

THE CONWAY-SLOANE TETRALATTICE PAIRS ARE NON-ISOMETRIC

JUAN MARCOS CERVIÑO AND GEORG HEIN

ABSTRACT. Conway and Sloane constructed a 4-parameter family of pairs of isospectral lattices of rank four. They conjectured that all pairs in their family are non-isometric, whenever the parameters are pairwise different, and verified this for classical integral lattices of determinant up to 10^4 . In this paper, we use our theory of lattice invariants to prove this conjecture.

1. INTRODUCTION

The isometry classes of unary, binary and ternary positive definite quadratic forms are determined by the representation numbers. That this fact does not hold in any dimension, was shown by E. Witt's example of two non-isometric, positive definite quadratic forms in dimension 16 with the same representation numbers.

If two positive definite quadratic forms have the same representation numbers, then we call them *isospectral*. A. Schiemann conducted a computer search to provide an example of two isospectral positive definite quaternary quadratic forms with integer coefficients which are not isometric (see [6]). Hence, already in rank 4, the *theta series*, which is the generating series for the representation numbers, does not determine the isometry class.

In [4], Conway and Sloane introduced a real 4-parameter family of pairs of isospectral lattices in the euclidean space \mathbb{E}^4 , where Schiemann's example is a member of. They conjectured that the lattice pairs are non-isometric whenever the parameter coordinates are pairwise different. They verified this for lattice pairs corresponding to classical integral quadratic forms of discriminant less than 10^4 .

In this article we prove the conjecture of Conway and Sloane using our theory of lattice invariants introduced in [1] and [2]. More precisely, for each tuple (m_1, \dots, m_k) of natural numbers, we associate in [2] a lattice invariant Θ_{m_1, \dots, m_k} . It is an analytic function on the upper half plane, which gives a modular form for integral lattices. For example, Θ_0 is the classical theta series of the lattice. In [1, Proposition 4.4], we showed that for Schiemann's example the invariants $\Theta_{1,1}$ are different, hence they are not isometric.

One observes that the function $\Theta_{1,1}$ is analytic in the four parameter coordinates of the Conway-Sloane family. This implies the Conway-Sloane conjecture on a dense open subset of the parameter domain. Motivated by this observation, we started a thorough investigation of the invariant $\Theta_{1,1}$ for the lattice pairs in the Conway-Sloane family. We show that for each pair the functions $\Theta_{1,1}$ are not equal, provided that the parameter coordinates are pairwise different – and so proving the full conjecture of Conway and Sloane in [4, Remark (v)].

The invariant $\Theta_{1,1}$ enables us to give the first example of non-isometric, isospectral lattices varying in a continuous family. So far, there were used only ad-hoc methods for proving

Date: October 12, 2009.

2000 Mathematics Subject Classification. 11F11, 11F27, 11E45.

Key words and phrases. Conway-Sloane conjecture, lattice invariants.

non-isometry of isospectral lattices – which usually can not be extended to such families with real parameters.

In Section 2 we start with an alternative description of the lattice pair (L_1, L_2) of Conway and Sloane. For this, we use an action of the Kleinian four group on the self-dual codes in \mathbb{F}_3^4 . This construction explains the term *tetralattice*, as already introduced in [3]. We repeat the definition of the invariants $\Theta_{1,1}(\tau, L_i)$ in Section 3. Furthermore, we develop an explicit formula for the q -expansion of $\delta(\tau) = \frac{1}{128}(\Theta_{1,1}(\tau, L_1) - \Theta_{1,1}(\tau, L_2))$. In the next section we determine those vectors contributing to the first coefficient of the q -expansion of δ . Finally, we prove our main result, Theorem 5.1, by computing this coefficient which turns out to be negative. Using our lattice invariant $\Theta_{1,1}$ this result reduces, in the end, to a simple computation.

Notation. In this article, \mathbb{E}^n denotes the euclidean n -dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. For any $v \in \mathbb{E}^n$, $\|v\|^2 = \langle v, v \rangle$ is called the *square norm* of v .

2. THE ISOSPECTRAL FAMILY OF CONWAY AND SLOANE

2.1. A lattice with an action of the Kleinian group K_4 . We start with a lattice $L \cong \mathbb{Z}^4$ together with its Gram matrix

$$G_L = \begin{pmatrix} r & \alpha & \beta & \gamma \\ \alpha & r & -\gamma & -\beta \\ \beta & -\gamma & r & -\alpha \\ \gamma & -\beta & -\alpha & r \end{pmatrix}.$$

We see that the Kleinian four group K_4 acts on L as isometries when given as:

$$K_4 = \left\{ g_0 = \text{id}, g_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, g_3 = g_2 \cdot g_1 \right\}.$$

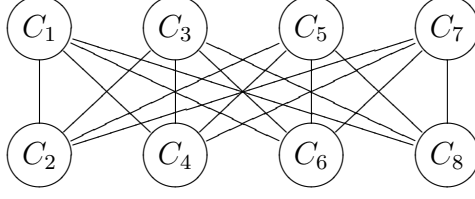
2.2. Sublattices of L from ternary codes. Using the above identification $L \cong \mathbb{Z}^4$ we obtain an isomorphism $L/3L \cong \mathbb{F}_3^4$, and a surjection $\pi : L \rightarrow \mathbb{F}_3^4$. For each linear subspace $C \subset \mathbb{F}_3^4$ we obtain a sublattice $L_C := \pi^{-1}(C)$ of L containing $3L$. Linear subspaces of \mathbb{F}_3^4 are called ternary codes. When we speak of a code C , we always mean a code $C \subset \mathbb{F}_3^4$. Since the above action of K_4 on L maps $3L$ to $3L$, we obtain an action of K_4 on \mathbb{F}_3^4 .

If two linear codes C and C' differ by an element $g \in K_4$, that is $C = g(C')$, then $L_{C'}$ and L_C are isometric because the elements of K_4 are isometries. On \mathbb{F}_3^4 we consider the non degenerate standard scalar product $\langle \cdot, \cdot \rangle : \mathbb{F}_3^4 \times \mathbb{F}_3^4 \rightarrow \mathbb{F}_3$. One easily verifies that the action of K_4 on \mathbb{F}_3^4 preserves this bilinear form. A code C is called self-dual when C is of dimension 2, and $\langle c, c' \rangle = 0$ for all $c, c' \in C$. A straightforward calculation shows that there are exactly eight self-dual codes. Here is the complete list:

$$\begin{aligned} C_1 &= \text{span}\{(1, 0, -1, -1)^t, (0, 1, +1, -1)^t\} & C_2 &= \text{span}\{(1, 0, -1, +1)^t, (0, 1, +1, +1)^t\} \\ C_3 &= \text{span}\{(1, 0, -1, +1)^t, (0, 1, -1, -1)^t\} & C_4 &= \text{span}\{(1, 0, +1, +1)^t, (0, 1, +1, -1)^t\} \\ C_5 &= \text{span}\{(1, 0, +1, -1)^t, (0, 1, +1, +1)^t\} & C_6 &= \text{span}\{(1, 0, -1, -1)^t, (0, 1, -1, +1)^t\} \\ C_7 &= \text{span}\{(1, 0, +1, +1)^t, (0, 1, -1, +1)^t\} & C_8 &= \text{span}\{(1, 0, +1, -1)^t, (0, 1, -1, -1)^t\}. \end{aligned}$$

The action of K_4 on the set $\{C_i\}_{i=1\dots 8}$ of self-dual codes has two orbits, namely $\{C_1, C_3, C_5, C_7\}$, and $\{C_2, C_4, C_6, C_8\}$. There is another description of the partition of the set $\{C_i\}_{i=1\dots 8}$. To see it, we draw the graph Γ with vertices the self-dual codes. We

connect two vertices C_i and C_j when $\dim(C_i \cap C_j) = 1$. We obtain the following picture.



Thus, Γ is the complete bipartite graph of type $(4, 4)$. The partition of the vertices is the above orbit partition.

2.3. The codes C_1 and C_2 . We write down the codes C_1 , and C_2 explicitly as

$$C_1 = \{0, \pm[v_0], \pm[v_1], \pm[v_2], \pm[v_3]\}, \text{ and } C_2 = \{0, \pm[w_0], \pm[w_1], \pm[w_2], \pm[w_3]\}$$

with

$$v_0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \text{ and}$$

$$w_0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

We observe that for each $v \in C_1$ different from zero there exists exactly one $g \in K_4$ such that $g(v) \in C_2$. We arranged the notation in such a way that $g_i(v_i) = w_i$, and $g_i(w_i) = v_i$ for all $i = 0, \dots, 3$.

2.4. The isospectral lattices L_1 and L_2 . We obtain two lattices $L_1 = \pi^{-1}(C_1)$ and $L_2 = \pi^{-1}(C_2)$. Both are sublattices of L of index 9 which contain $3L$. We show that L_1 and L_2 have the same length spectra. Any vector $l \in L_1$ has a unique form $l = 3l_1 + c_1$ with $l_1 \in L$ and $c_1 \in C$. Using this decomposition we give a map $\Psi : L_1 \rightarrow L_2$ by

$$\Psi(3l_1) = 3l_1, \text{ and } \Psi(3l_1 \pm v_i) = g_i(3l_1 \pm v_i) = 3g_i(l_1) \pm w_i.$$

It is easy to write down the inverse $\Phi : L_2 \rightarrow L_1$ of Ψ following the same recipe:

$$\Phi(3l_2) = 3l_2, \text{ and } \Phi(3l_2 \pm w_i) = g_i(3l_2 \pm w_i) = 3g_i(l_2) \pm v_i.$$

Since K_4 acts by isometries the lengths of $l \in L_1$ and $\Psi(l) \in L_2$ coincide. The bijection Ψ is not linear.

2.5. A new basis. We consider the four vectors

$$u_0 = \frac{1}{4} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_1 = \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, u_3 = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

These are common eigenvectors for the action of K_4 on $\mathbb{R}^4 = \mathbb{R} \otimes L$. Indeed, with respect to this basis the action of g_1 is given by the diagonal matrix $\text{diag}(-1, 1, -1, 1)$, and the action of g_2 corresponds to $\text{diag}(-1, -1, 1, 1)$. The Gram matrix with respect to $\mathcal{B} = \{u_0, u_1, u_2, u_3\}$ is given by

$$G_{\mathcal{B}} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \quad \text{with} \quad \begin{aligned} a &= \frac{1}{4}(r - \alpha - \beta - \gamma), \\ b &= \frac{1}{4}(r - \alpha + \beta + \gamma), \\ c &= \frac{1}{4}(r + \alpha - \beta + \gamma), \\ d &= \frac{1}{4}(r + \alpha + \beta - \gamma). \end{aligned}$$

Taking as lattice basis of L the column vectors of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}$$

with respect to the standard basis. We obtain as generators with respect to the basis \mathcal{B} the column vectors of

$$\begin{pmatrix} -1 & 1 & -1 & -1 \\ 3 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 3 & 3 & 3 \end{pmatrix}.$$

Denoting these lattice vectors by l_0 , l_1 , l_2 and l_3 , then L_1 is given by $L_1 = \text{span}\{l_0, l_1, 3l_2, 3l_3\}$, and L_2 can be described as $L_2 = \text{span}\{l_0, 3l_1, l_2, 3l_3\}$. From this description it is obvious that both lattices contain the lattice $L_{12} = L_1 \cap L_2 = \text{span}\{l_0, 3l_1, 3l_2, 3l_3\}$ as a sublattices of index three.

2.6. Conway and Sloane's description of L_1 and L_2 . Performing elementary operations with column vectors, we see that L_2 is generated by the columns of the matrix

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ -1 & -3 & -1 & 1 \\ -1 & 1 & -3 & -1 \\ -1 & -1 & 1 & -3 \end{pmatrix}$$

with respect to the basis \mathcal{B} . This is the original definition of the lattice L^- in [4]. For L_1 we find that its lattice generators with respect to \mathcal{B} are the columns of the matrix

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & -3 & 1 & -1 \\ -1 & 1 & 3 & -1 \\ -1 & -1 & 1 & 3 \end{pmatrix}.$$

Up to the diagonal matrix $\text{diag}(1, -1, 1, 1)$ which is an isometry with respect to the orthogonal basis \mathcal{B} this gives the lattice L^+ in [4]. We prefer the presented form to the one of Conway and Sloane. In our form both lattices contain the same index nine lattice $M = 3L$ spanned by the four vectors

$$m_0 = \begin{pmatrix} -3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, m_1 = \begin{pmatrix} 3 \\ -3 \\ 3 \\ 3 \end{pmatrix}, m_2 = \begin{pmatrix} 3 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \text{ and } m_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \end{pmatrix}.$$

2.7. The conjecture of Conway and Sloane. The lattices L_1 and L_2 (respectively L^+ and L^-) depend on the real numbers a , b , c , and d . To express this dependence we write $L_{1;a,b,c,d}$ and $L_{2;a,b,c,d}$. Considering a large (but finite) number of examples Conway and Sloane formulated the following

Conjecture 2.8. *For all real numbers $(a, b, c, d) \in \mathbb{R}^4$ subject to the condition $0 < a < b < c < d$ the lattices $L_{1;a,b,c,d}$ and $L_{2;a,b,c,d}$ are isospectral but not isomorphic.*

Remark 1. The above conjecture is a generalization of an example found by Schiemann in [6]. His example is the case $(a, b, c, d) = (1, 7, 13, 19)$.

Remark 2. It was shown by Conway and Sloane in [4] (and above in in 2.4) that L_1 and L_2 are isospectral.

Remark 3. As mentioned in [4, Remark (ii)] the condition $0 < a < b < c < d$ may be replaced by: $(a, b, c, d) \in \mathbb{R}_+^4$ and the four numbers are pairwise different.

3. THE DISCREPANCY OF A LATTICE PAIR

We will distinguish L_1 and L_2 using our invariant $\Theta_{1,1}$ introduced in [1]. We briefly review its definition and q -expansion. The discrepancy δ of the lattice pair (L_1, L_2) is defined to be the difference $2^{-7} (\Theta_{1,1}(L_1) - \Theta_{1,1}(L_2))$. We develop the q -expansion for the discrepancy.

3.1. The invariant $\Theta_{1,1,L}$. For a lattice $L \subset \mathbb{E}^n$ in the n dimensional Euclidean space \mathbb{E}^n , and a polynomial $h : \mathbb{E}^n \rightarrow \mathbb{C}$ we denote by $\Theta_{h,L}$ the weighted theta function

$$\Theta_{h,L}(\tau) := \sum_{l \in L} h(l) q^{\|l\|^2} \text{ with } q = \exp(2\pi i \tau).$$

This is an absolutely convergent power series for τ in the upper half plane (cf. [7, Section 3.2] and [5, Section 6]). While these functions depend on the embedding $L \subset \mathbb{E}^n$, there are algebraic combinations of them which are independent of the embedding:

Theorem 3.2. (cf. [1, Theorem 4.2]) *For a lattice $L \subset \mathbb{E}^4$, the analytic function*

$$\Theta_{1,1,L}(\tau) := \Theta_{1,1}(\tau, L) := 32 \left(\sum_{1 \leq i < j \leq 4} \Theta_{x_i x_j, L}^2(\tau) \right) + \sum_{i=1}^4 \Theta_{4x_i^2 - \sum_{j=1}^4 x_j^2, L}^2(\tau)$$

is an analytic function in τ which is independent of the embedding $L \rightarrow \mathbb{E}^4$. The function $\Theta_{1,1,L}$ can be expressed in terms of $q = \exp(2\pi i \tau)$. Its q -expansion is given by

$$\Theta_{1,1}(\tau, L) = \sum_{m \geq 0} a_m q^m \quad \text{with} \quad a_m = 4 \sum_{\substack{(l,k) \in L \times L \\ \|l\|^2 + \|k\|^2 = m}} (4 \cos^2(\angle(l, k)) - 1) \|l\|^2 \|k\|^2.$$

Proof. The defining equation gives $\Theta_{1,1}$ as a finite sum of products of analytic functions. Therefore $\Theta_{1,1}$ itself is analytic. It follows immediately from the second equality that $\Theta_{1,1}$ is independent of the chosen embedding. To show the equivalence of both expressions is a straightforward calculation:

$$\begin{aligned} \Theta_{1,1}(\tau, L) &= \sum_{(l,k) \in L \times L} \left(32 \sum_{1 \leq i < j \leq 4} l_i l_j k_i k_j + \sum_{i=1}^4 (4l_i^2 - \|l\|^2)(4k_i^2 - \|k\|^2) \right) q^{\|l\|^2 + \|k\|^2} \\ &= \sum_{(l,k) \in L \times L} \left(16 \sum_{i=1}^4 \sum_{j=1}^4 l_i l_j k_i k_j - 4 \|l\|^2 \|k\|^2 \right) q^{\|l\|^2 + \|k\|^2} \\ &= \sum_{(l,k) \in L \times L} (16 \langle l, k \rangle^2 - 4 \|l\|^2 \|k\|^2) q^{\|l\|^2 + \|k\|^2}. \end{aligned}$$

Now the definition of the cosine gives the formula for the q -expansion. □

3.3. The analytic function δ . We define the analytic function δ to be —up to a scaling factor— the difference of the two lattice invariants $\Theta_{1,1,L^+}$ and $\Theta_{1,1,L^-}$:

$$\delta(\tau, a, b, c, d) := \frac{1}{128} (\Theta_{1,1}(\tau, L_{1;a,b,c,d}) - \Theta_{1,1}(\tau, L_{2;a,b,c,d})).$$

Even though the four real parameters (a, b, c, d) are part of the definition we usually omit them for brevity.

Lemma 3.4. *We have the q -expansion*

$$\delta(\tau) = \frac{1}{8} \sum_{(l,k) \in L_1 \times L_1} (\langle l, k \rangle^2 - \langle \Psi(l), \Psi(k) \rangle^2) q^{\|k\|^2 + \|l\|^2}.$$

Proof. We have seen in the proof of Theorem 3.2 that

$$\Theta_{1,1,L_1}(\tau) = \sum_{(l,k) \in L_1 \times L_1} (16\langle l, k \rangle^2 - 4\|l\|^2\|k\|^2) q^{\|l\|^2 + \|k\|^2}.$$

Using the length preserving bijection $\Psi : L_1 \rightarrow L_2$ from 2.4 we can write

$$\begin{aligned} \Theta_{1,1,L_2}(\tau) &= \sum_{(l,k) \in L_1 \times L_1} (16\langle \Psi(l), \Psi(k) \rangle^2 - 4\|\Psi(l)\|^2\|\Psi(k)\|^2) q^{\|\Psi(l)\|^2 + \|\Psi(k)\|^2} \\ &= \sum_{(l,k) \in L_1 \times L_1} (16\langle \Psi(l), \Psi(k) \rangle^2 - 4\|l\|^2\|k\|^2) q^{\|l\|^2 + \|k\|^2}. \end{aligned}$$

Now the definition of δ implies the stated formula. \square

Next we define for $[v], [v'] \in L_1/M$ the analytic functions $\delta_{[v],[v']}$ by

$$\delta_{[v],[v']}(\tau) = \sum_{(l,k) \in [v] \times [v']} (\langle l, k \rangle^2 - \langle \Psi(l), \Psi(k) \rangle^2) q^{\|l\|^2 + \|k\|^2}.$$

Since every vector in L_1 lies in exactly one class of L_1/M we obtain from Lemma 3.4

$$(1) \quad \delta(\tau) = \frac{1}{8} \sum_{([v],[v']) \in L_1/M \times L_1/M} \delta_{[v],[v']}.$$

3.5. Recalling notation. Before we proceed, we give a system of representatives for L_1/M . We use the description from 2.3 as $L_1/M = \{[0], \pm[v_0], \pm[v_1], \pm[v_2], \pm[v_3]\}$ where

$$v_0 = \begin{pmatrix} -1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ -1 \\ -3 \\ -1 \end{pmatrix}$$

with respect to the basis \mathcal{B} . Furthermore, we recall that for $m \in M$ we have $\Psi(m) = m$ and $\Psi(m \pm v_i) = g_i(m \pm v_i)$. The g_i are isometries which are given with respect to \mathcal{B} by the diagonal matrices $g_0 = \text{id}$, $g_1 = \text{diag}(-1, 1, -1, 1)$, $g_2 = \text{diag}(-1, -1, 1, 1)$, and $g_3 = \text{diag}(1, -1, -1, 1)$.

Lemma 3.6. *The following relations among the $\delta_{[v],[v']}$ hold:*

- (1) $\delta_{[v],[v]} = 0$ for all $[v] \in L_1/M$.
- (2) $\delta_{[v],[v']} = \delta_{[v'],[v]}$ for all pairs $[v], [v'] \in L_1/M$.
- (3) $\delta_{[v],[v']} = \delta_{[v],[v']}$ for all pairs $[v], [v'] \in L_1/M$.
- (4) $\delta_{[0],[v]} = 0$ for all $[v] \in L_1/M$.

Proof. (1) We assume that $v = v_i$. The cases when $v = -v_i$ or $v = 0$ work similar. Now we rewrite the expression for $\delta_{[v_i],[v_i]}$ as follows

$$\delta_{[v_i],[v_i]}(\tau) = \sum_{(m,m') \in M \times M} \left(\langle m + v_i, m' + v_i \rangle^2 - \langle \Psi(m + v_i), \Psi(m' + v_i) \rangle^2 \right) q^{\|m + v_i\|^2 + \|m' + v_i\|^2}.$$

Since $\Psi(m + v_i) = g_i(m + v_i)$, $\Psi(m' + v_i) = g_i(m' + v_i)$, and g_i is an isometry, all summands are zero.

(2) follows immediately from the definition of $\delta_{[v],[v']}$ and $\delta_{[v'],[v]}$.

(3) First we expand the expression for $\delta_{[v],[v']}$.

$$\delta_{[v],[v']}(\tau) = \sum_{(m,m') \in M \times M} \left(\langle m + v, m' - v' \rangle^2 - \langle \Psi(m + v), \Psi(m' - v') \rangle^2 \right) q^{\|m + v\|^2 + \|m' - v'\|^2}.$$

Changing the summation parameter $m' = -m''$ we obtain

$$\delta_{[v],[v']}(v) = \sum_{(m,m'') \in M \times M} \left(\langle m+v, -m''-v' \rangle^2 - \langle \Psi(m+v), \Psi(-m''-v') \rangle^2 \right) q^{\|m+v\|^2 + \|-m''-v'\|^2}.$$

Since we have $\Psi(-m''-v') = -\Psi(m''+v')$, we obtain

$$\delta_{[v],[v']}(v) = \sum_{(m,m'') \in M \times M} \left(\langle m+v, m''+v' \rangle^2 - \langle \Psi(m+v), \Psi(m''+v') \rangle^2 \right) q^{\|m+v\|^2 + \|m''+v'\|^2}.$$

This gives the equality (3).

(4) From (2) we see that we may assume that $v \notin [0]$. By (3) we may assume that $v = v_i$ for some $i \in \{0, 1, 2, 3\}$. Before we show equality (4) we consider the action of the involution g_i on M . The orbits of length one correspond to the invariant vectors under g_i . We denote this set by M_i^1 . The orbits of length two we denote by M_i^2 . We use the disjoint union

$$M = M_i^1 \cup \bigcup_{\{m, g_i(m)\} \in M_i^2} \{m, g_i(m)\}.$$

Now we split up the summation over M into two parts due to this decomposition:

$$\delta_{[0],[v_i]} = \sum_{(m,m') \in M_i^1 \times M} \alpha_{m,m'} q^{\|m\|^2 + \|m'+v_i\|^2} + \sum_{(\{m, g_i(m)\}, m') \in M_i^2 \times M} \beta_{\{m, g_i(m)\}, m'} q^{\|m\|^2 + \|m'+v_i\|^2}$$

where the coefficients $\alpha_{m,m'}$ and $\beta_{\{m, g_i(m)\}, m'}$ are defined by

$$\begin{aligned} \alpha_{m,m'} &= \langle m, m'+v_i \rangle^2 - \langle \Psi(m), \Psi(m'+v_i) \rangle^2 \\ \beta_{\{m, g_i(m)\}, m'} &= \alpha_{m,m'} + \alpha_{g_i(m), m'} \end{aligned}$$

Now we consider the coefficients $a_{m,m'}$.

$$\begin{aligned} \alpha_{m,m'} &= \langle m, m'+v_i \rangle^2 - \langle \Psi(m), \Psi(m'+v_i) \rangle^2 \\ &= \langle m, m'+v_i \rangle^2 - \langle m, g_i(m'+v_i) \rangle^2 \\ &= \langle g_i(m), g_i(m'+v_i) \rangle^2 - \langle g_i(m), g_i(g_i(m'+v_i)) \rangle^2 && \text{since } g_i \text{ is an isometry} \\ &= \langle g_i(m), g_i(m'+v_i) \rangle^2 - \langle g_i(m), m'+v \rangle^2 && \text{since } g_i \text{ is an involution} \\ &= -\alpha_{g_i(m), m'}. \end{aligned}$$

We deduce that the coefficients $\beta_{\{m, g_i(m)\}, m'}$ are all zero. Furthermore, the coefficients $\alpha_{m,m'}$ are zero for $m = g_i(m)$. \square

Corollary 3.7. *The function $\delta(\tau)$ can be expressed as $\delta(\tau) = \sum_{0 \leq i < j \leq 3} \delta_{[v_i],[v_j]}(\tau)$.*

Proof. Starting with the formula of equation (1) and the set $\{0, \pm v_0, \pm v_1, \pm v_2, \pm v_3\}$ of representatives for L_1/M from 3.5 we get

$$\delta(\tau) = \frac{1}{8} \sum_{v, v' \in \{0, \pm v_0, \pm v_1, \pm v_2, \pm v_3\}} \delta_{[v],[v']}(v).$$

We may remove all the summands $\delta_{[v],[v']}$ with $[v] = \pm[v']$ by part (2) and (3) of the above lemma. Furthermore, we may remove the summands $\delta_{[v],[0]}$ and $\delta_{[0],[v']}$ by part (4) of Lemma 3.6. There remain 48 summands. For each $i < j$ we obtain from $\delta_{[\pm v_i],[\pm v_j]}$ and $\delta_{[\pm v_j],[\pm v_i]}$ eight times the summand $\delta_{[v_i],[v_j]}$ by (2) and (3) of Lemma 3.6. \square

4. MINIMAL VECTORS AND MINIMAL PAIRS

In this section we determine the first exponent in the q -expansion of δ . By Corollary 3.7 we have to search for the shortest lattice vectors in the equivalence classes $[v_i]$ only. The shortest vector in an equivalence class depends on the real parameters (a, b, c, d) . A vector is called minimal, if it is the shortest for a choice of the four parameters. It turns out that in each equivalence class there are at most two minimal vectors.

4.1. Minimal vectors. The square norm of a vector $v = \sum_{i=0}^3 \lambda_i u_i$ is given by $\|v\|^2 = a\lambda_0^2 + b\lambda_1^2 + c\lambda_2^2 + d\lambda_3^2$. We decompose the map assigning a vector $v \in L_1$ its square norm as follows

$$\begin{array}{ccc} L_1 & \xrightarrow{\varphi} & \mathbb{N}^4 \\ & \searrow \iota \mapsto \|v\|^2 & \downarrow \sigma \\ & & \mathbb{R} \end{array} \quad \text{with} \quad \varphi \left(\sum_{i=0}^3 \lambda_i u_i \right) = (\lambda_0^2, \lambda_1^2, \lambda_2^2, \lambda_3^2) \quad \text{and} \\ \sigma(n_0, n_1, n_2, n_3) = an_0 + bn_1 + cn_2 + dn_3.$$

Furthermore, we define a partial ordering \preceq on \mathbb{N}^4 by

$$(n_0, n_1, n_2, n_3) \preceq (n'_0, n'_1, n'_2, n'_3) \iff \sum_{i=0}^3 n_i \leq \sum_{i=0}^3 n'_i \text{ for all } i_0 \in \{0, 1, 2, 3\}.$$

As usual, we write $(n_0, n_1, n_2, n_3) \prec (n'_0, n'_1, n'_2, n'_3)$ when $(n_0, n_1, n_2, n_3) \preceq (n'_0, n'_1, n'_2, n'_3)$ but not $(n'_0, n'_1, n'_2, n'_3) \preceq (n_0, n_1, n_2, n_3)$ hold.

Lemma 4.2. *We have $(n_0, n_1, n_2, n_3) \prec (n'_0, n'_1, n'_2, n'_3)$ if and only if the inequality $an_0 + bn_1 + cn_2 + dn_3 < an'_0 + bn'_1 + cn'_2 + dn'_3$ holds for all real numbers (a, b, c, d) fulfilling $0 < a < b < c < d$.*

Proof. This equivalence is an obvious consequence of the equality $an_0 + bn_1 + cn_2 + dn_3 = (d - c)n_3 + (c - b)(n_2 + n_3) + (b - a)(n_1 + n_2 + n_3) + a(n_0 + n_1 + n_2 + n_3)$. \square

Using the map $\varphi : L_1 \rightarrow \mathbb{N}^4$, we may extend the relation \prec to the lattice L_1 by defining $l \prec l' \iff \varphi(l) \prec \varphi(l')$. For a subset $L' \subset L_1$ we say that $l' \in L'$ is minimal, when there is no $l'' \in L'$ with $l'' \prec l'$.

Lemma 4.3. *The following table gives all the minimal vectors in the equivalence classes $[v_i]$ for $i = 0, \dots, 3$.*

class	$[v_0]$	$[v_1]$	$[v_2]$	$[v_3]$
minimal vectors	v_0, v_4	v_1, v_5	v_2	v_3, v_6

$$\text{with } v_4 = \begin{pmatrix} -4 \\ 0 \\ 2 \\ -2 \end{pmatrix}, v_5 = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 0 \end{pmatrix}, v_6 = \begin{pmatrix} -4 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

Proof. The proof is similar in all four cases, so we consider here only the equivalence class $[v_0]$ leaving the remaining cases to the reader. First we remark that neither $v_0 \preceq v_4$ nor $v_4 \preceq v_0$ holds. So it is enough to show for any $w \in [v_0]$ at least one of the inequalities $v_0 \prec w$ or $v_4 \prec w$ is satisfied, unless $w \in \{v_0, v_4\}$. We take a vector $w = v_0 + \sum_{i=0}^3 \lambda_i m_i$ with m_i the lattice generators of M from 2.6 and $\lambda_i \in \mathbb{Z}$. This means

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \quad \text{with} \quad \begin{array}{l} w_1 = -1 + 3(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) \\ w_2 = 3 + 3(+\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) \\ w_3 = -1 + 3(+\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) \\ w_4 = 1 + 3(+\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3) \end{array}.$$

Suppose now that the inequality $v_0 \preceq w$ is not satisfied. By definition of the relation \preceq at least one of the following four inequalities hold:

$$\begin{aligned} (2) \quad & w_4^2 < 1 \\ (3) \quad & w_3^2 + w_4^2 < 2 \\ (4) \quad & w_2^2 + w_3^2 + w_4^2 < 11 \\ (5) \quad & w_1^2 + w_2^2 + w_3^2 + w_4^2 < 12 \end{aligned}$$

The integer w_4 is congruent to 1 modulo three. Thus $w_4^2 \geq 1$. This rules out (2). By the same argument we conclude that $w_3^2 \geq 1$ which makes inequality (3) impossible.

Assume now that (4) is fulfilled. Since $w_3^2 + w_4^2 \geq 2$, we deduce that $w_2^2 < 9$. However, w_3 is an integer multiple of 3, which implies $w_2 = 0$. We conclude that $\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3 = -1$. This way, we obtain $\lambda_1 = 1 + \lambda_0 + \lambda_2 + \lambda_3$. We obtain the following equations and inequality for w_3 and w_4 :

$$w_3 = 2 + 6(\lambda_0 + \lambda_3), \quad w_4 = 4 + 6(\lambda_0 + \lambda_2), \quad \text{and} \quad w_3^2 + w_4^2 < 11.$$

Since the λ_i are integers, we must have $w_3 = 2$ and $w_4 = -2$. We conclude that $\lambda_3 = -\lambda_0$, and $\lambda_2 = -1 - \lambda_0$. From $\lambda_1 = 1 + \lambda_0 + \lambda_2 + \lambda_3$, we deduce $\lambda_1 = -\lambda_0$. This yields $w = (-4 - 12\lambda_0, 0, 2, -2)^t$. So $v_4 \preceq w$ with equality only for $w = v_4$.

Finally we assume that inequality (5) holds. As before, we have $w_1^2 + w_3^2 + w_4^2 \geq 3$, and w_2 is divisible by three. So from $w_2^2 < 9$ we conclude $w_2 = 0$. As before we get $\lambda_1 = 1 + \lambda_0 + \lambda_2 + \lambda_3$. This yields the equations and inequality for w_1, w_3 , and w_4 :

$$w_1 = 2 + 6(\lambda_2 + \lambda_3), \quad w_3 = 2 + 6(\lambda_0 + \lambda_3), \quad w_4 = 4 + 6(\lambda_0 + \lambda_2), \quad \text{and} \quad w_1^2 + w_3^2 + w_4^2 < 12.$$

Since the λ_i are integers this implies the three equalities

$$\lambda_2 + \lambda_3 = 0, \quad \lambda_0 + \lambda_3 = 0, \quad \lambda_0 + \lambda_2 = -1.$$

From these equalities we deduce $\lambda_3 = \frac{1}{2}$. Thus inequality (5) is never fulfilled.

So we have seen that all $w \in v_0$ which are not of the form $w = (-4 - 12\lambda_0, 0, 2, -2)^t$ satisfy $v_0 \preceq w$. All vectors w of this form with $w \neq v_4$ satisfy $v_4 \prec w$. If $v_0 \preceq w$ and $w \preceq v_0$, then the squares of the coordinates of w coincide with those of v_0 . Thus, we have $w = (\pm 1, \pm 3, \pm 1, \pm 1)$. The only vector of this type in $[v_0]$ is v_0 . \square

Proposition 4.4. *Suppose we have two lattice vectors $v, v' \in L_1$ such that $v \in [v_i]$ and $v' \in [v_j]$ for $0 \leq i < j \leq 3$. If $(v, v') \notin \{(v_0, v_2), (v_5, v_2)\}$, then for all $0 < a < b < c < d$ we have*

$$\|v\|^2 + \|v'\|^2 > \min\{\|v_0\|^2 + \|v_2\|^2, \|v_5\|^2 + \|v_2\|^2\}.$$

Proof. First we note, that by Lemma 4.2 it is enough to show the statement of the proposition for the minimal vectors in each class. By Lemma 4.3 we can list all those pairs belonging to different classes modulo M :

i	j	$\varphi(v_i) + \varphi(v_j)$	i	j	$\varphi(v_i) + \varphi(v_j)$	i	j	$\varphi(v_i) + \varphi(v_j)$
0	1	(2, 10, 2, 10)	1	3	(2, 2, 10, 10)	2	6	(25, 5, 1, 5)
0	2	(10, 10, 2, 2)	1	4	(17, 1, 5, 13)	3	4	(17, 1, 13, 5)
0	3	(2, 10, 10, 2)	1	6	(17, 5, 1, 13)	3	5	(17, 5, 13, 1)
0	5	(17, 13, 5, 1)	2	3	(10, 2, 10, 2)	4	5	(32, 4, 8, 4)
0	6	(17, 13, 1, 5)	2	4	(25, 1, 5, 5)	4	6	(32, 4, 4, 8)
1	2	(10, 2, 2, 10)	2	5	(25, 5, 5, 1)	5	6	(32, 8, 4, 4)

There are two minimal 4-tuples among the $\varphi(v_i) + \varphi(v_j)$ with respect to the relation \preceq . These are the 4-tuples corresponding to the pairs $(i, j) \in \{(0, 2), (2, 5)\}$. To see that this is a complete list of minimal pairs, we check that for all pairs

$(i, j) \in \{(0, 1), (0, 3), (0, 6), (1, 2), (1, 3), (1, 4), (1, 6), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6)\}$ we

have $(\varphi(v_0) + \varphi(v_2)) \prec (\varphi(v_i) + \varphi(v_j))$. And for all those pairs of indices $(i, j) \in \{(0, 5), (1, 4), (2, 4), (2, 6), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6)\}$ we see that the inequality $(\varphi(v_2) + \varphi(v_5)) \prec (\varphi(v_i) + \varphi(v_j))$ is satisfied. So we have a complete list of minimal vectors. By Lemma 4.2 the minimum is attained by a minimal pair, which implies the proposition. \square

5. PROOF OF THE CONWAY-SLOANE CONJECTURE

Theorem 5.1. *For all real numbers (a, b, c, d) satisfying $0 < a < b < c < d$ the lattices $L^+ \cong L_{1,a,b,c,d}$ and $L^- = L_{2,a,b,c,d}$ are isospectral but not isometric.*

Proof. We have seen that both lattices are isospectral in 2.4. To show that they are not isomorphic it is enough by Theorem 3.2 to show that $\delta(\tau) = \frac{1}{128} (\Theta_{1,1,L^+}(\tau) - \Theta_{1,1,L^-}(\tau))$ is not zero. We have seen in Corollary 3.7 that

$$\delta(\tau) = \sum_{0 \leq i < j \leq 3} \delta_{[v_i],[v_j]}(\tau) \text{ with } \delta_{[v_i],[v_j]}(\tau) = \sum_{(l,k) \in [v_i] \times [v_j]} (\langle l, k \rangle^2 - \langle \Psi(l), \Psi(k) \rangle^2) q^{\|l\|^2 + \|k\|^2}.$$

By Proposition 4.4 the minimal value of $\|l\|^2 + \|k\|^2$ appearing in one of the $\delta_{[v_i],[v_j]}$ is $\min\{\|v_0\|^2 + \|v_2\|^2, \|v_5\|^2 + \|v_2\|^2\}$ and it can be attained only by the pairs (v_0, v_2) or (v_2, v_5) . Now we compute

$$\begin{aligned} \langle v_0, v_2 \rangle^2 - \langle \Psi(v_0), \Psi(v_2) \rangle^2 &= -12(b-a)(d-c) \\ \langle v_2, v_5 \rangle^2 - \langle \Psi(v_2), \Psi(v_5) \rangle^2 &= -96a(c-b). \end{aligned}$$

Since both numbers are negative by our assumption, we conclude that the coefficient of $q^{\min\{\|v_0\|^2 + \|v_2\|^2, \|v_5\|^2 + \|v_2\|^2\}}$ in $\delta(\tau)$ is negative. In particular it is not zero. Therefore $\delta(\tau) \neq 0$ which gives the result. \square

Acknowledgment. The authors thank Manuel Blicke for valuable remarks. This work has been supported by the SFB/TR 45 ‘‘Periods, moduli spaces and arithmetic of algebraic varieties’’.

REFERENCES

- [1] J. Cerviño, and G. Hein, *Lattice invariants from the heat kernel*, preprint 2009, math.NT/0906.1128.
- [2] J. Cerviño, and G. Hein, *Lattice invariants from the heat kernel II*, preprint 2009, math.NT/0909.0340.
- [3] J. H. Conway, *The sensual (quadratic) form*, Math. Assoc. America, Washington, DC, 1997. MR1478672 (98k:11035)
- [4] J. H. Conway, and N. J. A. Sloane, *Four-dimensional lattices with the same theta series*, Int. Math. Res. Notices **4** (1992) 93–96.
- [5] N. Elkies, *Theta functions and weighted theta functions of Euclidean lattices, with some applications*, draft from March 2009, 51 pages.
- [6] A. Schiemann, *Ein Beispiel positiv definitiver quadratischer Formen der Dimension 4 mit gleichen Darstellungszahlen*, Arch. Math. **54** (1990) 372–375.
- [7] D. Zagier, *Introduction to modular forms* in: M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson (eds.) From number theory to physics (Les Houches, 1989), 238–291, Springer, Berlin, 1992.

FB MATHEMATIK, UNIVERSITY DUISBURG-ESSEN, 45117 ESSEN, GERMANY
E-mail address: juan.cervino@uni-due.de

FB MATHEMATIK, UNIVERSITY DUISBURG-ESSEN, 45117 ESSEN, GERMANY
E-mail address: georg.hein@uni-due.de