{Harm}_{m-2k}.$$

- (6) For $h_1, h_2 \in \text{Harm}_{2m-2k}$ we have the equality

$$\langle r^{2k} h_1, r^{2k} h_2 \rangle = a_{k,m} \langle h_1, h_2 \rangle \quad \text{with } a_{k,m} = 2^k k! \prod_{l=1}^k (n + 4m - 2k - 2l)$$

- (7) For $h \in \text{Harm}_m$, and natural numbers $k, d \in \mathbb{N}$ we have

$$r^{2k} \Delta^k (r^{2d} h) = b_{k,d,m} r^{2d} h \quad \text{with } b_{k,d,m} = \prod_{l=0}^{k-1} (2l - 2d)(n - 2 + 2d - 2l + 2m).$$

Note that $b_{0,d,m} = 1$, and $b_{k,d,m} = 0 \iff k > d$.

- (8) Using the above numbers $b_{k,d,m}$ we define for $m \geq 0$, and $k = 0, \dots, \lfloor m/2 \rfloor$ rational numbers $d_{k,m}$ by the assignment $r_{0,m} := 1$, and

$$r_{d,m} := \frac{-1}{b_{d,d,m-2d}} \sum_{k=0}^{d-1} r_{k,m} b_{k,d,m-2d} \quad \text{for all integers } d = 1, \dots, \lfloor m/2 \rfloor.$$

The use of the integers $b_{k,d,m}$ is not necessary. Using their definition we obtain:

$$r_{0,m} = 1 \quad \text{and by } r_{d,m} = - \sum_{k=0}^{d-1} \left(\prod_{l=k}^{d-1} \frac{1}{(2l - 2d)(n - 2 + 2m - 2d - 2l)} \right) r_{k,m}.$$

From this, we derive in Lemma 5.3 the explicit formula:

$$r_{k,m}^{-1} = 2^k k! \prod_{l=0}^{k-1} (n + 2m - 4 - 2l).$$

However, the first definition implies immediately that for all $d \geq 1$ we have:

$$\sum_{k=0}^d r_{k,m} b_{k,d,m-2d} = 0.$$

Therefore we conclude, that the linear map $P_{\text{harm},m} = \sum_{k \geq 0}^{m/2} r_{k,m} r^{2k} \Delta^k$ operates on A_m as the harmonic projection. Indeed, for a homogeneous harmonic function h of degree $m - 2d$ we find that $P_{\text{harm},m}(r^{2d} h) = \left(\sum_{k=0}^d r_{k,m} b_{k,d,m-2d} \right) r^{2d} h$.

- (9) Let us explicitly give the harmonic projections in degree two, four and six:

$$\begin{aligned} P_{\text{harm},2} &= \text{id} + \frac{1}{2n} r^2 \Delta \\ P_{\text{harm},4} &= \text{id} + \frac{1}{2(n+4)} r^2 \Delta + \frac{1}{8(n+2)(n+4)} r^4 \Delta^2 \\ P_{\text{harm},6} &= \text{id} + \frac{1}{2(n+8)} r^2 \Delta + \frac{1}{8(n+6)(n+8)} r^4 \Delta^2 + \frac{1}{48(n+4)(n+6)(n+8)} r^6 \Delta^3 \end{aligned}$$

2.3. Harmonic Taylor coefficients. We consider the homogeneous parts of the Taylor expansion of f_Λ at the point $x = 0$. Since f_Λ is symmetric in x , only the even parts appear. We set

$$f_{\Lambda,m} = \sum_{I \subset \mathbb{N}^n, |I|=2m} a_I \frac{x^I}{I!} \text{ where for } I = (i_1, i_2, \dots, i_n) \quad I! := \prod_{m=1}^n i_m! , \quad x^I := \prod_{m=1}^n x_m^{i_m} , \text{ and}$$

$$a_I := \langle x^I, f_\Lambda \rangle = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} f_\Lambda|_{\mathbb{R}^+ \times \{0\}} .$$

The $f_{\Lambda,m}$ are homogeneous polynomials of degree $2m$ in the x_i over the ring of functions in t . We may write

$$f_{\Lambda,m} = \sum_{I \subset \mathbb{N}^n, |I|=2m} \left\langle \frac{x^I}{\sqrt{I!}}, f_\Lambda \right\rangle \frac{x^I}{\sqrt{I!}} .$$

Indeed, the operator $f \mapsto \sum_{I \subset \mathbb{N}^n, |I|=2m} \left\langle \frac{x^I}{\sqrt{I!}}, f \right\rangle \frac{x^I}{\sqrt{I!}}$ is the identity on A_{2m} the space of homogeneous polynomials of degree $2m$. So we can replace the orthonormal basis $\left\{ \frac{x^I}{\sqrt{I!}} \right\}_{I \subset \mathbb{N}^n, |I|=2m}$ by any other orthonormal basis. If $\mathcal{B}_{2m}^{\text{harm}}$ is an orthonormal basis of Harm_{2m} , the space of harmonic polynomials of degree $2m$, then the projection of $f_{\Lambda,m}$ to Harm_{2m} is called the *harmonic Taylor coefficient* of f_Λ and given by

$$f_{\Lambda,m}^{\text{harm}} = \sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} \langle h, f_\Lambda \rangle h .$$

We derive more formulas for $f_{\Lambda,m}^{\text{harm}}$ which we will use in the sequel. Taking any orthonormal basis \mathcal{B}_{2m} of A_{2m} we obtain $f_{\Lambda,m}^{\text{harm}} = \sum_{g \in \mathcal{B}_{2m}} \langle g, f_\Lambda \rangle P_{\text{harm}}(g)$ where $P_{\text{harm}} : A_{2m} \rightarrow A_{2m}$ denotes the orthogonal projection to the space of harmonic polynomials. Therefore, we conclude $f_{\Lambda,m}^{\text{harm}} = \sum_{g \in \mathcal{B}_{2m}} \langle P_{\text{harm}}^*(g), f_\Lambda \rangle g$ with P_{harm}^* the adjoint operator of P_{harm} . Since an orthogonal projection is self adjoint we find that $f_{\Lambda,m}^{\text{harm}} = \sum_{g \in \mathcal{B}_{2m}} \langle P_{\text{harm}}(g), f_\Lambda \rangle g$. Using the formula for the harmonic projection developed in 2.2.(8) we derive the next

Proposition 2.4. *Let \mathcal{B} be any orthonormal basis of A_{2m} . We have:*

$$\begin{aligned} f_{\Lambda,m}^{\text{harm}} &= \sum_{g \in \mathcal{B}_{2m}} \langle P_{\text{harm}}(g), f_\Lambda \rangle g = \sum_{g \in \mathcal{B}_{2m}} \langle g, P_{\text{harm}}(f_\Lambda) \rangle g = \\ &= \sum_{g \in \mathcal{B}_{2m}} \left\langle g, \sum_{k=0}^m p_{k,2m} r^{2k} \Delta^k f_\Lambda \right\rangle g = \sum_{g \in \mathcal{B}_{2m}} \left\langle g, \sum_{k=0}^m p_{k,2m} r^{2k} (-1)^k \frac{\partial^k}{\partial t^k} f_\Lambda \right\rangle g \end{aligned}$$

with the rational numbers $p_{k,2m} = \frac{1}{2^k k!} \prod_{l=0}^{k-1} (n + 4m - 4 - 2l)^{-1}$ from 2.2.(8).

Proof. We have shown all equalities but the last one. This is a consequence of the identity $\Delta^k f_\Lambda = (-1)^k \frac{\partial^k}{\partial t^k} f_\Lambda$ (see 2.1 in [1]). \square

2.5. The invariant harmonic system $p_{m_1, \dots, m_k, \Lambda}$. We define for any set of integers m_1, \dots, m_k with $m_i \geq 0$ the function $p_{m_1, \dots, m_k, \Lambda}$ by

$$p_{m_1, \dots, m_k, \Lambda} := \int_{S^{n-1}} f_{\Lambda, m_1}^{\text{harm}} \cdot f_{\Lambda, m_2}^{\text{harm}} \cdots f_{\Lambda, m_k}^{\text{harm}} d\bar{\mu} .$$

If $\varphi : \mathbb{E}^n \rightarrow \mathbb{E}^n$ is any isometry, then φ commutes with the multiplication with r^2 as well as with Δ . Whence it commutes with the harmonic projection, which can be described in

terms of r^2 and Δ . In consequence $p_{m_1, \dots, m_k, \varphi(\Lambda)} = p_{m_1, \dots, m_k, \Lambda}$. Using the defining equation of f_{Λ, m_i} we can write

$$p_{m_1, \dots, m_k, \Lambda} = \sum_{h_1 \in \mathcal{B}_{2m_1}^{\text{harm}} \dots h_k \in \mathcal{B}_{2m_k}^{\text{harm}}} \left(\prod_{l=1}^k \langle h_l, f_{\Lambda} \rangle \right) \int_{S^{n-1}} h_1 \dots \cdot h_k d\bar{\mu},$$

where $\mathcal{B}_{2m_i}^{\text{harm}}$ is an orthonormal basis of harmonic polynomials of degree $2m_i$. Since the $\int_{S^{n-1}} h_1 \dots \cdot h_k d\bar{\mu}$ are merely real numbers we obtain from this equation and the equality $p_{m_1, \dots, m_k, \varphi(\Lambda)} = p_{m_1, \dots, m_k, \Lambda}$ for all isometries $\varphi \in O(n)$, that $p_{m_1, \dots, m_k, \Lambda}$ is an invariant harmonic system for lattices $\Lambda \in \mathbb{E}^n$ (see [1, 2.8]). The Proposition 2.9 from [1] implies:

Theorem 2.6. *For any integral lattice $\Lambda \subset \mathbb{E}^n$ the modular form*

$$\Theta_{m_1, \dots, m_k, \Lambda} = \sum_{h_1 \in \mathcal{B}_{2m_1}^{\text{harm}} \dots h_k \in \mathcal{B}_{2m_k}^{\text{harm}}} \left(\prod_{l=1}^k \Theta_{h_l, \Lambda} \right) \int_{S^{n-1}} h_1 \dots \cdot h_k d\bar{\mu}$$

is a modular form of weight $\frac{nk}{2} + 2 \sum_{l=1}^k m_l$. The modular form is of level N , the level of the lattice Λ . Furthermore, $\Theta_{m_1, \dots, m_k, \Lambda}$ is independent from the chosen embedding $\Lambda \rightarrow \mathbb{E}^n$. If k is an odd number, then $\Theta_{m_1, \dots, m_k, \Lambda}$ has character $\left(\frac{D}{\cdot}\right)$. For k an even integer $\Theta_{m_1, \dots, m_k, \Lambda}$ is a modular form for the trivial character. \square

The functions $\left\{ \frac{x^I}{\sqrt{I!}} \right\}_{I \subset \mathbb{N}^n, |I|=2m}$ form an orthonormal basis of A_{2m} . Whereas an orthonormal basis of the subspace $\text{Harm}_{2m} \subset A_{2m}$ is more difficult. However by Proposition 2.4 we can compute $p_{m_1, \dots, m_k, \Lambda}$ in a different manner:

$$p_{m_1, \dots, m_k, \Lambda} = \sum_{g_1 \in \mathcal{B}_{2m_1} \dots g_k \in \mathcal{B}_{2m_k}} \left(\prod_{l=1}^k \langle P_{\text{harm}}(g_l), f_{\Lambda} \rangle \right) \int_{S^{n-1}} g_1 \dots \cdot g_k d\bar{\mu},$$

with the \mathcal{B}_{2m_i} orthonormal basis of A_{2m_i} . Applying [1, Proposition 2.9] to this presentation of $p_{m_1, \dots, m_k, \Lambda}$ we obtain the next

Proposition 2.7. *The modular form $\Theta_{m_1, \dots, m_k, \Lambda}$ can be computed using orthonormal basis \mathcal{B}_{2m_i} of A_{2m_i} , and the orthogonal harmonic projections $P_{\text{harm}} : A_{2m_i} \rightarrow A_{2m_i}$ as follows*

$$\Theta_{m_1, \dots, m_k, \Lambda} = \sum_{g_1 \in \mathcal{B}_{2m_1} \dots g_k \in \mathcal{B}_{2m_k}} \left(\prod_{l=1}^k \Theta_{P_{\text{harm}}(g_l), \Lambda} \right) \int_{S^{n-1}} g_1 \dots \cdot g_k d\bar{\mu}.$$

3. THE MODULAR FORMS $\Theta_{m, m, \Lambda}$ FOR INTEGERS $m \geq 0$

3.1. Definition of $\Theta_{m, m, \Lambda}$. On the real polynomials on \mathbb{E}^n we have two $O(n)$ -invariant scalar products. The one defined in 2.2, and the integral scalar product $\langle f, g \rangle_2 := \int_{S^{n-1}} f g d\bar{\mu}$. The first has the advantage that $A_m \perp A_k$ for $m \neq k$. However, when we restrict to the irreducible subspace Harm_{2m} the two scalar products agree up to a constant c_{2m} , i.e. $\langle f, g \rangle_2 = c_{2m} \langle f, g \rangle$ for all $f, g \in \text{Harm}_{2m}$. The formula from Theorem 2.6 yields

$$\begin{aligned} \Theta_{m, m, \Lambda} &= \sum_{h_1 \in \mathcal{B}_{2m}^{\text{harm}}} \sum_{h_2 \in \mathcal{B}_{2m}^{\text{harm}}} \Theta_{h_1, \Lambda} \Theta_{h_2, \Lambda} \int_{S^{n-1}} h_1 h_2 d\bar{\mu} \\ &= \sum_{h_1 \in \mathcal{B}_{2m}^{\text{harm}}} \sum_{h_2 \in \mathcal{B}_{2m}^{\text{harm}}} \Theta_{h_1, \Lambda} \Theta_{h_2, \Lambda} \langle h_1, h_2 \rangle_2 \\ &= c_{2m} \sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} \Theta_{h, \Lambda}^2. \end{aligned}$$

A straightforward computation using [1, Corollary A.2] yields $c_{2m} = \prod_{k=0}^{2m-1} \frac{1}{n+2k}$. However, to ease notation we define $\Theta_{m,m,\Lambda} := \sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} \Theta_{h,\Lambda}^2$.

Lemma 3.2. *Let v and w to vectors in \mathbb{E}^n , $\mathcal{B}_{2m}^{\text{harm}}$ be a orthonormal basis of Harm_{2m} , and $c = \cos(\angle(v, w)) = \frac{\langle v, w \rangle}{\|v\| \|w\|}$ be the cosine of the angle between v and w . We have*

$$\sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} h(v)h(w) = p_m(c) \|v\|^{2m} \|w\|^{2m}$$

where p_m is the even polynomial of degree $2m$ from Lemma 5.4.

Proof. We proceed by induction on m . For $m = 0$ the statement is obvious. Let \mathcal{B}_{2m} be a orthonormal basis of A_{2m} . We want to compute $D_m(v, w) = \sum_{h \in \mathcal{B}_{2m}} h(v)h(w)$. Since $O(n)$ acts orthogonal on A_{2m} , this number is independent of the chosen basis. In particular, we can take $\mathcal{B}_{2m} = \left\{ \frac{x^I}{\sqrt{I!}} \right\}_{|I|=2m}$. With this choice we find $D_m(v, w) = \frac{\langle v, w \rangle^{2m}}{(2m)!}$. Another choice for an orthonormal basis is by 2.2.(6)

$$\mathcal{B}'_{2m} = \mathcal{B}_{2m}^{\text{harm}} \cup \frac{1}{\sqrt{a_{1,m}}} r^2 \mathcal{B}_{2m-2}^{\text{harm}} \cup \dots \cup \frac{1}{\sqrt{a_{m,m}}} r^{2m} \mathcal{B}_0^{\text{harm}},$$

where the numbers a_i, m are those defined in 2.2.(6). This basis corresponds to the irreducible decomposition $A_{2m} = \bigoplus_{k=0}^m r^{2k} \text{Harm}_{2m-2k}$. Working with this orthonormal basis we deduce

$$D_m(v, w) = \sum_{k=0}^m \frac{1}{a_{k,m}} \sum_{h \in \mathcal{B}_{2m-2k}^{\text{harm}}} h(v)h(w).$$

The defining equation for the polynomials p_m , Lemma 5.4, and the induction hypothesis yield the stated formula for p_m . \square

Theorem 3.3. *For an integer lattice $\Lambda \subset \mathbb{E}^n$ the modular form*

$$\Theta_{m,m,\Lambda} = \sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} \Theta_{h,\Lambda}^2$$

is of weight $4m + n$, has level $N(\Lambda)$, and is independent of the chosen embedding. Its q -expansion is given by

$$\Theta_{m,m,\Lambda}(\tau) = \sum_{k \geq 0} \left(\sum_{(v,w) \in \Lambda^2, \|v\|^2 + \|w\|^2 = k} p_m(\cos(\angle(v, w))) \|v\|^{2m} \|w\|^{2m} \right) q^k.$$

For $m > 0$ we have, $\frac{(2m)! 2^{2m} \prod_{l=0}^{m-1} (n+4m-4-2l)}{q^{2l}} \Theta_{m,m,\Lambda} \in \mathbb{Z}[[q]]$ where l denotes the minimum of $\|v\|^2$ for all nonzero $v \in \Lambda$.

Proof. We take the function $\Theta_{m,m,\Lambda} = \sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} \Theta_{h,\Lambda}^2$. As a sum of squares of modular forms of weight $2m + \frac{n}{2}$ it is a modular form of weight $4m + n$. Let us calculate the q -expansion:

$$\Theta_{m,m,\Lambda}(\tau) = \sum_{h \in \mathcal{B}_{2m}^{\text{harm}}} \sum_{(v,w) \in \Lambda \times \Lambda} h(v)h(w) q^{\|v\|^2 + \|w\|^2} \quad \text{with } q = \exp(2\pi i \tau).$$

This yields by Lemma 3.2 the stated q -expansion. From the q -expansion we directly deduce that $\Theta_{m,m,\Lambda}$ is independent from the chosen embedding $\Lambda \rightarrow \mathbb{E}^n$. Likewise we see

that the coefficients of q^k in $\Theta_{m,m,\Lambda}$ vanish for all $k < 2l$. Now we consider for two vectors $v, w \in \Lambda$ the number

$$\delta_m(v, w) = (2m)! 2^{2m} \prod_{l=0}^{m-1} (n + 4m - 4 - 2l) p_m(\cos(\angle(v, w))) \|v\|^{2m} \|w\|^{2m}.$$

This number is by Lemma 5.4 and the definition of the cosine given by

$$\delta_m(v, w) = \sum_{k=0}^m (-1)^k \frac{(2m)!}{(2m-2k)!k!} 2^{2m-k} \langle v, w \rangle^{2m-2k} \|v\|^{2k} \|w\|^{2k} \prod_{l=k}^{m-1} (n + 4m - 4 - 2l).$$

Since Λ is integral $\langle v, w \rangle \in \frac{1}{2}\mathbb{Z}$. Thus, $\delta_m(v, w)$ is a sum of integers. This completes the proof. \square

3.4. Example: The modular forms Θ_{m,m,E_8} . Let $v \in E_8$ be any lattice vector of length one. Basic combinatorics yield, that the possible values of $\langle v, w \rangle^2$ for the 240 lattice vectors $w \in E_8$ of length one are: one (2 times), $\frac{1}{4}$ (112 times), and 0 (126 times). This allows by Theorem 3.3 the computation of the coefficient $a_{m,m,2}$ of q^2 in the q -expansion of Θ_{m,m,E_8} for all integers $m \geq 1$. We list the first:

m	$a_{m,m,2}$
$m \in \{1, 2, 3, 5\}$	0
4	3/896
6	7/316293120
7	1/30057431040
8	1/22235892940800
9	1/21727643959296000

Now Θ_{m,m,E_8} is a cusp form of weight $4m + 8$ for $\mathrm{SL}_2(\mathbb{Z})$ which starts with $\Theta_{m,m,E_8} = a_{m,m,2}q^2 + \dots$. Since we know the dimensions of the spaces of cusp forms we can determine Θ_{m,m,E_8} from $a_{m,m,2}$ for $m \leq 6$, and obtain:

$$\Theta_{m,m,E_8}(\tau) = 0 \quad \text{for } m \in \{1, 2, 3, 5\}$$

$$\Theta_{4,4,E_8}(\tau) = \frac{3}{896} \Delta^2(\tau)$$

$$\Theta_{6,6,E_8}(\tau) = \frac{7}{658944} G_8(\tau) \Delta^2(\tau),$$

where $\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}$ is the discriminant function, and $G_8(\tau) = \frac{1}{480} + \sum_{n \geq 1} \sigma_7(n) q^n$ is the Eisenstein series of weight eight (see Chapter 0 in [7]). The vanishing of $\Theta_{m,m,E_8}(\tau)$ for $m \in \{1, 2, 3, 5\}$ can be deduced alternatively: These modular forms are sums of squares of cusp forms of weight $2m + 4$ which do not exist for those values of m . The same argument can be applied for $m \in \{7, 8, 9\}$. Here Θ_{m,m,E_8} is a sum of squares of cusp forms of weight $2m + 4$. Those cusp forms form a one-dimensional vector space with generator, say $G_{2m-8} \Delta$, where G_{2m-8} is the Eisenstein series of weight $2m - 8$. We deduce that Θ_{m,m,E_8} is a scalar multiple of $G_{2m-8}^2 \Delta^2$. We can use the coefficient $a_{m,m,2}$

to deduce this scalar. After all, this yields:

$$\Theta_{7,7,E_8}(\tau) = \frac{9}{1064960} G_6(\tau)^2 \Delta(\tau)^2 \quad \text{with } G_6(\tau) = \frac{-1}{504} + \sum_{n \geq 1} \sigma_5(n) q^n,$$

$$\Theta_{8,8,E_8}(\tau) = \frac{1}{96509952} G_8(\tau)^2 \Delta(\tau)^2,$$

$$\Theta_{9,9,E_8}(\tau) = \frac{11}{3429236736000} G_{10}(\tau)^2 \Delta(\tau)^2 \quad \text{with } G_{10}(\tau) = \frac{-1}{264} + \sum_{n \geq 1} \sigma_9(n) q^n.$$

4. THE MODULAR FORM $\Theta_{1,1,1,\Lambda}$

4.1. The invariant harmonic datum $p_{1,1,1,\Lambda}$. We consider the harmonic Taylor coefficient $f_{\Lambda,1}^{\text{harm}}$ of the function f_Λ . By Proposition 2.4 it is given by

$$f_{\Lambda,1}^{\text{harm}} = \sum_{i=1}^n \left\langle x_i^2 - \frac{1}{n} \sum_{j=1}^n x_j^2, f_\Lambda \right\rangle \frac{x_i^2}{2} + \sum_{1 \leq i < j \leq n} \langle x_i x_j, f_\Lambda \rangle x_i x_j.$$

When introducing the shorthand $h_i = \left\langle n x_i^2 - \sum_{j=1}^n x_j^2, f_\Lambda \right\rangle$, and $b_{ij} = \langle x_i x_j, f_\Lambda \rangle$ we obtain

$$2n f_{\Lambda,1}^{\text{harm}} = \sum_{i=1}^n h_i x_i^2 + 2n \sum_{1 \leq i < j \leq n} b_{ij} x_i x_j.$$

We consider the invariant harmonic datum:

$$p_{1,1,1,\Lambda} = \int_{S^{n-1}} (f_{\Lambda,1}^{\text{harm}})^3 d\bar{\mu}.$$

We need the following spherical integrals $\int_{S^{n-1}} x_i^6 d\bar{\mu} = \frac{15}{n(n+2)(n+4)}$, $\int_{S^{n-1}} x_i^4 x_j^2 d\bar{\mu} = \frac{3}{n(n+2)(n+4)}$, and $\int_{S^{n-1}} x_i^2 x_j^2 x_k^2 d\bar{\mu} = \frac{1}{n(n+2)(n+4)}$. Furthermore $\int_{S^{n-1}} \prod_i x_i^{n_i} d\bar{\mu} = 0$ when at least one of the exponents n_i is an odd integer (see [1, Corollary A.2]). After these preparation we compute:

$$\begin{aligned} n(n+2)(n+4)(2n)^3 p_{1,1,1,\Lambda} &= n(n+2)(n+4) \int_{S^{n-1}} 2n (f_{\Lambda,1}^{\text{harm}})^3 d\bar{\mu} \\ &= n(n+2)(n+4) \int_{S^{n-1}} \left(\sum_{i=1}^n h_i x_i^2 + 2n \sum_{1 \leq i < j \leq n} b_{ij} x_i x_j \right)^3 d\bar{\mu} \\ &= \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{i_3=1}^n h_{i_1} h_{i_2} h_{i_3} + 6 \sum_{i_1=1}^n \sum_{i_2=1}^n h_{i_1}^2 h_{i_2} + 8 \sum_{i=1}^n h_i^3 + 12n^2 \sum_{i=1}^n \sum_{1 \leq j < k \leq n} h_i b_{jk}^2 + \\ &\quad + 24n^2 \sum_{1 \leq j < k \leq n} (h_k + h_j) b_{jk}^2 + 48n^3 \sum_{1 \leq i < j < k \leq n} b_{ij} b_{ik} b_{jk} \end{aligned}$$

Having in mind that $\sum_{i=1}^n h_i = 0$ we obtain

Lemma 4.2. *With the notation from 4.1 we have*

$$\begin{aligned} n^4(n+2)(n+4) p_{1,1,1,\Lambda} &= n^4(n+2)(n+4) \int_{S^{n-1}} (f_{\Lambda,1}^{\text{harm}})^3 d\bar{\mu} \\ &= \sum_{i=1}^n h_i^3 + 3n^2 \sum_{1 \leq i < j \leq n} (h_i + h_j) b_{ij}^2 + 6n^3 \sum_{1 \leq i < j < k \leq n} b_{ij} b_{ik} b_{jk}. \end{aligned}$$

4.3. Definition of $\Theta_{1,1,1,\Lambda}$. Again we rescale our definition and find by Theorem 2.6 for any integral lattice Λ the modular form

$$\Theta_{1,1,1,\Lambda} = \sum_{i=1}^n \Theta_{h_i,\Lambda}^3 + 3n^2 \sum_{1 \leq i < j \leq n} (\Theta_{h_i,\Lambda} + \Theta_{h_j,\Lambda}) \Theta_{x_i x_j,\Lambda}^2 + 6n^3 \sum_{1 \leq i < j < k \leq n} \Theta_{x_i x_j,\Lambda} \Theta_{x_i x_k,\Lambda} \Theta_{x_j x_k,\Lambda}$$

where h_i denotes the harmonic polynomial $h_i = nx_i^2 - \sum_{j=1}^m x_j^2$. To determine the q -expansion of $\Theta_{1,1,1,\Lambda}$ we need

Lemma 4.4. *Let $u, v, w \in \mathbb{E}^n$ be three vectors in euclidean space. We fix with $\mathcal{B}_2 = \left\{ \frac{x^I}{\sqrt{|I|}} \right\}_{I \subset \mathbb{N}^n, |I|=2}$ a basis of the homogeneous polynomials of degree 2 on \mathbb{E}^n . Denote by $P = P_{\text{harm},2}$ be the harmonic projection of degree two. Then we have an equality*

$$\begin{aligned} \Xi(u, v, w) &:= n^3(n+2)(n+4) \sum_{g_1, g_2, g_3 \in \mathcal{B}_2} P(g_1(u))P(g_2(v))P(g_3(w)) \int_{S^{n-1}} g_1 g_2 g_3 d\bar{\mu} = \\ &= 2\|u\|^2\|v\|^2\|w\|^2 - n(\|u\|^2\langle v, w \rangle^2 + \|v\|^2\langle u, w \rangle^2 + \|w\|^2\langle u, v \rangle^2) + n^2\langle v, w \rangle\langle u, w \rangle\langle u, v \rangle. \end{aligned}$$

Furthermore, when $\|u\|^2$, $\|v\|^2$, and $\|w\|^2$ are integers, and the scalar products $\langle v, w \rangle$, $\langle u, w \rangle$, and $\langle u, v \rangle$ are in $\frac{1}{2}\mathbb{Z}$, then $8\Xi(u, v, w)$ is an integer.

Proof. This is a straightforward calculation along the lines of computing $p_{1,1,1,\Lambda}$ in 4.1. \square

Theorem 4.5. *For any integral lattice Λ of level N and discriminant D , the modular form $\Theta_{1,1,1,\Lambda}$ defined in 4.3 has level N , character $\left(\frac{D}{\cdot}\right)$, and weight $\frac{n+12}{2}$. $\Theta_{1,1,1,\Lambda}$ is independent from the embedding $\Lambda \rightarrow \mathbb{E}^n$. Its q -expansion is given by*

$$\Theta_{1,1,1,\Lambda}(\tau) = n \sum_{k \geq 0} \left(\sum_{(u,v,w) \in \Lambda^3, \|u\|^2 + \|v\|^2 + \|w\|^2 = k} \Xi(u, v, w) \right) q^k.$$

Furthermore, $\frac{8}{n}\Theta_{1,1,1,\Lambda}(\tau) \in \mathbb{Z}[[q]]$.

Proof. If we define $\Theta_{1,1,1,\Lambda}$ by

$$\Theta_{1,1,1,\Lambda} = n^4(n+2)(n+4) \sum_{h_1, h_2, h_3 \in \mathcal{B}_2^{\text{harm}}} \Theta_{h_1,\Lambda} \Theta_{h_2,\Lambda} \Theta_{h_3,\Lambda} \int_{S^{n-1}} h_1 h_2 h_3 d\bar{\mu},$$

then we obtain by Theorem 2.6 an invariant modular form of the given weight, character, and level. Lemma 4.2 show that this definition coincides with the definition in 4.3. By Proposition 2.7 we can use the basis $\mathcal{B}_2 = \left\{ \frac{x^I}{\sqrt{|I|}} \right\}_{I \subset \mathbb{N}^n, |I|=2}$ of the homogeneous polynomials of degree 2 and the harmonic projection P to compute $\Theta_{1,1,1,\Lambda}$ as

$$\Theta_{1,1,1,\Lambda} = n^4(n+2)(n+4) \sum_{h_1, h_2, h_3 \in \mathcal{B}_2} \Theta_{P(h_1),\Lambda} \Theta_{P(h_2),\Lambda} \Theta_{P(h_3),\Lambda} \int_{S^{n-1}} g_1 g_2 g_3 d\bar{\mu}.$$

Now we deduce the q -expansion and $\frac{8}{n}\Theta_{1,1,1,\Lambda} \in \mathbb{Z}[[q]]$ from Lemma 4.4. \square

5. SOME COMBINATORICS

Lemma 5.1. *We define for all integers $d \geq 1$, and w the quantity*

$$q_{d,w} = \sum_{k=0}^d (-1)^k \binom{d}{k} \binom{w+k}{d-1}.$$

For all integers $d \geq 1$, we have $q_{d,-1} = 1$, and $q_{d,w} = 0$ for $w \neq -1$.

Proof. We consider the formal Laurent series $f_d, g_{d-1,w} \in \mathbb{Q}((t))$, given by

$$f_d := \sum_{k=0}^d (-t^{-1})^k \binom{d}{k} = \left(\frac{t-1}{t}\right)^d, \quad \text{and} \quad g_{a,b} := \sum_{k \in \mathbb{Z}} \binom{b+k}{a} t^k.$$

We have $g_{0,b} = t^{-b}(1-t)^{-1}$ by definition. From the formula $\binom{b+k}{a} = \binom{b-1+k}{a} + \binom{b-1+k}{a-1}$ we deduce that $g_{a,b} = t(g_{a,b} + g_{a-1,b})$. Hence $g_{a,b} = \frac{t}{1-t} g_{a-1,b}$ which gives by induction:

$$g_{a,b} = t^{a-b}(1-t)^{-1-a}.$$

The number $q_{d,w}$ is the coefficient of t^0 in $f_d g_{d-1,w}$. Eventually, we conclude from the above calculation that $f_d g_{d-1,w} = t^{1+w}$. \square

Lemma 5.2. *For all integers w , and $r \geq 0$ we have an equality*

$$(1) \quad \sum_{p=0}^r (-1)^{r-p} (w+2p-2r) \binom{w}{r-p} \binom{w+p-2r-1}{p} = \begin{cases} w & \text{for } r=0 \\ 0 & \text{for } r \geq 1 \end{cases}$$

Proof. First we denote the left hand side of equation (1) by $\xi_{r,w}$. We consider the formal power series $f \in \mathbb{Z}[[t]]$ given by:

$$f = \sum_{p \geq 0} (w-2p) \binom{w}{p} (-t)^p = w \sum_{p \geq 0} \binom{w}{p} (-t)^p + 2 \sum_{p \geq 0} \binom{w}{p} (-p) (-t)^p.$$

Using the binomial equation $(1-t)^w = \sum_{p \geq 0} \binom{w}{p} (-t)^p$, and its derivative with respect to t we obtain $\sum_{p \geq 0} \binom{w}{p} (-p) (-t)^p = (-t) \frac{\partial}{\partial t} (1-t)^w = wt(1-t)^{w-1}$. Therefore we have

$$f = w(1-t)^w + 2wt(1-t)^{w-1} = w(1+t)(1-t)^{w-1}.$$

Next we consider the formal power series $h \in \mathbb{Z}[[t]]$ which we define to be

$$h = \sum_{p \geq 0} \binom{(w-2r-1)+p}{p} t^p = g_{w-2r-1, w-2r-1} = (1-t)^{2r-w},$$

where $g_{w-2r-1, w-2r-1}$ is the function defined in the proof of Lemma 5.1. Now we are able to compute $\xi_{r,w}$. Indeed, the number $\xi_{r,w}$ is the coefficient of t^r in $f \cdot h$. Since $f \cdot h = w(1+t)(1-t)^{2r-1}$, we deduce from the binomial formula that for $r \geq 1$ we have

$$\xi_{r,w} = w \left((-1)^r \binom{2r-1}{r} + (-1)^{r-1} \binom{2r-1}{r-1} \right) = 0.$$

Since for $r=0$ the assertion also holds, we are done. \square

Lemma 5.3. *The rational numbers $r_{k,m} = \frac{1}{2^k k! \prod_{l=0}^{k-1} (n+2m-4-2l)}$ fulfill the equation*

$$\sum_{k=0}^d \left(\prod_{l=k}^{d-1} \frac{1}{(2l-2d)(n-2+2m-2d-2l)} \right) r_{k,m} = 0 \quad \text{for all } d \geq 1.$$

Proof. We obtain from Lemma 5.1 that $q_{d,w+d-2} = 0$ for all integers $w \geq 2-d$. We may write

$$q_{d,w+d-2} = d \sum_{k=0}^d \frac{(-1)^k}{(d-k)! k!} \prod_{l=k}^{k+d-2} (w+l).$$

Considered as a polynomial in w of degree at most $d-1$, $q_{d,w+d-2}$ can have at most $d-1$ zeros. Hence it is identically zero for all $w \in \mathbb{Q}$. Setting $w = 2-m-\frac{n}{2}$ we obtain therefore

that

$$\begin{aligned}
0 &= \frac{(-2)^{d-1}}{(-2)^d \prod_{l=0}^{2d-2} (n+2m-4-2l)} \sum_{k=0}^d \frac{(-1)^k}{(d-k)!k!} \prod_{l=k}^{k+d-2} (w+l) \\
&= \frac{1}{(-2)^d \prod_{l=0}^{2d-2} (n+2m-4-2l)} \sum_{k=0}^d \frac{(-1)^k}{(d-k)!k!} \prod_{l=k}^{k+d-2} (n+2m-4-2l) \\
&= \frac{1}{(-2)^d} \sum_{k=0}^d \frac{(-1)^k}{(d-k)!k!} \frac{1}{\prod_{l=k+d-1}^{2d-2} (n+2m-4-2l)} \frac{1}{\prod_{l=0}^{k-1} (n+2m-4-2l)} \\
&= \frac{1}{(-2)^d} \sum_{k=0}^d \frac{(-1)^k}{(d-k)!k!} \frac{1}{\prod_{l=k}^{d-1} (n-2+2m-2d-2l)} \frac{1}{\prod_{l=0}^{k-1} (n+2m-4-2l)} \\
&= \sum_{k=0}^d \left(\prod_{l=k}^{d-1} \frac{1}{(2l-2d)(n-2+2m-2d-2l)} \right) \left(\frac{1}{2^k k! \prod_{l=0}^{k-1} (n+2m-4-2l)} \right)
\end{aligned}$$

This is the stated assertion. \square

Lemma 5.4. *The polynomials $p_m \in \mathbb{Q}[c^2]$ which are implicitly defined by*

$$\sum_{k=0}^m \frac{p_{m-k}}{a_{k,m}} = \frac{c^{2m}}{(2m)!} \quad \text{with the integers} \quad a_{k,m} = 2^k k! \prod_{l=1}^k (n+4m-2k-2l)$$

from 2.2 (6) are explicitly given by

$$(2) \quad p_m(c) = \sum_{k=0}^m \frac{(-1)^k c^{2m-2k}}{(2m-2k)!k!2^k \prod_{l=0}^{k-1} (n+4m-4-2l)}.$$

Proof. We take the formula (2) as the definition of p_m and compute the sum $s_m := \sum_{k=0}^m \frac{p_{m-k}}{a_{k,m}}$. By definition s_m is a polynomial in $\mathbb{Q}[c^2]$ of degree at most $2m$. We write $s_m = \sum_{r=0}^m \frac{t_r}{4^r (2m-2r)!} c^{2m-2r}$. We find that

$$t_r = \sum_{p=0}^r \frac{(-1)^p}{p!(r-p)! \left(\prod_{l=1}^{r-p} (w+p-r-l) \right) \left(\prod_{q=0}^{p-1} (w+2p-2r-2-q) \right)},$$

with $w := \frac{n}{2} + 2m - 1$. Up to the factor $(w+2p-2r)$ the two products in the denominator give $\prod_{l=0}^r (w+p-r-l)$. So we get

$$t_r = \sum_{p=0}^r \frac{(-1)^p (w+2p-2r)}{p!(r-p)! \prod_{l=0}^r (w+p-r-l)}$$

Multiplying both side with the factor $\prod_{q=0}^{2r} (w-q) \neq 0$ we get

$$\left(\prod_{q=0}^{2r} (w-q) \right) t_r = \sum_{p=0}^r \frac{(-1)^p (w+2p-2r) \prod_{q=0}^{r-p-1} (w-q) \prod_{q=0}^{p-1} (w+p-2r-1-q)}{p!(r-p)!}$$

As usual, we define for a non-negative integer k for all complex numbers z the binomial coefficient $\binom{z}{k} = \frac{\prod_{a=0}^{k-1} (z-a)}{k!}$. Using this notation we obtain

$$\left(\prod_{q=0}^{2r} (w-q) \right) t_r = \sum_{p=0}^r (-1)^p \binom{w}{r-p} \binom{w+p-2r-1}{p} (w+2p-2r).$$

Both side of this equation are polynomials of degree at most $2r$. By Lemma 5.2 the right hand side is zero for $r \geq 1$ and all integers w . So it is zero for all w . We conclude that $t_r = 0$ for all $r \geq 1$. We finish the proof by checking $t_0 = 1$ which is obvious. \square

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