

# Upper Bounds on the Slope of a Genus 3 Fibration

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ABSTRACT. We divide families of non-hyperelliptic curves of genus 3 into 6 types, and for each type we give an upper bound on their slopes.

## 1. Introduction

Let  $f : S \rightarrow C$  be a fibration of a complex smooth projective surface  $S$  onto a smooth curve  $C$  of genus  $b$ , i.e., a holomorphic map with connected fibres. Suppose that the fibration  $f$  is relatively minimal, i.e.,  $S$  has no  $(-1)$ -curves contained in a fibre of  $f$ . A fibration is called of genus  $g$  if the genus of its general fibre equals  $g$ . Similarly, a fibration is said to be hyperelliptic or non-hyperelliptic according to the type of its general fibre. We have the following basic relative numerical invariants of  $f$

$$\begin{aligned} K_{S/C}^2 &= K_S^2 - 8(g-1)(b-1), \\ \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1). \end{aligned}$$

Whenever  $\chi_f \neq 0$ , i.e.,  $f$  is locally non-trivial, the slope of the fibration  $f$  is defined as

$$\lambda_f = K_{S/C}^2 / \chi_f.$$

The slope  $\lambda_f$  is an important invariant for a fibration. In 1987, G. Xiao [17] proved that for a relatively minimal fibration  $f$  of genus  $g \geq 2$  (see also [4] for semistable fibrations), one has

$$4 - 4/g \leq \lambda_f \leq 12,$$

and  $\lambda_f = 12$  if and only if every fibre of  $f$  is smooth and reduced, i.e.,  $f$  is a Kodaira fibration. For a genus 2 fibration  $f$ , Xiao [16] proved that

$$2 \leq \lambda_f \leq 7.$$

In general, if  $f$  is a hyperelliptic fibration of genus  $g$ , Xiao [18] obtained an upper bound:

$$4 - 4/g \leq \lambda_f \leq \begin{cases} 12 - (8g+4)/g^2, & g \text{ even,} \\ 12 - (8g+4)/(g^2-1), & g \text{ odd.} \end{cases}$$

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In particular, for a hyperelliptic fibration  $f$  of genus 3, we have

$$8/3 \leq \lambda_f \leq 17/2$$

As for the relatively minimal non-hyperelliptic fibration  $f$  of genus  $g$ , one has:

$$\lambda_f \geq \begin{cases} 3, & g = 3, \text{ E. Horikawa [8] and [5],} \\ 24/7, & g = 4, \text{ Z. Chen [3] and K. Konno [6];} \\ 4, & g = 5, \text{ K. Konno [6];} \\ 96/25, & g = 6, \text{ K. Konno [7].} \end{cases}$$

Stankova-Frenkel [11] proved that if  $f$  is a semistable trigonal fibration, then

$$\lambda_f \geq \frac{24(g-1)}{5g+1}.$$

See [11, 13] for some other lower bounds. [2] is a very good survey on the study of slopes.

In this paper, we will investigate non-hyperelliptic fibrations  $f : S \rightarrow C$  of genus 3. We can not hope to get a better absolute upper bound on the slope, because theoretically there are Kodaira fibrations of genus 3 by Satake's compactification of the moduli space of genus 3 curves, although no explicit examples have been found. On the other hand, there is a close relationship between the geometry of curves on  $S$  and the slope  $\lambda_f$ .

First, for any irreducible horizontal curve  $\Gamma$  in  $S$ , we associate it with an integer  $\alpha(\Gamma)$  between 0 and 5. Let  $p$  be a generic point on  $\Gamma$  such that the fibre  $F$  of  $f$  over  $f(p)$  is a smooth non-hyperelliptic curve intersecting  $\Gamma$  transversely at  $\Gamma F$  distinct points. Denote by  $K_F$  the canonical divisor of  $F$ . Then the linear system  $|K_F - p|$  on  $F$  defines a triple cover  $F \rightarrow \mathbb{P}^1$ . Let  $\alpha_p$  be the degree of the totally ramified locus. One can prove that this number is fixed for generic points  $p$  on  $\Gamma$  (using base changes, see §3). We denote this integer by  $\alpha(\Gamma)$ .

Now we define

$$\alpha := \max \{ \alpha(\Gamma) \mid \Gamma \text{ is an irreducible horizontal curve} \}.$$

Therefore, we divide non-hyperelliptic fibrations of genus 3 into 6 types according to the value of  $\alpha$ . Our main purpose is to give an upper bound on the slope  $\lambda_f$  for each type.

**THEOREM 1.1.** *Let  $f : S \rightarrow C$  be a non-hyperelliptic fibration of genus 3. If  $f$  is not locally trivial, then*

$$\lambda_f \leq \begin{cases} 9, & \text{if } \alpha = 3, 4, 5; \\ 21/2, & \text{if } \alpha = 2. \end{cases}$$

*Therefore, Kodaira fibrations occur only when  $\alpha \leq 1$ .*

Furthermore, we obtain a formula to calculate the Horikawa number of a fibre. As a byproduct, examples of local trigonal fibration with a smooth hyperelliptic central fibre are given in this paper.

The paper is organized as follows: In §2, we recall the basic notions and the canonical resolution of a triple cover. Then we introduce a formula to calculate the Horikawa number of a fibre by the canonical resolution in §4. The proof of the main theorem is given in §5 by using a key lemma which is proved in §6 by checking case by case. In the last section §7 we give some examples of local smooth hyperelliptic

fibre within a trigonal fibration and investigate the properties of the branch locus of a trigonal Kodaira fibration.

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## 2. Preliminaries on the triple covers

In this section we recall some facts about triple covers. The details are referred to [15, 14]. (See [9] for a general theory based on rank two vector bundles).

**2.1. Triple cover data.** Let  $X$  be a smooth algebraic surface over  $\mathbb{C}$ , and let  $\pi : Y \rightarrow X$  be a normal triple cover. A triple cover data is a pair of sections  $(s, t, \mathcal{L})$ , where  $\mathcal{L}$  is an invertible sheaf,  $s \in H^0(X, \mathcal{L}^2)$  and  $0 \neq t \in H^0(X, \mathcal{L}^3)$ .  $Y$  is the normalization of the surface defined by  $z^3 + sz + t = 0$  in the line bundle of  $\mathcal{L}$ . Any triple cover  $\pi$  is determined by some triple cover data  $(s, t, \mathcal{L})$ .

If  $s = 0$ , then the triple cover is cyclic and everything is known. In fact, for numerical computations, the case when  $s = 0$  can be viewed as a special case of  $s \neq 0$  where  $\pi$  is totally ramified over the branch locus, i.e., there is no simple ramification. So we always assume that  $s \neq 0$ , which implies  $\mathcal{L} = \mathcal{O}_X(\text{div}(t/s))$  is determined by  $s$  and  $t$ .

Let

$$a = \frac{4s^3}{\gcd(s^3, t^2)}, \quad b = \frac{27t^2}{\gcd(s^3, t^2)}, \quad c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

Then  $a$ ,  $b$  and  $c$  are coprime sections of an invertible sheaf such that  $a + b = c$ .

Conversely, any coprime triples  $(a, b, c)$  with  $a + b = c$  can determine a triple cover over  $X$ . Assume that we have decompositions (according to the decompositions of their divisors)

$$a = 4a_1a_2^2a_0^3, \quad b = 27b_1b_0^2, \quad c = c_1c_0^2,$$

where  $a_1, a_2, b_1, c_1$  are square-free and  $\gcd(a_1, a_2) = 1$ . Then the data  $(s, t)$  determined by  $(a, b, c)$  is given as follows:

$$s = a_1a_2^2b_1a_0, \quad t = a_1a_2^2b_1^2b_0.$$

Denote the corresponding divisors by

$$A_i = \text{Div}(a_i), \quad B_i = \text{Div}(b_i), \quad C_i = \text{Div}(c_i).$$

Let  $D_1 = B_1 + C_1$ ,  $D_2 = A_1 + A_2$ . Then the branch locus of the triple cover  $\pi$  is

$$R = 2D_2 + D_1 = 2A_1 + 2A_2 + B_1 + C_1.$$

$\pi$  is totally ramified over  $D_2 = A_1 + A_2$ , hence  $D_2$  is called *totally ramified branch locus*.  $D_1$  is called *simply ramified branch locus*. Let  $\mathcal{E}_\pi$  denote the trace-free subsheaf of  $\pi_*\mathcal{O}_Y$ , then  $c_1(\mathcal{E}_\pi) = -D_2 - \frac{1}{2}D_1$ .

It is known that  $X$  is smooth if and only if  $D_2$  is smooth,  $D_2$  and  $D_1$  have no common points, and all of the singular points of  $D_1$  are ‘‘cusps’’ (i.e., locally defined by  $y^2 + x^{3n} = 0$ ) where  $\pi$  is totally ramified.

**2.2. Canonical resolution.** The *canonical resolution*  $\tau : \tilde{Y} \rightarrow Y$  of the singularities of  $Y$  is the following commutative diagrams.

$$\begin{array}{ccccccccc} \tilde{Y} = Y_k & \xrightarrow{\tau_k} & Y_{k-1} & \xrightarrow{\tau_{k-1}} & \cdots & \longrightarrow & Y_2 & \xrightarrow{\tau_2} & Y_1 & \xrightarrow{\tau_1} & Y_0 = Y \\ \tilde{\pi} = \pi_k \downarrow & & \downarrow \pi_{k-1} & & & & \downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 = \pi \\ \tilde{X} = X_k & \xrightarrow{\sigma_k} & X_{k-1} & \xrightarrow{\sigma_{k-1}} & \cdots & \longrightarrow & X_2 & \xrightarrow{\sigma_2} & X_1 & \xrightarrow{\sigma_1} & X_0 = X \end{array}$$

- (1)  $\sigma_{i+1}$  is the blow-up of  $X_i$  at a singular point  $p_i$  of the branch locus of  $\pi_i$ .  $Y_{i+1}$  is the normalization of  $X_{i+1} \times_{X_i} Y_i$ .  $\tilde{\pi} = \pi_k$  has a smooth branch locus. So  $\tilde{Y} = Y_k$  is smooth.
- (2) The corresponding data  $(a^{(i)}, b^{(i)}, c^{(i)})$  of  $\pi_i$  is obtained from

$$\left( \sigma_i^* a^{(i-1)}, \sigma_i^* b^{(i-1)}, \sigma_i^* c^{(i-1)} \right)$$

by eliminating the common factors.

(See also [1] for the resolution of some special singularities).

**2.3. Determination of the branch locus.** Put

$$d_i = \min \left\{ \mu_{p_i}(A^{(i)}), \mu_{p_i}(B^{(i)}), \mu_{p_i}(C^{(i)}) \right\},$$

where  $\mu_p(D)$  is the multiplicity of a divisor  $D$  at  $p$ . Let

$$(2.1) \quad m_i = \left\lfloor \frac{\mu_{p_i}(D_1^{(i)})}{2} \right\rfloor,$$

$$(2.2) \quad n_i = \begin{cases} \mu_{p_i}(D_2^{(i)}), & \text{if } d_i \equiv \mu_{p_i}(A^{(i)}) \pmod{3}; \\ \mu_{p_i}(D_2^{(i)}) - 1, & \text{otherwise.} \end{cases}$$

Let  $E_i$  be the exceptional curve of  $\sigma_i$ ,  $\mathcal{E}_i$  be the total transform of  $E_i$  in  $\tilde{X}$ , and let  $\sigma = \sigma_1 \cdots \sigma_k$ . Then

$$(2.3) \quad \tilde{D}_1 = \sigma^*(D_1) - 2 \sum_{i=0}^{k-1} m_i \mathcal{E}_{i+1},$$

$$(2.4) \quad \tilde{D}_2 = \sigma^*(D_2) - \sum_{i=0}^{k-1} n_i \mathcal{E}_{i+1}.$$

Hence we get the following criterion:

LEMMA 2.1. *We use also  $E_i$  to denote the strict transform of  $E_i$  in  $\tilde{X}$ .*

- (i)  $E_i \subset \tilde{D}_1 \iff \mu_{p_i}(D_1^{(i)})$  is odd;
- (ii)  $E_i \not\subset \tilde{D}_1$  and  $E_i \not\subset \tilde{D}_2 \iff \mu_{p_i}(D_1^{(i)})$  is even and  $d_i \equiv \mu_{p_i}(A^{(i)}) \pmod{3}$ ;
- (iii)  $E_i \subset \tilde{D}_2 \iff \mu_{p_i}(D_1^{(i)})$  is even and  $d_i \not\equiv \mu_{p_i}(A^{(i)}) \pmod{3}$ . Furthermore,
  - (a) if  $\mu_{p_i}(A^{(i)}) - d_i \equiv 1 \pmod{3}$ , then  $E_i$  is a component of  $\tilde{A}_1$ ;
  - (b) if  $\mu_{p_i}(A^{(i)}) - d_i \equiv 2 \pmod{3}$ , then  $E_i$  is a component of  $\tilde{A}_2$ .

LEMMA 2.2. *Let  $(D_1 D_2)_p$  be the intersection number of the divisors  $D_1$  and  $D_2$  at an intersecting point  $p \in P$ , then  $(D_1 D_2)_p$  is an even number.*

PROOF. Since  $\tilde{D}_1\tilde{D}_2 = 0$ , by (2.3), (2.4), one has

$$\begin{aligned} 0 = \tilde{D}_1\tilde{D}_2 &= \left( \tilde{\sigma}^*(D_1) - 2 \sum_{i=0}^{k-1} m_i \mathcal{E}_{i+1} \right) \left( \tilde{\sigma}^*(D_2) - \sum_{i=0}^{k-1} n_i \mathcal{E}_{i+1} \right) \\ &= D_1 D_2 - 2 \sum_{i=0}^{k-1} m_i n_i. \end{aligned}$$

Hence

$$(2.5) \quad D_1 D_2 = 2 \sum_{i=0}^{k-1} m_i n_i.$$

Thus

$$(D_1 D_2)_p = 2 \sum_{\sigma(\mathcal{E}_{i+1})=p} m_i n_i.$$

□

REMARK 2.3. This lemma is very useful in the canonical resolution when we do not know the defining equations. In this processes, the exceptional curves of  $E_i$  meet transversely. If  $E_i$  meets  $E_j$  at  $p$ , and there is no other component of the branch locus passing through  $p$ , then the case where  $E_i \subset D_1$  and  $E_j \subset D_2$  can not occur.

**2.4. Computation of invariants.** Now we have the formulas for the triple cover:

$$(2.6) \quad \begin{aligned} \chi(\mathcal{O}_{\tilde{Y}}) &= 3\chi(\mathcal{O}_X) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1K_X + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_X \\ &\quad - \sum_{i=0}^{k-1} \frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1} \frac{n_i(5n_i-9)}{18}, \end{aligned}$$

$$(2.7) \quad \begin{aligned} K_{\tilde{Y}}^2 &= 3K_X^2 + \frac{1}{2}D_1^2 + 2D_1K_X + \frac{4}{3}D_2^2 + 4D_2K_X \\ &\quad - \sum_{i=0}^{k-1} 2(m_i-1)^2 - \sum_{i=0}^{k-1} \frac{4n_i(n_i-3)}{3} - k, \end{aligned}$$

### 3. On the number $\alpha$

Let  $\Gamma$  be a horizontal curve on  $S$  and  $p$  be a generic point on  $\Gamma$ . In order to prove that  $\alpha_p$  (defined as in the introduction) is independent of  $p$ , we consider the base change  $\pi : \tilde{C} \rightarrow C$ , where  $\tilde{C}$  is the normalization of  $\Gamma$  and  $\pi$  is induced by  $f$ . Let  $\tilde{f} : \tilde{S} \rightarrow \tilde{C}$  be the pullback fibration of  $f$  under the base change  $\pi$ . Then  $\tilde{f}$  admits a section  $\tilde{\Gamma}$  whose image in  $S$  is  $\Gamma$ . A generic fibre of  $f$  with respect to  $\Gamma$  is isomorphic to a generic fibre of  $\tilde{f}$  with respect to  $\tilde{\Gamma}$ . So we only need to prove the independence for  $\tilde{\Gamma}$ . Namely we can assume that  $\Gamma$  is a section of  $f$ .

Now we let  $\mathcal{L} = \omega_{S/C}(-\Gamma)$ , and let  $\mathcal{E} = f_*\mathcal{L}$ . It is easy to check that  $h^0(F, \mathcal{L}|_F) = h^0(F, \omega_F(-p)) = 2$  for any fibre  $F$ , where  $\{p\} = F \cap \Gamma$ . Hence  $\mathcal{E}$  is a rank 2 vector bundle over  $C$ . We can construct a rational map

$$\phi_{\mathcal{L}} : S \dashrightarrow P_0 = \mathbb{P}(\mathcal{E}).$$

If  $F$  is a general fibre, then rational map  $\phi_{\mathcal{L}}|_F$  is a triple cover of  $\mathbb{P}^1$  defined by  $\omega_F(-p)$ . Hence  $\phi_{\mathcal{L}}$  is generally a triple cover, and the fibration is induced by the ruling of  $P_0$ . Let  $\alpha(\Gamma)$  be the degree of the totally ramified branch locus on a general fibre. Then  $\alpha_p = \alpha(\Gamma)$  for a generic point  $p$  on  $\Gamma$ . Note that  $\alpha$  is independent of the existence of the global triple cover.

#### 4. Horikawa numbers

Let  $f : S \rightarrow C$  be a relatively minimal non-hyperelliptic fibration of genus 3. Assume that  $f$  has a section  $\Gamma$  and  $\phi_{\mathcal{L}} : S \dashrightarrow P_0$  is the rational map defined as above.

In the following commutative diagrams,  $\hat{\pi}$  is the elimination of indeterminacy of  $\phi_{\mathcal{L}}$  such that  $\phi = \phi_{\mathcal{L}} \circ \hat{\pi} : \hat{S} \rightarrow P_0$  is a morphism.  $\phi = \pi_0 \circ \varepsilon$  is the Stein factorization of  $\phi$ , so  $\pi_0$  is a triple cover. The square on the right hand side is the canonical resolution of  $S_0$ .

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\varepsilon} & S_0 & \longleftarrow & \tilde{S} \\ \hat{\pi} \downarrow & & \pi_0 \downarrow & & \downarrow \tilde{\pi} \\ S & \xrightarrow{\phi_{\mathcal{L}}} & P_0 & \xleftarrow{\tilde{\sigma}} & \tilde{P} \end{array}$$

In fact,  $f : S \rightarrow C$  is the relatively minimal model of  $\tilde{S} \rightarrow C$ . So  $\tilde{\tau} : \tilde{S} \rightarrow S$  is a birational morphism.

$$(4.1) \quad \begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\pi}} & \tilde{P} \\ \tilde{\tau} \downarrow & & \downarrow \tilde{\sigma} \\ S & \xrightarrow{\phi_{\mathcal{L}}} & P_0 \\ & \searrow f & \swarrow \varphi_0 \\ & & C \end{array}$$

Let  $\tilde{R}$  be the branch locus of  $\tilde{\pi}$ ,  $R = \tilde{\sigma}(\tilde{R})$ . Then  $\tilde{\sigma}$  is the embedded resolution of singularities of the branch locus  $R$  appeared in the canonical resolution,  $\tilde{\pi}$  is the canonical resolution of the triple cover. Let  $C_0$  be a section of the ruled surface  $\varphi_0 : P_0 \rightarrow C$  such that the self-intersection number  $C_0^2 = -e$  is minimal. Let

$$D_1 = B_1 + C_1, \quad D_2 = A_1 + A_2, \quad R = D_1 + 2D_2.$$

Here  $D_1 \equiv 2\eta$  is the branch locus of double ramification, and  $D_2$  is the branch locus of triple ramification. Since the genus of a general fibre is equal to 3,  $RF = D_1F + 2D_2F = 10$ . Let

$$D_2 \sim \alpha C_0 + \beta F, \quad D_1 \sim (10 - 2\alpha)C_0 + 2\gamma F.$$

By (2.6), (2.7), we have

$$\begin{aligned} \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1) = \chi(\mathcal{O}_{\tilde{S}}) - (g-1)(b-1) \\ &= \left(\frac{5\alpha}{9} - 1\right) \left(\beta - \frac{\alpha}{2}e\right) + (4-\alpha) \left(\gamma - \frac{5-\alpha}{2}e\right) \\ &\quad - \sum_{i=0}^{k-1} \frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1} \frac{n_i(5n_i-9)}{18}, \end{aligned}$$

$$\begin{aligned}
K_{S/C}^2 &= K_S^2 - 8(g-1)(b-1) = K_S^2 - 8(g-1)(b-1) + \varepsilon \\
&= 8\left(\frac{\alpha}{3} - 1\right)\left(\beta - \frac{\alpha}{2}e\right) + 4(3-\alpha)\left(\gamma - \frac{5-\alpha}{2}e\right) \\
&\quad - \sum_{i=0}^{k-1} 2(m_i - 1)^2 - \sum_{i=0}^{k-1} \frac{4n_i(n_i - 3)}{3} - k + \varepsilon.
\end{aligned}$$

where  $\varepsilon$  is the number of  $(-1)$ -curves blown down by  $\tilde{\tau}$ . Then

$$\begin{aligned}
K_{S/C}^2 - 3\chi_f &= (\alpha - 5)\left(\beta - \frac{\alpha}{2}e\right) - \alpha\left(\gamma - \frac{5-\alpha}{2}e\right) \\
&\quad + \frac{1}{2}\sum_{i=0}^{k-1} (m_i - 1)(4 - m_i) + \frac{1}{2}\sum_{i=0}^{k-1} n_i(5 - n_i) - k + \varepsilon \\
&= -\frac{1}{2}D_1D_2 + \frac{1}{2}\sum_{i=0}^{k-1} (m_i - 1)(4 - m_i) + \frac{1}{2}\sum_{i=0}^{k-1} n_i(5 - n_i) - k + \varepsilon.
\end{aligned}$$

By (2.5)

$$(4.2) \quad D_1D_2 = 2\sum_{i=0}^{k-1} m_i n_i.$$

Let  $w_i = m_i + n_i$ , we have

$$\begin{aligned}
(4.3) \quad K_{S/C}^2 - 3\chi_f &= -\frac{1}{2}\sum_{i=0}^{k-1} (m_i + n_i - 2)(m_i + n_i - 3) + \varepsilon \\
&= -\frac{1}{2}\sum_{i=0}^{k-1} (w_i - 2)(w_i - 3) + \varepsilon.
\end{aligned}$$

For any point  $p \in C$ ,  $F = f^{-1}(p)$ , we may define

$$(4.4) \quad \tilde{H}_F = -\frac{1}{2}\sum_j (w_j - 2)(w_j - 3) + \varepsilon_F,$$

where  $w_j$  is the invariant occurred during the canonical resolution of the fibre  $\varphi_0^{-1}(p)$  and  $\varepsilon_F$  is the number of exceptional curves in  $\tilde{\tau}^{-1}(F)$  contracted by  $\tilde{\tau}$ . It is obvious that for any smooth non-hyperelliptic fibre  $F$  we have  $\tilde{H}_F = 0$ . Furthermore, we can prove (not in this paper) that  $\tilde{H}_F$  is a non-negative integer. Then

$$(4.5) \quad K_{S/C}^2 - 3\chi_f = \sum_F \tilde{H}_F.$$

Let  $F$  be a fibre of a relatively minimal non-hyperelliptic fibration  $f : S \rightarrow C$  of genus 3 and let  $p = f(F)$ . The Horikawa number of  $F$  is defined as (cf. [10])

$$H_F = \text{length coker} \left( S^2 f_* \omega_{S/C} \hookrightarrow f_* \left( \omega_{S/C}^{\otimes 2} \right) \right)_p.$$

(Ashikaga and Konno define in [2] the Horikawa number in more general situations.)

The global invariants of  $f$  depend on this number. In fact, Reid [10] shows that

$$(4.6) \quad K_{S/C}^2 - 3\chi_f = \sum_F H_F.$$

Compare (4.6) with (4.5), we believe that  $H_F = \tilde{H}_F$ , but we could not prove it rigorously now. By abuse of language, we still call  $\tilde{H}_F$  a Horikawa number and denote it by  $H_F$  in this paper.

Similarly, we can define the Horikawa number for a singular point  $p$  in the branch locus  $R$ :

$$(4.7) \quad H_p = -\frac{1}{2} \sum_j (w_j - 2)(w_j - 3) + \varepsilon_p,$$

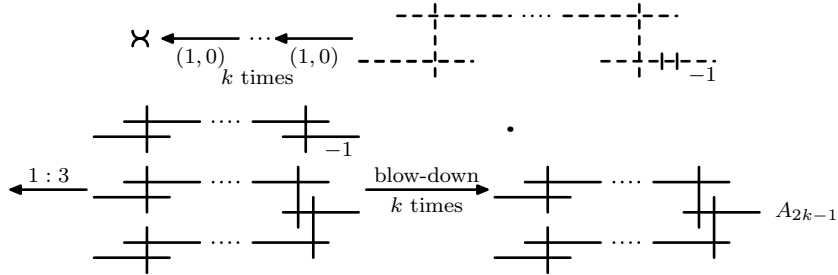
where  $w_j$  is the invariants occurred during the canonical resolution of the point  $p \in R$  and  $\varepsilon_p$  is the number of exceptional curves in  $(\tilde{\sigma}\tilde{\pi})^{-1}(p)$  contracted by  $\tilde{\tau}$ . Then

$$(4.8) \quad H_F = \sum_{\varphi_0(p)=f(F)} H_p + \varepsilon'_F,$$

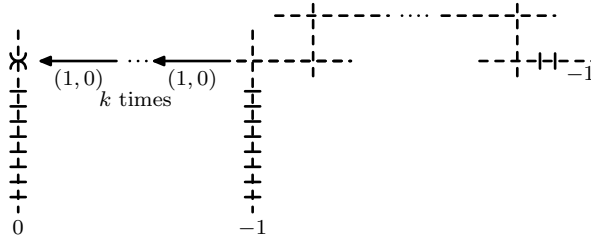
where  $\varepsilon'_F$  is the number of extra contractions.

Now we give an example to show the calculation of Horikawa numbers and the number of extra contractions.

EXAMPLE 4.1. Let  $p \in D_1$  be a double point (i.e.,  $\mu_p(D_1) = 2, \mu_p(D_2) = 0$ ) which will be resolved after  $k$  blow-ups. To illustrate the canonical resolution of the singularities, a *thick line* (resp. *thin line*) is used to represent a component contained in the divisor  $D_2$  (resp.  $D_1$ ), and a *dashed line* is used to represent a component not contained in the branch locus. The self-intersection number is marked near the component. In most cases the self-intersection number  $-2$  is omitted. The pair of numbers under an arrow  $\xleftarrow{(m_i, n_i)}$  represents the invariants  $m_i$  and  $n_i$  associated with this blow-up. And  $\xleftarrow{1:3}$  represents a triple cover.

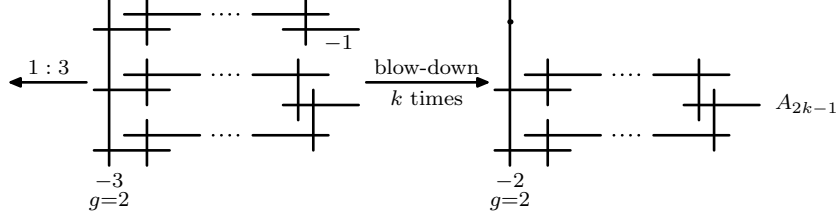


Here we have  $\varepsilon_p = k, H_p = (-1) \cdot k + k = 0$ . Now let us look at a fibre of  $\phi_0$ :

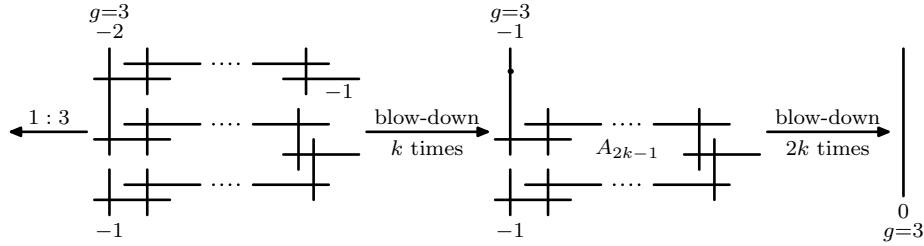


Assume that this fibre meets  $D_1$  also in 8 mutually distinct points other than  $p$ . Then there are 2 cases for the triple cover of this fibre according to the local behavior of the covering data  $(s, t, \mathcal{L})$ . In the first case, the triple cover of the strict

transform of the fibre is an irreducible curve of genus 2. In this case, the number of extra contractions  $\varepsilon'_F = 0$ .  $H_F = H_p = 0$ .



In the second case, the triple cover of the strict transform of the fibre has 2 components. One is a smooth hyperelliptic curve of genus 3, and the other is a  $(-1)$ -curve. In this case,  $\varepsilon'_F = 1$ ,  $\varepsilon_p = 2k - 1$ ,  $H_F = H_p + 2k = 2k$ .



To simplify the calculation, in what follows, we will modify the definition of the Horikawa number of a point  $p$ . For example, in the second case, we define  $H_p = 2k$ . Like this, the formula (4.8) will be replaced by the following formula.

$$(4.9) \quad H_F = \sum_{\varphi_0(p)=f(F)} H_p.$$

## 5. Proof of Theorem 1.1

We recall the following theorem about the slope of a fibration  $f$ :

**THEOREM 5.1** ([12], Corollary 4.3 and Corollary 4.4). *Let  $\tilde{f}$  be the semistable reduction of  $f$  with  $\lambda_{\tilde{f}} > 8$ .*

- (1) *If  $f$  is not semistable, then  $\lambda_{\tilde{f}} > \lambda_f$ .*
- (2) *If the slope of  $f$  is maximal, then  $f$  is semistable.*

Note that our upper bounds in the main theorem are bigger than 8, by this theorem, we can assume that  $f$  is semistable. So under any base change, the pullback fibration  $\tilde{f}$  has the same slope as  $f$ . In particular, we can assume that  $f$  admits a section  $\Gamma$ . Thus the fibration  $f$  is induced by a triple cover as in diagram (4.1). Our next purpose is to choose a better minimal model of the ruled surface  $\tilde{P} \rightarrow C$  for our triple cover.

**LEMMA 5.2.**  *$\tilde{P}$  can be contracted to a relatively minimal model  $P$  with a ruling  $\varphi : P \rightarrow C$  satisfying the following conditions.*

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{\psi}} & P \\ & \searrow \tilde{\varphi} & \swarrow \varphi \\ & C & \end{array}$$

- (1) Let  $\tilde{R}$  be the branch locus of  $\tilde{\pi}$ , and  $R$  be the image of  $\tilde{R}$  in  $P$ . Then  $\tilde{\psi} : \tilde{P} \rightarrow P$  is the canonical resolution of  $R$ .
- (2) Let  $R_h$  be the horizontal part of  $R$  (i.e.,  $R_h$  does not contain any fibres of  $\varphi$  and  $R_v = R - R_h$  is the sum of some fibres), then the multiplicities of the singular points of  $R_h$  (resp.  $R$ ) are less or equal to  $g+2$  (resp.  $g+4$ ).

The induced triple cover over  $P$  ramified over  $R$  is called *normalized*.

PROOF. Let  $\tilde{F}$  be a singular fibre of  $\tilde{\varphi} : \tilde{P} \rightarrow C$ . Then  $\tilde{F}$  contains a vertical  $(-1)$ -curve  $E$ . Since  $K_{\tilde{P}}\tilde{F} = -2$  and  $K_{\tilde{P}}E = -1$ , it is easy to prove that either the multiplicity of  $E$  in  $\tilde{F}$  is at least 2, or  $\tilde{F}$  contains another  $(-1)$ -curve  $E'$ .

As  $\tilde{R}\tilde{F} = \tilde{R}_h\tilde{F} = 2g+4$ , where  $\tilde{R}_h$  is the horizontal part of  $\tilde{R}$ , we can take a  $(-1)$ -curve  $E_1$  in  $\tilde{F}$  such that

$$\tilde{R}_h E_1 \leq \frac{1}{2} \tilde{R}_h \tilde{F} = g+2,$$

Let  $\psi_1 : \tilde{P} \rightarrow P_1$  be the contraction of  $E_1$ ,  $R_{h,1}$  be the image of  $\tilde{R}_h$  in  $P_1$ , and  $p_1 = \psi_1(E_1)$ . Then the multiplicity of  $R_{h,1}$  at  $p_1$  is at most  $g+2$ . Similarly, we do the same procedure until the ruled surface is relative minimal.

Therefore, we can get  $\tilde{\psi} : \tilde{P} \rightarrow P$  such that  $P$  is relatively minimal over  $C$ , and the multiplicities of the singular points of the image  $R_h$  of  $\tilde{R}_h$  in  $P$  is at most  $g+2$ .  $\square$

Now we use the normalized triple cover over  $P$  to replace the original triple cover  $\pi_0$  over  $P_0$  to obtain our fibration  $f : S \rightarrow C$ .

LEMMA 5.3. *Let  $\tilde{\varphi} : \tilde{P} \rightarrow C$  be a ruled surface over  $C$  (not necessarily minimal), let  $(s, t, \mathcal{L})$  be the triple cover data of a triple cover  $\theta : \tilde{S} \rightarrow \tilde{P}$  with a smooth branch locus, and let  $\tilde{f} : \tilde{S} \rightarrow C$  be the induced fibration. Assume that the genus  $g$  of  $\tilde{f}$  is nonzero. Let  $E$  be a vertical  $(-1)$ -curve in  $\tilde{S}$  and  $D = \theta(E)$ . Then we have*

- (1)  $D$  is a vertical  $(-3)$ -curve contained in  $D_2$ ;
- (2)  $D$  is a vertical  $(-2)$ -curve contained in  $D_1$ ;
- (3)  $D$  is a vertical  $(-1)$ -curve disjoint from the branch locus.
- (4)  $D$  is a vertical  $(-1)$ -curve such that near  $D$ ,  $\theta$  is factorized as a double cover and a one-to-one cover of  $\tilde{P}$ . So  $D$  is disjoint from  $D_2$ , but has intersection with  $D_1$ .
- (5)  $D \cong \mathbb{P}^1$  is a fibre disjoint from  $D_2$ , tangent to  $D_1$  with order 2 at one point  $p$ . Outside of  $p$ , the triple cover over  $D$  is decomposed as a double cover and an isomorphism.

PROOF. It is obvious that there is a vertical  $(-1)$ -curve over the above curves. Conversely, we know that  $D$  is a smooth rational curve in a fibre. Then there are 5 cases:

- 1)  $D$  is contained in  $D_2$ ;
- 2)  $D$  is contained in  $D_1$ ;
- 3)  $D$  is disjoint from the branch locus;
- 4)  $D$  is not a fibre, not disjoint from the branch locus and not a component of it.
- 5)  $D$  is a fibre, but not a component of the branch locus.

In case 1), it is easy to see that  $D^2 = 3E^2 = -3$ . In case 2), there is only simple ramification over  $D$ . Thus the two curves over  $D$  have no intersection. The

self-intersection number of the ramified curve is  $D^2/2$ , that of the other one is  $D^2$ . Thus  $D$  is a  $(-2)$ -curve. In case 3),  $D$  has the same self-intersection number as  $E$ , so  $D$  is a  $(-1)$ -curve.

In what follows, we consider cases 4) and 5). Note that

$$\theta^*(D_2) = 3\tilde{D}_2, \quad \theta^*(D_1) = 2\tilde{D}_1 + D'_1.$$

We have  $K_{\tilde{S}} = \theta^*(K_{\tilde{P}}) + R_\theta$ , where

$$R_\theta = 2\tilde{D}_2 + \tilde{D}_1.$$

Thus

$$0 \leq R_\theta E = K_{\tilde{S}} E - \theta^*(K_{\tilde{P}}) E = -1 - K_{\tilde{P}}(\theta_* E) = -1 - dK_{\tilde{P}} D,$$

where  $\theta_* E = dD$  and  $d$  is the degree of  $\theta|_E : E \rightarrow D$ . So  $K_{\tilde{P}} D < 0$ . Since  $D$  is a smooth rational curve contained in a fibre,  $D^2 \leq 0$  and  $K_{\tilde{P}} D + D^2 = -2$ . Thus  $D$  is a  $(-1)$ -curve or a smooth fibre.  $d \neq 3$  because otherwise  $E = \theta^*(D)$  and  $-1 = E^2 = 3D^2$ , a contradiction.

First we assume that  $D$  is a  $(-1)$ -curve. Then  $R_\theta E = d - 1 \leq 1$ , so  $E\tilde{D}_2 = 0$ . It is easy to see that  $DD_2 = 0$ . If  $d = 2$ , then  $\theta^*(D) = E + E'$ . Thus  $EE' = \theta^*(D)E + 1 = -d + 1 < 0$ , a contradiction. Hence  $d = 1$  and  $R_\theta E = 0$ . This is the case (4).

Now we assume that  $D$  is a fibre. Then  $F = \theta^*(D)$  is a fibre and  $K_{\tilde{P}} D = -2$ . If  $d = 2$ , then  $F = E + E'$ . We have  $E'E = 1$  and  $E'^2 = -1$ . Because  $E'$  is a smooth rational curves,  $g = p_a(F) = 0$ , which contradicts our assumption. Thus  $d = 1$  and  $R_\theta E = 1$ . Then we know that  $D$  has no intersection with  $D_2$ . We have also  $E'^2 = -1$ ,  $EE' = 1$ . Denote by  $q$  the intersection point of  $E$  and  $E'$ , and let  $p = \theta(q)$ . By local computation, we see that  $D$  must be tangent to  $D_1$  at  $p$  with order 2. Outside of  $p$ , the triple cover is locally composed of a double cover and a one to one cover. This is the case (5).  $\square$

REMARK 5.4. In Example 7.1, we can find a  $(-1)$ -curve of type (5).

LEMMA 5.5. *In the normalized triple cover, we can assume that*

- (1) *the branch locus  $R = D_1 + 2D_2$  contains no fibres of the ruling;*
- (2) *each component of  $D_1$  or  $D_2$  is a section of  $\varphi : P \rightarrow C$ .*

PROOF. Note first that we can use base change freely because  $f$  is semistable. Look at the following base change (which induces a base change of  $f$ ):

$$\begin{array}{ccc} \overline{P} & \xrightarrow{\overline{\eta}} & P \\ \overline{\varphi} \downarrow & & \downarrow \varphi \\ \overline{C} & \xrightarrow{\eta} & C \end{array}$$

Let  $(s, t, \mathcal{L})$  be the triple cover data on  $P$ . Then  $(\overline{\eta}^* s, \overline{\eta}^* t, \overline{\eta}^* \mathcal{L})$  is the triple cover data of the pullback triple cover over  $\overline{P}$ . Let  $\overline{s} = \overline{\eta}^* s$ ,  $\overline{t} = \overline{\eta}^* t$ ,  $\overline{\mathcal{L}} = \overline{\eta}^* \mathcal{L}$ , and we denote the corresponding triple cover data on  $\overline{P}$  by adding a bar. Since the greatest common divisor is independent of the base change, we have

$$\gcd(\overline{s}^3, \overline{t}^2) = \overline{\eta}^*(\gcd(s^3, t^2)).$$

Hence

$$\overline{a} = \overline{\eta}^* a, \quad \overline{b} = \overline{\eta}^* b, \quad \overline{c} = \overline{\eta}^* c.$$

Suppose that the fibres  $F_1, \dots, F_s$  are contained in the branch locus  $D_1 + D_2$ . We consider a base change  $\eta$  of degree  $6n$  totally ramified over  $p_1 = \varphi(F_1), \dots, p_s = \varphi(F_s)$  and some other generic points. Then the multiplicity of the pullback fibre  $\bar{\eta}^{-1}(F_i)$  in  $\bar{a}$  or  $\bar{b}$  or  $\bar{c}$  is divided by 6. So it can not be in the branch locus  $\bar{D}_1 + \bar{D}_2$ . Now we see that the branch locus of the pullback triple cover contains no fibres. So we can assume that the original branch locus contains no fibres.

On the other hand, there exists a base change  $\bar{\eta}$  such that  $\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{C}_1$  are composed of sections of  $\bar{\varphi}$ . Thus the branch locus of the pullback triple cover consists of sections.

The pullback of a normalized triple cover is still normalized. This completes the proof.  $\square$

There are 2 invariants associated with the canonical resolution of a singular point  $p$  (this may be an infinitely near point) in the branch locus, i.e., Horikawa number  $H_p$  and the contribution to  $\chi(\mathcal{O}_{\tilde{S}})$ , denoted by  $\delta_p$ . Their computation formulas are as follows:

$$H_p = \frac{1}{2} \sum_i (w_i - 2)(3 - w_i) + \varepsilon_p,$$

$$\delta_p = \sum_i \frac{m_i(m_i - 1)}{2} + \sum_i \frac{n_i(5n_i - 9)}{18}.$$

From these 2 invariants, we will define a slope function

$$s_p(\lambda) = H_p + \delta_p \lambda,$$

Our goal is to find the upper bound of the slope function, especially under the condition  $9 \geq \lambda \geq 6$ .

Let  $D$  be a horizontal effective divisor in the ruled surface  $\varphi : P \rightarrow C$ . In what follows, we always denote a fibre of the minimal ruled surface by  $F$ . Then the relative ramification index of  $D$  is defined as

$$r_D = D(D + K_{P/C}) \geq 0.$$

In fact, if  $\tilde{D}$  is the normalization of  $D$ , then

$$r_D = 2 \cdot (\text{geometric genus of the singular points of } D) \\ + (\text{ramification index of the finite morphism } \tilde{D} \rightarrow C).$$

Denote the contribution to  $r_D$  of each singular point  $p$  of  $D$  by  $r_p$ , then

$$r_D = \sum_p r_p.$$

It is obvious that  $r_p$  is the sum of 2 times of the geometric genus of  $(D, p)$  and the contribution of the inverse image of  $p$  to the ramification index of  $\tilde{D} \rightarrow C$ .

In what follows, we use the following notations:

$$r_1 = r_{D_{1,h}}, \quad r_2 = r_{D_{2,h}}, \\ r_{1,p} = r_{D_{1,h,p}}, \quad r_{2,p} = r_{D_{2,h,p}}.$$

LEMMA 5.6. *Assume that the triple cover data  $(s, t, \mathcal{L})$  over  $\varphi : P \rightarrow C$  is normalized (cf. Lemma 5.2), and that the induced fibration  $f : S \rightarrow C$  is semistable and of maximal slope. We assume also that the branch locus  $D_1$  and  $D_2$  consist of sections (without vertical components). Let  $\lambda$  be a number such that*

$$\lambda \geq \begin{cases} 6, & \text{if } \alpha = D_2F \geq 3, \\ 27/4, & \text{if } \alpha = D_2F = 2. \end{cases}$$

Then for any singular point  $p$  of the branch locus, we have

$$(5.1) \quad s_p(\lambda) \leq M_1(\lambda)r_{1,p} + M_2(\lambda)r_{2,p} + M_3(\lambda)(D_1D_2)_p,$$

where

$$\begin{aligned} M_1(\lambda) &= \lambda/9, \\ M_2(\lambda) &= \lambda/6 + 1/3, \\ M_3(\lambda) &= \begin{cases} -\lambda/9 + 1/3, & \text{if } D_2F \geq 3, \\ -\lambda/9 + 5/12, & \text{if } D_2F = 2. \end{cases} \end{aligned}$$

PROOF OF THEOREM 1.1. By Lemma 5.5, the hypothesis is satisfied. Note that  $\alpha = D_2F$ . Let

$$D_2 \sim \alpha C_0 + \beta F, \quad D_1 \sim (10 - 2\alpha)C_0 + 2\gamma F.$$

By calculation,

$$r_1 = 4(9 - 2\alpha) \left( \gamma - \frac{5 - \alpha}{2} e \right), \quad r_2 = 2(\alpha - 1) \left( \beta - \frac{\alpha}{2} e \right),$$

$$D_1D_2 = 2(5 - \alpha) \left( \beta - \frac{\alpha}{2} e \right) + 2\alpha \left( \gamma - \frac{5 - \alpha}{2} e \right).$$

Since  $D_1$  and  $D_2$  are horizontal effective divisors,

$$\beta - \frac{\alpha}{2} e \geq 0, \quad \gamma - \frac{5 - \alpha}{2} e \geq 0.$$

Let

$$\chi_s = \left( \frac{5\alpha}{9} - 1 \right) \left( \beta - \frac{\alpha}{2} e \right) + (4 - \alpha) \left( \gamma - \frac{5 - \alpha}{2} e \right).$$

We have

$$\begin{aligned} \lambda\chi_f - \sum_F H_F &= \lambda \left( \chi_s - \sum_p \delta_p \right) - \sum_p H_p \\ &= \lambda\chi_s - \sum_p (H_p + \lambda\delta_p) \\ &\geq \lambda\chi_s - \sum_p (M_1(\lambda)r_{1,p} + M_2(\lambda)r_{2,p} + M_3(\lambda)(D_1D_2)_p) \\ &= \lambda\chi_s - M_1(\lambda)r_1 - M_2(\lambda)r_2 - M_3(\lambda)(D_1D_2). \end{aligned}$$

If  $\alpha \geq 3$ , then

$$\lambda\chi_f - \sum_F H_F \geq \frac{4}{9}(\lambda - 6) \left( \beta - \frac{\alpha}{2} e \right) + \frac{\alpha}{9}(\lambda - 6) \left( \gamma - \frac{5 - \alpha}{2} e \right).$$

We put  $\lambda = 6$ . Then  $6\chi_f \geq \sum_F H_F = K_{S/C}^2 - 3\chi_f$ . Hence  $K_{S/C}^2 \leq 9\chi_f$ .

If  $\alpha = 2$ , then

$$\lambda\chi_f - \sum_F H_F \geq \frac{4}{9} \left( \lambda - \frac{57}{8} \right) (\beta - e) + \frac{2}{9} \left( \lambda - \frac{15}{2} \right) \left( \gamma - \frac{3}{2}e \right).$$

Let  $\lambda = 15/2$ . Then  $\lambda\chi_f \geq \sum_F H_F = K_{S/C}^2 - 3\chi_f$ , i.e.,  $K_{S/C}^2 \leq (21/2)\chi_f$ .  $\square$

## 6. Proof of Lemma 5.6

Under the hypothesis of Lemma 5.6, we have  $\mu_p(D_1) + 2\mu_p(D_2) \leq 5$ . Hence there are 9 types:

- (i)  $\mu_p(D_1) = 2, \mu_p(D_2) = 0$ ;
- (ii)  $\mu_p(D_1) = 0, \mu_p(D_2) = 2$ ;
- (iii)  $\mu_p(D_1) = 1, \mu_p(D_2) = 1$ ;
- (iv)  $\mu_p(D_1) = 3, \mu_p(D_2) = 0$ ;
- (v)  $\mu_p(D_1) = 2, \mu_p(D_2) = 1$ ;
- (vi)  $\mu_p(D_1) = 1, \mu_p(D_2) = 2$ ;
- (vii)  $\mu_p(D_1) = 4, \mu_p(D_2) = 0$ ;
- (viii)  $\mu_p(D_1) = 3, \mu_p(D_2) = 1$ ;
- (ix)  $\mu_p(D_1) = 5, \mu_p(D_2) = 0$ .

At first, we will investigate the case of extra contractions (cf. §4).

LEMMA 6.1. *Under the hypothesis of Lemma 5.6, if  $D_2F \geq 2$ , then the extra contractions occur only in the following cases:*

- (1) Case (ii) with  $D_2F = 2$ ;
- (2) Case (vi) with  $D_2F = 2$ .

PROOF. An extra contraction occurs only if the triple cover of the strict transform of the fibre  $F$  contains a  $(-1)$ -curve. Hence the self-intersection number of the strict transform of  $F$  must be  $-1$  and there is no triple ramification on it (case (4) of Lemma 5.3). This implies that there exists only one singular point  $p$  of  $R$  on  $F$ , and after a blow-up, there is no singular point on the strict transform of  $F$ . On the other hand,  $D_2 \cap F = \{p\}$ . Since  $\mu_p(D_2) \leq 2$ , we must have  $\mu_p(D_2) = 2$  under the condition  $D_2F \geq 2$ . The only possible cases are (ii) and (vi).  $\square$

REMARK 6.2. If we omit the condition  $D_2F \geq 2$ , then extra contractions can occur in all the types (i)–(ix).

LEMMA 6.3. *Under the hypothesis of Lemma 5.6, during the canonical resolution, if an exceptional curve is a double ramification component, then it cannot have more than one singular point on it. Similarly, if an exceptional curve is a triple ramification component, then it cannot have more than 2 singular points on it. If the self-intersection number of a triple ramification exceptional curve is less than  $-1$ , then it cannot have more than one singular points on it.*

PROOF. Since the components of  $D_1, D_2$  are sections, after blowing up, their strict transforms cannot be tangent to the exceptional curve. If an exceptional curve  $E$  is contained in  $D_1$  or  $D_2$ , then the ramification divisor over  $E$  is a multiple component of the fibre in  $f$ . Because  $f$  is semistable, this ramification divisor must be contracted.

Note that if the exceptional curve is not coming from the blow-ups of a good cusp, then the contraction of the inverse image curve of type (2) in Lemma 5.3

is isolated, i.e., it does not induce the contraction of the curves connected with it. Type (1) of Lemma 5.3 is similar. So the exceptional curve  $E$  will become a  $(-2)$ -curve in  $D_1$  or a  $(-3)$ -curve in  $D_2$  as in Lemma 5.3.

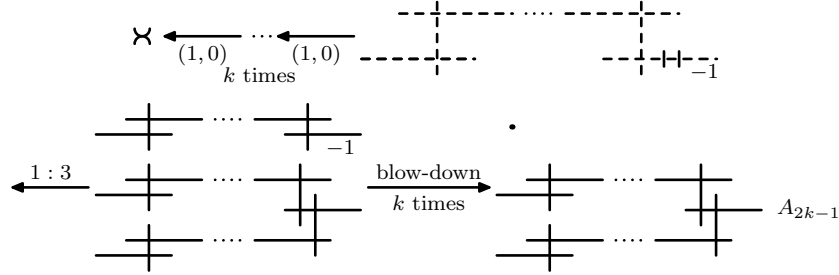
The  $(-2)$ -curve of type (2) in Lemma 5.3 comes from a  $(2k + 1 \rightarrow 2k + 1)$  type singularity of  $D_1$ , so there is only one singular point on  $E$ .

The proof of the totally ramified case is similar. □

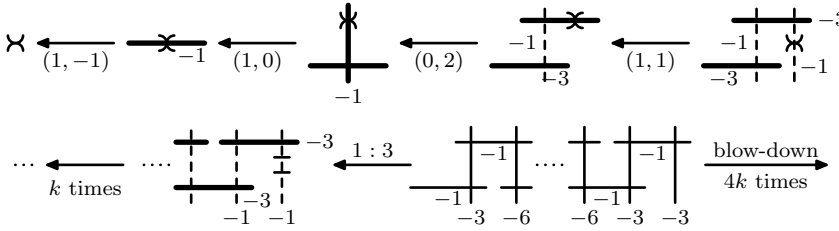
Now let's investigate type by type.

**Type (i)**  $\mu_p(D_1) = 2, \mu_p(D_2) = 0$ . It has 2 subtypes:

**(i-a)** Hyperelliptic type (cf. Example 4.1):  $H_p = \delta_p = 0, s_p(\lambda) = 0$ . In fact this is a rational double point  $A_{2k-1}$  plus a smooth point, we call it hyperelliptic.



**(i-b)** Good cusp:



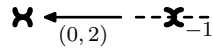
The local equation of this point  $p$  is  $x^2 + y^{6k}$  (it has two smooth components). The triple cover of this point is a smooth point. The invariants of the canonical resolution are as follows:

$$\begin{aligned} H_p &= [(-3) + (-1) + 0 + 0]k + 4k = 0; \\ \delta_p &= [7/9 + 0 + 1/9 - 2/9]k = 2k/3; \\ r_{1,p} &= 6k; \\ s_p(\lambda) &= (\lambda/9)r_{1,p}. \end{aligned}$$

In what follows, we will investigate a segment of the series of blow-ups. If after some blow-ups the obtained singular points are the same type or previously known low-multiplicity singularities, we will stop the process. If we can check that in this segment the inequality (5.1) is true, then by induction, it is true for this point.

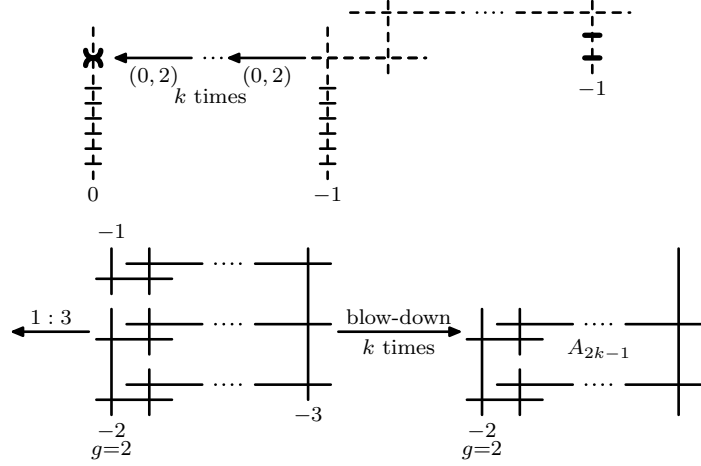
**Type (ii)**  $\mu_p(D_1) = 0, \mu_p(D_2) = 2$ .

**(ii-a)**



$$\begin{aligned} H_p &= 0; & \delta_p &= 1/9; & r_{2,p} &= 2. \\ s_p(\lambda) &= \lambda/9 \leq (\lambda/18)r_{2,p} < (\lambda/6 + 1/3)r_{2,p}. \end{aligned}$$

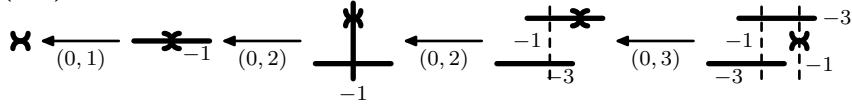
(ii-b) If  $\alpha = D_2F = 2$ , we may have extra contractions.



$$H_p = k; \quad \delta_p = k/9; \quad r_{2,p} = 2k.$$

$$s_p(\lambda) = (\lambda/18 + 1/2)r_{2,p} \leq (\lambda/6 + 1/3)r_{2,p}, \quad \text{if } \lambda \geq 1.5.$$

(ii-c)

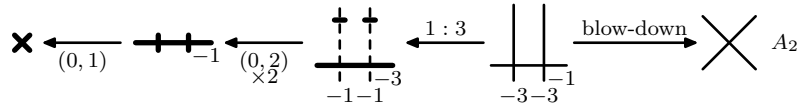


In this subtype,  $\varepsilon_p = 3$  (two  $(-1)$ -curves totally ramified over the two  $(-3)$ -curves, and one  $(-3)$ -curve over the  $(-1)$ -curve).

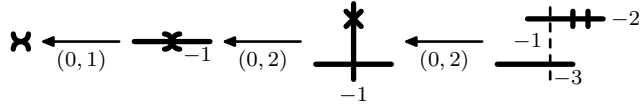
$$H_p = -1 + 0 + 0 + 0 + 3 = 2; \quad \delta_p = -2/9 + 1/9 + 1/9 + 1 = 1; \quad r_{2,p} = 6.$$

$$s_p(\lambda) = \lambda + 2 \leq (\lambda/6 + 1/3)r_{2,p}.$$

(ii-d) Degenerate case.

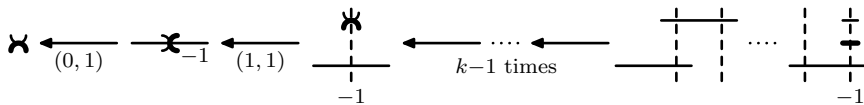


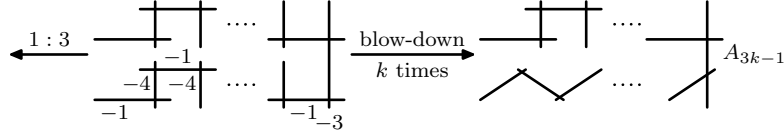
$$H_p = -1 + 0 + 0 + 0 + 1 = 0; \quad \delta_p = -2/9 + 1/9 + 1/9 = 0; \quad s_p(\lambda) = 0.$$



The last exceptional curve is a triple ramification component, its self-intersection number is  $-2$  and has 2 singular points on it. By Lemma 6.3, this is impossible. In fact it is a rational double point  $E_6$ .

**Type (iii)**  $\mu_p(D_1) = 1, \mu_p(D_2) = 1$ .



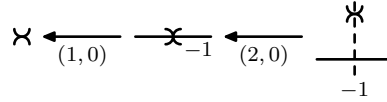


$$H_p = (-1)k + k = 0; \quad \delta_p = (-2/9 - 2/9)k = -4k/9; \quad (D_1 D_2)_p = 2k.$$

$$s_p(\lambda) = -2\lambda/9(D_1 D_2)_p < (-\lambda/9 + 1/3)(D_1 D_2)_p.$$

**Type (iv)**  $\mu_p(D_1) = 3$ ,  $\mu_p(D_2) = 0$ .

At first, we investigate the non-degenerate case.



If the singular point is hyperelliptic, then  $\varepsilon_p = 3$ . If not,  $\varepsilon_p = 1$ . Hence we have

$$H_p \leq -1 + 0 + 3 = 2; \quad \delta_p = 0 + 1 = 1; \quad r_{1,p} = 12.$$

$$s_p(\lambda) \leq (\lambda/12 + 1/6)r_{1,p} \leq (\lambda/9)r_{1,p}, \quad \text{if } \lambda \geq 6.$$

If after a blow-up the singular point becomes of lower multiplicity (degenerate case), then there are 2 possible cases as follows:

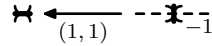


This is impossible by Lemma 6.3. In fact, in the degenerate case, the singular point is rational double point of type  $D_l$  or  $E_l$ .

In the general case, after some blow-ups, the infinitely near singular points will be of lower multiplicity or in the degenerate case. In any case, Lemma 5.6 is true.

**Type (v)**  $\mu_p(D_1) = 2$ ,  $\mu_p(D_2) = 1$ .

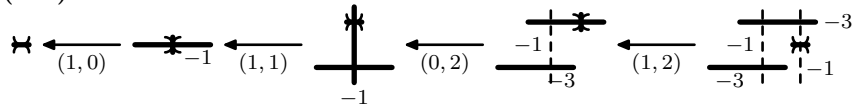
(v-a)



$$H_p = 0; \quad \delta_p = -2/9; \quad r_{1,p} = 2; \quad (D_1 D_2)_p = 2.$$

$$s_p(\lambda) = -2\lambda/9 < (\lambda/9)r_{1,p} + (-\lambda/9 + 1/3)(D_1 D_2)_p.$$

(v-b)

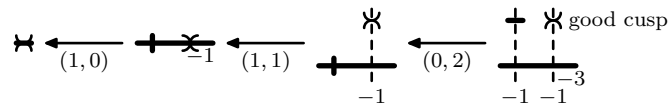


In this case  $\varepsilon_p = 3$ . Hence

$$H_p = -1 + 0 + 0 + 0 + 3 = 2; \quad \delta_p = 0 - 2/9 + 1/9 + 1/9 = 0; \quad r_{1,p} = 6; \quad (D_1 D_2)_p = 6.$$

$$s_p(\lambda) = 2 = (\lambda/9)r_{1,p} + (-\lambda/9 + 1/3)(D_1 D_2)_p.$$

(v-c) The degenerate case. Only one case induces a semistable fibre.



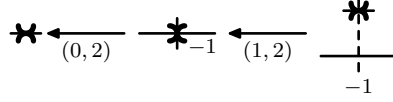
In this case  $\varepsilon_p = 3$ . Hence

$$H_p = -1+0+0+0+3 = 2; \quad \delta_p = 0-2/9+1/9+1/9 = 0; \quad r_{1,p} = 6; \quad (D_1 D_2)_p = 6.$$

$$s_p(\lambda) = 2 = (\lambda/9)r_{1,p} + (-\lambda/9 + 1/3)(D_1 D_2)_p.$$

**Type (vi)**  $\mu_p(D_1) = 1, \mu_p(D_2) = 2$ .

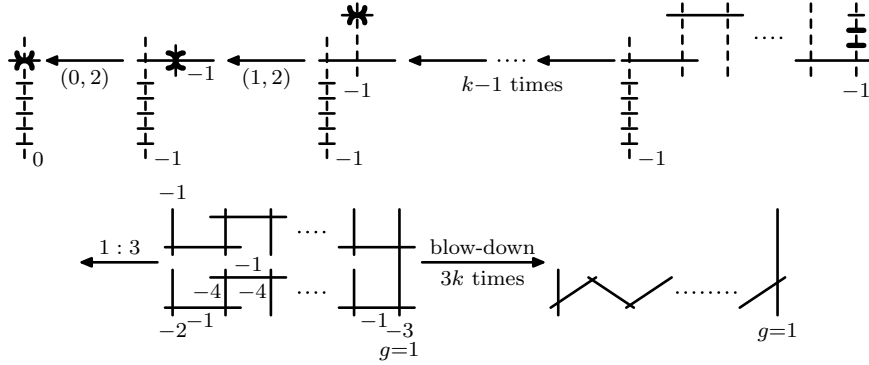
(vi-a)



$$H_p = 0 + 0 + 1 = 1; \quad \delta_p = 1/9 + 1/9 = 2/9; \quad r_{2,p} = 4; \quad (D_1 D_2)_p = 4.$$

$$s_p(\lambda) = 2\lambda/9 + 1 < (\lambda/6 + 1/3)r_{2,p} + (-\lambda/9 + 1/3)(D_1 D_2)_p.$$

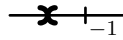
(vi-b) If  $\alpha = D_2 F = 2$ , we may have extra contractions.



$$H_p = 3k; \quad \delta_p = 2k/9; \quad r_{2,p} = 4k; \quad (D_1 D_2)_p = 4k.$$

$$s_p(\lambda) = 2k\lambda/9 + 3k = (\lambda/6 + 1/3)r_{2,p} + (-\lambda/9 + 5/12)(D_1 D_2)_p.$$

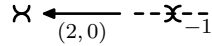
By Lemma 6.3, the degenerate case



is impossible.

**Type (vii)**  $\mu_p(D_1) = 4, \mu_p(D_2) = 0$ .

(vii-a)

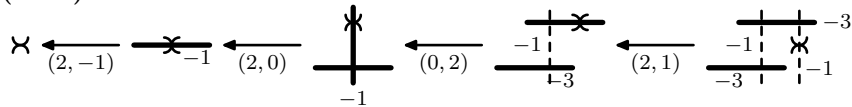


If the singular point is hyperelliptic, then  $\varepsilon_p = 1$ . Otherwise,  $\varepsilon_p = 0$ . Hence we have

$$H_p \leq 0 + 1 = 1; \quad \delta_p = 1; \quad r_{1,p} = 12.$$

$$s_p(\lambda) \leq (\lambda/12 + 1/12)r_{1,p} \leq (\lambda/9)r_{1,p}, \quad \text{if } \lambda \geq 3.$$

(vii-b)

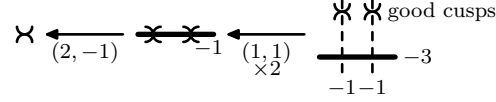


In this case we have  $\varepsilon_p = 3$ .

$$H_p = -1 + 0 + 0 + 0 + 3 = 2; \quad \delta_p = 16/9 + 1 + 1/9 + 7/9 = 11/3; \quad r_{1,p} = 36.$$

$$s_p(\lambda) = (11\lambda/108 + 1/18)r_{1,p} \leq (\lambda/9)r_{1,p}, \quad \text{if } \lambda \geq 6.$$

(vii-c) Degenerate case, note that  $(D_1D_2)_p$  should be even.

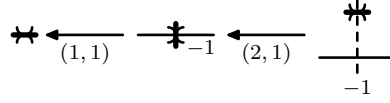


In this case we have  $\varepsilon_p = 1$ .

$$H_p = -1 + 0 \times 2 + 1 = 0; \quad \delta_p = 16/9 - 2/9 - 2/9 = 4/3; \quad r_{1,p} = 16.$$

$$s_p(\lambda) = (\lambda/12)r_{1,p} \leq (\lambda/9)r_{1,p}.$$

**Type (viii)**  $\mu_p(D_1) = 3, \mu_p(D_2) = 1$ .



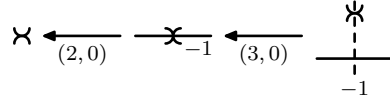
In this case we have  $\varepsilon_p = 1$ .

$$H_p = 0 + 0 + 1 = 1; \quad \delta_p = -2/9 + 7/9 = 5/9; \quad r_{1,p} = 12; \quad (D_1D_2)_p = 6.$$

$$s_p(\lambda) = 5\lambda/9 + 1 < (\lambda/9)r_{1,p} + (-\lambda/9 + 1/3)(D_1D_2)_p.$$

**Type (ix)**  $\mu_p(D_1) = 5, \mu_p(D_2) = 0$ .

At first, we investigate the non-degenerate case.



If the singular point is hyperelliptic, then  $\varepsilon_p = 3$ . Otherwise  $\varepsilon_p = 1$ . Hence we have

$$H_p \leq 0 + 0 + 3 = 3; \quad \delta_p = 1 + 3 = 4; \quad r_{1,p} = 40.$$

$$s_p(\lambda) \leq (\lambda/10 + 3/40)r_{1,p} \leq (\lambda/9)r_{1,p}, \quad \text{if } \lambda \geq 27/4.$$

In the degenerate case, after a blow-up there would be more than one singular point, this is impossible by Lemma 6.3.

## 7. Examples of smooth hyperelliptic central fibre

In this section we will give some examples to show how to construct local fibration by triple cover such that its central fibre is a smooth hyperelliptic curve of genus 3. Let  $P = \mathbb{P}_{\mathbb{C}[[t]]}^1 = \mathbb{P}_{\mathbb{C}}^1 \times_{\mathbb{C}} \text{Spec}(\mathbb{C}[[t]])$ . Then  $\varphi : P \rightarrow \text{Spec}(\mathbb{C}[[t]])$  is a local  $\mathbb{P}^1$ -bundle whose central fibre is  $F_0 = \varphi^{-1}(0) \cong \mathbb{P}^1$ . Let  $y$  denote the affine coordinate in  $\mathbb{P}_{\mathbb{C}}^1$ . Let  $P = U \cup V$  be an affine open cover of  $P$  where  $P \setminus U$  is the line at infinity  $\infty \times_{\mathbb{C}} \text{Spec}(\mathbb{C}[[t]])$ ,  $P \setminus V = Z(y)$ . Let  $U_y = U - Z(y) = \text{Spec}(\mathbb{C}[[t]][y, y^{-1}])$ .

EXAMPLE 7.1. Let

$$\mathbf{s} = (-9t^3 + 9t^2 - 3)y^4 + 12ty^2 - 3t^2 \in \Gamma(P, \mathcal{O}_P(4)),$$

$$\mathbf{t} = (9t^3 - 9t^2 + 2)y^6 + (9t^4 + 18t^3 - 12t)y^4 + 15t^2y^2 + 2t^3 \in \Gamma(P, \mathcal{O}_P(6)).$$

and  $\mathcal{L} = \mathcal{O}_P(2)$ . By using the following polynomial equation in  $\mathcal{L}^3$

$$p(z) = z^3 + \mathbf{s}z + \mathbf{t},$$

we can define the triple cover  $f : Y \rightarrow P$  determined by the triple cover data  $(\mathbf{s}, \mathbf{t}, \mathcal{L})$ .

Then we have

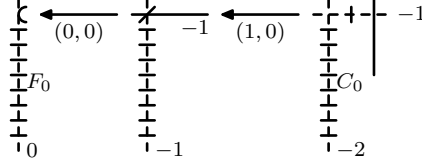
$$a_0 = \mathbf{s} = (-9t^3 + 9t^2 - 3)y^4 + 12ty^2 - 3t^2,$$

$$b_0 = \mathbf{t} = (9t^3 - 9t^2 + 2)y^6 + (9t^4 + 18t^3 - 12t)y^4 + 15t^2y^2 + 2t^3,$$

$$a_1 = a_2 = b_1 = 1, \quad c_0 = 27t^2y,$$

$$c_1 = (-4t^5 + 12t^4 - 12t^3 + 3t^2 + 2t - 1)y^{10} + (22t^3 - 26t^2 + 4t + 4)y^8 \\ + (-t^4 + 20t^3 + 8t^2 - 22t - 2)y^6 + (22t^2 + 8t)y^4 + (4t^2 - 1)y^2 + 4t.$$

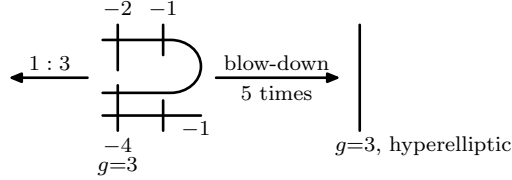
The discriminant of  $c_1$  is a polynomial in  $t$ , hence it has 10 simple roots in an infinitely small neighborhood of  $t = 0$ . When  $t = 0$ ,  $c_1$  has a double root  $y = 0$  and 8 simple roots. Thus this triple cover has only double ramification. The following diagram shows the resolution of the singular points of the branch locus.



Note that  $U_y$  is invariant during the resolution,  $F_0 \cap U_y \cong C_0 \cap U_y$ . Since  $F_0$  is contained in the zero set of  $c_0$ ,  $Y$  is not normal over  $f^{-1}(F_0)$  (cf. [15]). But the restriction of the defining polynomial  $p(z)$  to  $F \cap U_y$  is

$$p(z) \equiv z^3 - 3y^4z + 2y^6 = (z + 2y^2)(z - y^2)^2 \pmod{t}$$

So  $p(z)$  is reducible in  $\mathbb{C}[[t]][y, y^{-1}]$ . This implies that after the normalization  $\tilde{Y} \rightarrow Y$ , the triple cover of  $C_0$  has 2 components. By the connectedness of the fibre, we can obtain the smooth fibre bundle.



The Horikawa number  $H_F = -3 - 1 + 5 = 1$ .

EXAMPLE 7.2. Similarly to Example 7.1, let

$$\mathbf{s} = -3y^4 + 9(-t^4 + t^3)y^3 + 18t^2y \in \Gamma(P, \mathcal{O}_P(4)),$$

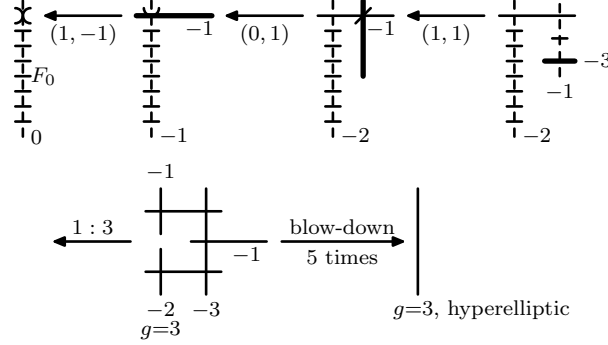
$$\mathbf{t} = 2y^6 + 9(t^4 - t^3)y^5 - 18t^2y^3 + 27t^5y^2 + 27t^4 \in \Gamma(P, \mathcal{O}_P(6)).$$

and  $\mathcal{L} = \mathcal{O}_P(2)$ . Then we have

$$a_0 = \mathbf{s} = -3y^4 + 9(-t^4 + t^3)y^3 + 18t^2y,$$

$$\begin{aligned}
 b_0 = \mathbf{t} &= 2y^6 + 9(t^4 - t^3)y^5 - 18t^2y^3 + 27t^5y^2 + 27t^4, \\
 a_1 = a_2 = b_1 &= 1, \quad c_0 = 27t^3, \\
 c_1 &= -(t^2 - 2t + 1)y^{10} - (4t^6 - 12t^5 + 12t^4 - 4t^3)y^9 + 4y^8 \\
 &\quad + (24t^4 - 30t^3 + 6t^2)y^7 - (30t^2 + 6t)y^5 + 27t^4y^4 - 4y^3 + 54t^3y^2 + 27t^2
 \end{aligned}$$

The following diagram shows the resolution of the singular points of the branch locus.



The Horikawa number  $H_F = -3 - 1 + 0 + 5 = 1$ .

In these two examples, if we replace  $t$  by  $t^n$ , namely we consider a base change of degree  $n$  totally ramified over the central fibre, then we get a new hyperelliptic central fibre  $F_n$ . By the formula in Theorem 4.8 of [12], the Horikawa number of  $F_n$  is

$$H_{F_n} = nH_F = n.$$

If a fibre in a trigonal fibration is a smooth hyperelliptic curve, it is called *bad smooth fibre*. In the preceding examples, some bad smooth fibre have been constructed. Their Horikawa number can take any positive integers.

**PROPOSITION 7.3.** *Let  $F_0$  be a fibre of a minimal ruled surface  $\varphi : P \rightarrow C$ , and let  $f : S \rightarrow C$  be a relatively minimal fibration obtained by a triple cover of  $P$ . If the fibre of  $f$  over  $F_0$  is a bad smooth fibre, then*

- (1)  $\alpha = D_2F \leq 1$ ;
- (2) *There is only one singular point  $p \in F_0$  of branch locus. If  $D_2F = 0$ , then  $\mu_p(D_1) \leq 3$ . If  $D_2F = 1$ , then  $\mu_p(D_1) = \mu_p(D_2) = 1$ . Hence the other intersecting points of branch locus with  $F_0$  are all of double ramification.*

**PROOF.** Obviously  $F_0$  could not be a component of the branch locus. So there must be singular points (good cusps are excluded) of branch locus in  $F_0$ . After the canonical resolution, there are many components in the inverse image of  $F_0$ . If there is no extra contractions, the obtained fibre of the relative minimal fibration cannot be smooth. And the hyperelliptic curve must be a component of the strict transform of  $F_0$ . This smooth hyperelliptic component is a double cover of  $F_0 \cong \mathbb{P}^1$ . To insure that its genus is  $g$ , there should be  $2g + 2$  distinct double ramification points in  $F_0$ . But note that  $RF_0 = 2g + 4$ .

Let  $p \in F_0$  be a singular point of the branch locus. If  $D_2F_0 \geq 2$ , then there are at most  $2g$  double ramifications in  $F_0$ , a contradiction. If  $\mu_p(D_2) = 1$ , then  $\mu_p(D_1) \geq 1$  and  $\mu_p(D_1)$  must be odd. But by the same reason as previous,  $\mu_p(D_1) \geq$

3 will not do. So we have  $\mu_p(D_1) = \mu_p(D_2) = 1$ . If  $\mu_p(D_2) = 0$  and  $\mu_p(D_1) \geq 4$ , then the remaining double ramifications are not sufficient.

The examples above imply that  $\mu_p(D_2) = 0$ ,  $\mu_p(D_1) \leq 3$  may locally induce a smooth hyperelliptic fibre of genus 3. But we have not found an example for the case  $\mu_p(D_2) = \mu_p(D_1) = 1$ . Since any singular point will decrease the number of double ramifications, there cannot exist more than one singular points in  $F_0$ .  $\square$

The examples above imply that bad smooth fibres may exist when  $\alpha = D_2F \leq 1$ . As we know the Kodaira fibration do exist when  $g \geq 3$ , so the slope may reach the upper bound 12 when  $\alpha \leq 1$ . At last we will investigate the behavior of the branch locus if  $f$  is Kodaira fibration.

**COROLLARY 7.4.** *If  $f$  is a Kodaira fibration, then the branch locus must satisfy the following conditions:*

- (1)  $D_2F = 0$ : A singular point  $p$  of the branch locus (good cusp is excluded) must be one of following type. If a fibre has a singular point as follows, it can have neither second singular point nor good cusps.
  - (a) Double point not tangent to the fibre;
  - (b) Triple point not tangent to the fibre;
  - (c) Smooth point tangent to the fibre with order 2.
- (2)  $D_2F = 1$ : A singular point  $p$  of the branch locus (good cusp is excluded) must be of following type. If a fibre has a singular point as follows, it can have neither second singular points nor good cusps.
  - (a)  $\mu_p(D_1) = \mu_p(D_2) = 1$  and the intersection number  $(D_1D_2)_p$  is even.  $D_1, D_2$  are not tangent to the fibre.

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