CONGRUENCE FOR RATIONAL POINTS OVER FINITE FIELDS AND CONIVEAU OVER LOCAL FIELDS

HÉLÈNE ESNAULT AND CHENYANG XU

Abstract. If the ℓ-adic cohomology of a projective smooth variety, defined over a local field K with finite residue field k, is supported in codimension ≥ 1, then every model over the ring of integers of K has a k-rational point. For K a p-adic field, this is [8, Theorem 1.1]. If the model $\mathcal{X}$ is regular, one has a congruence $|\mathcal{X}(k)| \equiv 1$ modulo $|k|$ for the number of k-rational points ([7, Theorem 1.1]). The congruence is violated if one drops the regularity assumption.

1. Introduction

Let X be a projective variety defined over a local field K with finite residue field $k = \mathbb{F}_q$. Let R be the ring of integers of K. A *model* of X/K is a flat projective morphism $\mathcal{X} \to \mathrm{Spec}(R)$, with $\mathcal{X}$ an integral scheme, such that tensored with K over R, it is $X \to \mathrm{Spec}(K)$. As in [7] and [8], we consider ℓ-adic cohomology $H^i(\bar{X})$ with $\mathbb{Q}_\ell$-coefficents. Recall briefly that one defines the first coniveau level

$$N^1H^i(\bar{X}) = \{\alpha \in H^i(\bar{X}), \exists \text{ divisor } D \subset X \text{ s.t. } 0 = \alpha|_{X\setminus D} \in H^i(\overline{X \setminus D})\}.$$

As $H^i(\bar{X})$ is a finite dimensional $\mathbb{Q}_\ell$-vector space, one has by localization

$$\exists D \subset X \text{ s.t. } N^1H^i(\bar{X}) = \mathrm{Im}\big(H^i_{\bar{D}}(\bar{X}) \to H^i(\bar{X})\big),$$

where $D \subset X$ is a divisor. One says that $H^i(\bar{X})$ is *supported in codimension 1* if $N^1H^i(\bar{X}) = H^i(\bar{X})$. The purpose of this note is twofold. We show the following theorem.

Theorem 1.1. *Let X be a smooth, projective, absolutely irreducible variety defined over a local field K with finite residue field k. Assume that ℓ-adic cohomology $H^i(\bar{X})$ is supported in codimension ≥ 1 for all $i \geq 1$. Let $\mathcal{X}$ be a model of X over the ring of integers R of K. Then there is a projective surjective morphism $\sigma : \mathcal{Y} \to \mathcal{X}$ of R-schemes such that*

$$|\mathcal{Y}(k)| \equiv 1 \bmod |k|.$$

In particular, any model $\mathcal{X}/R$ of X/K has a k-rational point.

Date: June 6, 2007.

Partially supported by the DFG Leibniz Preis and the Amercian Institute for Mathematics.

This generalizes [8, Theorem 1.1] where the theorem is proven under the assumption that K has characteristic 0. On the other hand, assuming that $\mathcal{X}$ is regular, we showed in [7, Theorem 1.1] that the number of k-rational points $|\mathcal{X}(k)|$ is congruent to 1 modulo $|k|$. It was in fact the way to show that k-rational points exist on $\mathcal{X}$, as surely $|k|$, being a p-power, where p is the characteristic of k, is > 1. We show that if we drop the regularity assumption, there are models which, according to Theorem 1.1, have a rational point, but do not satisfy the congruence.

Theorem 1.2. *Let* $X_0 = \mathbb{P}^2$ *over* $K_0 := \mathbb{Q}_p$ *or* $\mathbb{F}_p((t))$*. Then there is a finite field extension* $K \supset K_0$*, which can be chosen to be unramified, and there is a normal model* $\mathcal{X}/R$ *of* $X := X_0 \otimes_{K_0} K$*, such that* $|\mathcal{X}(k)|$ *is not congruent to* 1 *modulo* $|k|$*.*

The proof of Theorem 1.1 follows closely the one in unequal characteristic in [8, Theorem 1.1], and, aside of Deligne's integrality theorem [5, Corollaire 5.5.3] and [7, Appendix] and purity [9], relies strongly on de Jong's alteration theorem as expressed in [4]. However, we have to replace the trace argument we used there by a more careful analysis of the Leray spectral sequence stemming from de Jong's construction. The construction of the examples in Theorem 1.2 uses Artin's contraction theorem as expressed in [1] and is somewhat inspired by Kollár's construction exposed in [2, Section 3.3].

Acknowledgements: We thank Johan de Jong for his interest.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let K be a local field with finite residue field k. Let $R \subset K$ be its valuation ring. Let $\mathcal{X} \to \operatorname{Spec} R$ be a model of a projective variety $X \to \operatorname{Spec} K$. We do not assume here that X is absolutely irreducible, nor do we assume that X/K is smooth. Then by [4, Corollary 5.15], there is a diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\pi"] \arrow[drr] & \mathcal{Y} \arrow[r, "\sigma"] \arrow[dr] & \mathcal{X} \arrow[d] \\
 & & \operatorname{Spec} R
\end{tikzcd} \tag{2.1}$$

and a finite group G acting on $\mathcal{Z}$ over $\mathcal{Y}$ with the properties

(i) $\mathcal{Z} \to \operatorname{Spec} R$ and $\mathcal{Y} \to \operatorname{Spec} R$ are flat,
(ii) σ is projective, surjective, $K(\mathcal{X}) \subset K(\mathcal{Y})$ is a purely inseparable field extension,
(iii) $\mathcal{Y}$ is the quotient of $\mathcal{Z}$ by G,
(iv) $\mathcal{Z}$ is regular.

We want to show that this $\mathcal{Y}$ does it. Let us set

$$Y = \mathcal{Y} \otimes K,\ Z = \mathcal{Z} \otimes K.$$

The only difference with [8, (2.1)] is that $K(\mathcal{X}) \subset K(\mathcal{Y})$ may be a purely inseparable extension rather than an isomorphism. Thus, the argument there breaks down as one does not have traces as in [8, (2.3), (2.4)]. We do not have [8, (2.5)] a priori, and we can't conclude [8, Claim 2.1].

Let us overtake the notations of *loc. cit.*: we endow all schemes considered (which are R-schemes) with the upper subscript ^{u} to indicate the base change $\otimes_R R^u$ or $\otimes_K K^u$, where $K^u \supset K$ is the maximal unramified extension, and $R^u \supset R$ is the normalization of R in K^u. Likewise, we write $\bar{?}$ to indicate the base change $\otimes_R \bar{R}$, $\otimes_K \bar{K}$, $\otimes_k \bar{k}$, where $\bar{K} \supset K$, $\bar{k} \supset k$ are the algebraic closures and $\bar{R} \supset R$ is the normalization of R in $\bar{K}$. We consider as in [7, (2.1)] the F-equivariant exact sequence ([6, 3.6(6)])

$$\ldots \to H^i_{\bar{B}}(\mathcal{Y}^u) \xrightarrow{\iota} H^i(\bar{B}) = H^i(\mathcal{Y}^u) \xrightarrow{sp^u} H^i(Y^u) \to \ldots, \tag{2.2}$$

where $F \in \mathrm{Gal}(\bar{k}/k)$ is the geometric Frobenius, and $B = \mathcal{Y} \otimes k$. We have [8, Claim 2.2] unchanged:

Claim 2.1. The eigenvalues of the geometric Frobenius $F \in \mathrm{Gal}(\bar{k}/k)$ acting on $\iota(H^i_{\bar{B}}(\mathcal{Y}^u)) \subset H^i(\bar{B})$ lie in $q \cdot \bar{\mathbb{Z}}$ for all $i \geq 1$.

So the problem is to show that the eigenvalues of F acting on $\mathrm{Im}(sp^u) \subset H^i(Y^u)$ lie in $q \cdot \bar{\mathbb{Z}}$ as well. Let us decompose the morphism σ as

$$\sigma : Y \xrightarrow{\tau} X_1 \xrightarrow{\epsilon} X \tag{2.3}$$

where X_1 is the normalization of X in $K(Y)$. Thus in particular, τ is birational, ϵ is finite and purely inseparable. Let us denote by $U \subset X$ a non-empty open such that $\tau|_{\epsilon^{-1}(U)}$ is an isomorphism, and let us set $D := X \setminus U$. We define

$$\mathcal{C} := \mathrm{cone}(\mathbb{Q}_\ell \to R\tau_* \mathbb{Q}_\ell)[-1] \tag{2.4}$$

as an object in the bounded derived category of $\mathbb{Q}_\ell$-constructible sheaves on X_1. Since $\tau_* \mathbb{Q}_\ell = \mathbb{Q}_\ell$, the cohomology sheaves of $\mathcal{C}$ are in degree ≥ 1, and have support in $D_1 := D \times_X X_1$. We conclude

$$H^i_{D_1^u}(X_1^u, \mathcal{C}) = H^i(X_1^u, \mathcal{C})\ \forall i \geq 0. \tag{2.5}$$

One has the commutative diagram of exact sequences

(2.6)
$$\begin{tikzcd}
H^{i+1}_{D_1^u}(X_1^u) & \\
H^i_{D_1^u}(X_1^u, \mathcal{C}) \arrow[u] \arrow[r, "=(2.5)"] & H^i(X_1^u, \mathcal{C}) \\
H^i_{E^u}(Y^u) \arrow[u] \arrow[r] & H^i(Y^u) \arrow[u] \\
H^i_{D_1^u}(X_1^u) \arrow[u] \arrow[r] & H^i(X_1^u) \arrow[u]
\end{tikzcd}$$

where $E = \sigma^{-1}(D)$. By [7, Theorem 1.5 and Appendix] the eigenvalues of F on $H^i(X^u) = H^i(X_1^u)$ and on $H^{i+1}_{D_1^u}(X_1^u) = H^{i+1}_{D^u}(X^u)$ lie in $q \cdot \bar{\mathbb{Z}}$. For the latter cohomology, one has to argue again by purity on X^u before applying *loc. cit.*: by purity one is reduced to considering cohomology of the type $H^a(\Sigma^u)(-1)$ for a regular scheme Σ over K and $a \geq 0$. It remains to consider the eigenvalues of F on $H^i_{E^u}(Y^u) = H^i_{L^u}(Z^u)^G$ where $L = D \times_X Z$. This is again the argument by purity and then *loc. cit.* So we conclude

Claim 2.2. The eigenvalues of the geometric Frobenius $F \in \mathrm{Gal}(\bar{k}/k)$ acting on $H^i(Y^u)$, and therefore on $\mathrm{Im}(sp^u) \subset H^i(Y^u)$, lie in $q \cdot \bar{\mathbb{Z}}$ for all $i \geq 1$.

So we conclude now as usual that all the eigenvalues of F acting on $H^i(\bar{B})$ lie in $q \cdot \bar{\mathbb{Z}}$ for $i \geq 1$, thus the Grothendieck-Lefschetz trace formula applied to $H^*(\bar{B})$, together with the absolute irreducibility of B, imply the congruence. This finishes the proof of Theorem 1.1.

3. Construction of examples

This section is devoted to the proof of Theorem 1.2.

Let us first recall that if E is a smooth genus 1 curve over a finite field $\mathbb{F}_q$, it is always an elliptic curve, which means that it always carries a $\mathbb{F}_q$-rational point. Furthermore one has

Claim 3.1. Given an elliptic curve $E/\mathbb{F}_q$, there is a finite field extension $\mathbb{F}_{q^n} \supset \mathbb{F}_q$ such that $|E(\mathbb{F}_{q^n})|$ is not congruent to 1 modulo q^n.

Proof. By the trace formula, $|E(\mathbb{F}_{q^n})|$ being congruent to 1 modulo q^n for all $n \geq 1$ is equivalent to saying that the eigenvalues of F^n acting on $H^i(\bar{E})$ lie in $q^n \cdot \bar{\mathbb{Z}}$ for all $n \geq 1$ and $i \geq 1$. By purity (which in dimension 1 is Weil's theorem), this is equivalent to saying that the eigenvalues of F^n acting on $H^1(\bar{E})$ lie in $q^n \cdot \bar{\mathbb{Z}}$ for all $n \geq 1$. On the other hand, by duality, if λ is an eigenvalue, then $\frac{q^n}{\lambda}$ is also an eigenvalue. This is then impossible that both λ and $\frac{q^n}{\lambda}$ be q^n-divisible as algebraic integers.

□

We now construct the following scheme. Let us set $\mathcal{P}_0 := \mathbb{P}^2$ over $R_0 := \mathbb{Z}_p$ or over $\mathbb{F}_p[[t]]$. Choose an elliptic curve $E_0 \subset \mathcal{P} \otimes \mathbb{F}_p = \mathbb{P}^2_{\mathbb{F}_p}$ defined over $\mathbb{F}_p$. Let $k \supset \mathbb{F}_p$ be a finite field extension such that $|E_0(k)|$ is not k-divisible (Claim 3.1). Set $E := E_0 \otimes_{\mathbb{F}_p} k,\ \mathcal{P} := \mathcal{P}_0 \otimes_{R_0} R$, with $R = W(k)$ or $\mathbb{F}_q[[t]]$, and $K = \mathrm{Frac}(R)$. Choose a smooth projective curve $\mathcal{C} \subset \mathcal{P}$ over R, of degree ≥ 4, such that $C := \mathcal{C} \otimes k$ is transversal to E. Define $\Sigma = E \cap C \subset E$ to be the 0-dimensional intersection subscheme. It has degree ≥ 12, thus in particular > 9. Let $b : \mathcal{Y} \to \mathcal{P}$ be the blow up of $\Sigma \subset \mathcal{P}$. We denote by P_Σ the exceptional locus, which is a trivial $\mathbb{P}^2$ bundle over Σ, by Y the strict transform of $\mathbb{P}^2_k$, and we still denote by $E \subset Y$ the strict transform of the elliptic curve. Then the conormal bundle $N^\vee_{E/\mathcal{Y}}$ of E in $\mathcal{Y}$ is an extension of the conormal bundle $N^\vee_{E/Y}$ of E in Y by the restriction to E of the conormal bundle $N^\vee_{Y/\mathcal{Y}}$ of Y in $\mathcal{Y}$, both ample line bundles on E by the condition on the degree of Σ.

Let $I \subset \mathcal{O}_\mathcal{Y}$ be the ideal sheaf of E. For a coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$, we denote by $I^n/I^{n+1} \cdot \mathcal{F}$ the image of $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}$ in $\mathcal{F}$, where $n \in \mathbb{N}$.

Claim 3.2. For every coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$, one has $H^1(E, I^n/I^{n+1} \cdot \mathcal{F}) = 0$ for all $n \in \mathbb{N}$ large enough.

Proof. As by definition one has a surjection $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F} \twoheadrightarrow I^n/I^{n+1} \cdot \mathcal{F}$, it is enough to show $H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}) = 0$ for n large enough. As I^n/I^{n+1} is locally free, $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}$ is an extension of $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}_0$ by $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{T}$, where $\mathcal{T} \subset \mathcal{F}$ is the maximal torsion subsheaf and $\mathcal{F}_0 = \mathcal{F}/\mathcal{T}$ is locally free. As $H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{T}) = 0$, we may assume that $\mathcal{F}$ is locally free. As I^n/I^{n+1} is a locally free filtered sheaf, with associated graded a sum of ample line bundles of strictly increasing degree as n grows, we have $H^1(E, \mathrm{gr}(I^n/I^{n+1}) \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}) = 0$ for n large enough, and thus $H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}) = 0$ as well.

□

Artin's contraction criterion [1, Theorem 6.2] applied to $E \to \mathrm{Spec}(k)$, together with Artin's existence theorem [1, Theorem 3.1] show the existence of a contraction

$$a_1 : \mathcal{Y} \to \mathcal{X}_1 \tag{3.1}$$

where $\mathcal{X}_1$ is an algebraic space over R, $a_1|_{\mathcal{Y}\setminus E}$ is an isomorphism and $a_1(E) = \mathrm{Spec}(k)$. Let $\mathcal{X} \xrightarrow{\nu} \mathcal{X}_1$ be the normalization of $\mathcal{X}_1$ in $K(\mathcal{Y}) = K(\mathcal{P})$. This is a normal algebraic space over R. One has a diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "a"'] \arrow[rr, bend left, "a_1"] \arrow[d, "b"'] & \mathcal{X} \arrow[r, "\nu"'] & \mathcal{X}_1 \\ \mathcal{P} & & \end{tikzcd} \tag{3.2}$$

Claim 3.3. $|\mathcal{X}(k)|$ is not congruent to 1 modulo $|k|$.

Proof. By [7, Theorem 1.1] (or by a simple computation in this case), $|\mathcal{Y}(k)|$ is congruent to 1 modulo $|k|$. By Claim 3.1 and the choice of E, $|\mathcal{X}_1(k)|$ is not congruent to 1 modulo $|k|$. On the other hand, as the fibers of a_1 are absolutely irreducible, ν has to be a homeomorphism. Thus $|\mathcal{X}(k)| = |\mathcal{X}_1(k)|$. This finishes the proof.

□

In order to finish the proof of Theorem 1.2, it remains to show

Claim 3.4. $\mathcal{X} \to \mathrm{Spec}(R)$ is a model of $X = \mathbb{P}^2/K$.

Proof. We have to show that $\mathcal{X} \to \mathrm{Spec}(R)$ is a flat projective morphism. Since $\mathcal{X}$ is reduced, $\mathrm{Spec}(R)$ is regular of dimension 1, then [10, IV Proposition 14.3.8] allows to conclude that $\mathcal{X}/R$ is flat. Thus we just have to show that $\mathcal{X}/R$ is projective. To this aim, we want to decend a line bundle from $\mathcal{Y}$ to $\mathcal{X}$. Let us define the line bundle $\mathcal{M} := b^*\mathcal{O}_{\mathcal{P}}(\mathcal{C})(-P_\Sigma)$ on $\mathcal{Y}$. By definition, one has

$$\mathcal{M}|_E \cong \mathcal{O}_E. \tag{3.3}$$

Claim 3.5. The line bundle $\mathcal{M}$ descends to $\mathcal{X}$, that is there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $a^*\mathcal{L} = \mathcal{M}$.

Proof of Claim 3.5. The proper morphism of algebraic spaces $a : \mathcal{Y} \to \mathcal{X}$, with $a_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$, has the property that $a^{-1}a(E) = E$ set-theoritically, that $a|_{\mathcal{Y}\setminus E} : \mathcal{Y}\setminus E \to \mathcal{X}\setminus a(E)$ is an isomorphism, and that $H^1(E, I^n/I^{n+1}) = 0$ for $n \geq 1$. So Keel's theorem [11, Lemma 1.10] asserts that some positive power $\mathcal{M}^{\otimes r}$ descends to $\mathcal{X}$ if the following condition is fulfilled

$$\forall m > 0, \exists r(m) > 0 \text{ s.t } \mathcal{M}^{\otimes r(m)}|_{E_m} \text{ descends to } a(E_m) \tag{3.4}$$
$$\text{where } E_m := \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}}/I^{m+1}).$$

So we just have to check that (3.4) is fulfilled with $r = 1$ in our situation. The scheme $a(E_m)$ has Krull dimension 0. Thus by Hilbert 90's theorem (see e.g. [12, Corollary 11.6]) one has

$$\mathrm{Pic}(a(E_m)) = 0. \tag{3.5}$$

We conclude that to check (3.4) is equivalent to checking that $\mathcal{M}^{\otimes r(m)}|_{E_m} \cong \mathcal{O}_{E_m}$ for some positive power $r(m)$. In fact one has

$$\mathcal{M}|_{E_m} \cong \mathcal{O}_{E_m}\ \forall m \geq 1. \tag{3.6}$$

For $m = 1$, this is (3.3). We argue by induction and assume that for $m > 1$, we have a trivializing section $s_m : \mathcal{O}_{E_m} \xrightarrow{\cong} \mathcal{M}|_{E_m}$. We want to show that it lifts to a trivializing section $s_{m+1} : \mathcal{O}_{E_{m+1}} \xrightarrow{\cong} \mathcal{M}|_{E_{m+1}}$. One has an exact sequence

$$0 \to I^{m+1}/I^{m+2} \to \mathcal{M}|_{E_{m+1}} \to \mathcal{M}|_{E_m} \to 0. \tag{3.7}$$

Since $H^1(E, I^{m+1}/I^{m+2}) = 0$, as $m \geq 0$, the trivializing section of $s_m : \mathcal{O}_{E_m} \xrightarrow{\cong} \mathcal{M}|_{E_m}$ lifts to a section $s_{m+1} : \mathcal{O}_{E_{m+1}} \to \mathcal{M}|_{E_{m+1}}$, and likewise, its inverse $t_m : \mathcal{M}|_{E_m} \xrightarrow{\cong} \mathcal{O}_{E_m}$ lifts to $t_{m+1} : \mathcal{M}|_{E_{m+1}} \to \mathcal{O}_{E_{m+1}}$. The composite $t_{m+1} \circ s_{m+1} : \mathcal{O}_{E_{m+1}} \to \mathcal{O}_{E_{m+1}}$ lifts the identity of $\mathcal{O}_{E_m}$. Therefore it is invertible. This shows that s_{m+1} trivializes. The proof of Keel's theorem (see (2) after [11, (1.10.1)]) shows then that one can take $r = 1$.

□

In order the finish the proof of Claim 3.4, it remains to see that $\mathcal{L}$ on $\mathcal{X}$ is ample. First, $\mathcal{L}|_{\mathcal{X}\otimes k}$ is ample because by [11, Corollary 0.3], this is enough to see that the linear system associated to $\mathcal{L}|_{\mathcal{X}\otimes k}$ does not contract any curve, which is true by construction. So by Serre vanishing theorem, for sufficiently large m, $H^1(\mathcal{X} \otimes k, \mathcal{L}|^{\otimes m}_{\mathcal{X}\otimes k}) = 0$. Base change implies $H^1(\mathcal{X}, \mathcal{L}^{\otimes m}) \otimes k = 0$ ([10, III Theorem 7.7.5]), thus by Nakayama's lemma, one has

$$H^1(\mathcal{X}, \mathcal{L}^{\otimes m}) = 0 \text{ for } m \text{ large enough.} \tag{3.8}$$

As $\mathcal{L}$ is invertible, multiplication $\mathcal{L}^{\otimes m} \xrightarrow{\pi} \mathcal{L}^{\otimes m}$ by the uniformizer π is injective, with quotient $\mathcal{L}|^{\otimes m}_{\mathcal{X}\otimes k}$. Thus (3.8) implies surjectivity $H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \twoheadrightarrow H^0(\mathcal{X} \otimes k, \mathcal{L}|^{\otimes m}_{\mathcal{X}\otimes k})$ for m large enough. Thus $H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$ is a free R-module, and the linear system $H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$ maps base point freely $\mathcal{X}$ to $\mathbb{P}^N_R$, with $N + 1 = \mathrm{rank}_R H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$. As it embedds $\mathcal{X} \otimes k$, it embedds $\mathcal{X}$ as well. This finishes the proof.

□

4. Dimension 1

Remark 4.1. In Theorem 1.1, if X/K has dimesnion 1, which means concretely if $X/K = \mathbb{P}^1/K$, then any normal model $\mathcal{X}/R$ satisfies the congruence $|\mathcal{X}(k)| \equiv 1$ modulo $|k|$. Thus the examples of Theorem 1.2 have the smallest possible dimension.

Proof. Indeed, using (2.1), the only thing to check is that $H^1(\bar{A})$, which is equal to $H^1(\mathcal{X}^u)$, injects via σ^* into $H^1(\bar{B}) = H^1(\mathcal{Y}^u)$. Here $A := \mathcal{X} \otimes_R k$. Let us denote by $\mathcal{X}'$ the normalization of $\mathcal{X}$ in $K(\mathcal{Y})$, with factorization

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\sigma'"'] \arrow[rr, bend left, "\sigma"] & \mathcal{X}' \arrow[r, "\nu"'] & \mathcal{X} \end{tikzcd} \tag{4.1}$$

and set $A' := A \times_{\mathcal{X}} \mathcal{X}'$. Then σ' induces an isomorphism $K(\mathcal{X}') \xrightarrow{\cong} K(\mathcal{Y})$. Furthermore, $\mathcal{X}' \xrightarrow{\nu} \mathcal{X}$ and and $A' \xrightarrow{\nu|_A} A$ are homeomorphisms. Thus $H^1(\mathcal{X}^u) = H^1(\bar{A}) \xrightarrow{\nu^*} H^1((\mathcal{X}')^u) = H^1(\bar{A}')$ is an isomorphism. On the other hand, since $\sigma'_*\mathbb{Q}_\ell = \mathbb{Q}_\ell$, the Leray spectral sequence for σ' applied to $H^1(\mathcal{Y}^u)$ yields an inclusion $H^1((\mathcal{X}')^u) = H^1(\bar{A}') \xrightarrow{\text{inj}} H^1(\mathcal{Y}^u) = H^1(\bar{B})$. This finishes the proof. □

REFERENCES

[1] Artin, M.: Algebraization of Formal Moduli: II. Existence of Modifications, Ann. of Math. (2) **91** (1970), 88–135.

[2] Blickle, M., Esnault, H.: Rational singularities and rational points, preprint 2006, 12 pages, appears in the volume dedicated to F. Bogomolov, Pure and Applied Math. Quarterly.

[3] de Jong, A. J.: Smoothness, semi-stability and alterations, Publ. Math. IHES **83** (1996), 51–93.

[4] de Jong, A. J.: Families of curves and alterations, Ann. Inst. Fourier **47** no2 (1997), 599–621.

[5] Deligne, P.: Théorème d'intégralité, Appendix to Katz, N.: Le niveau de la cohomologie des intersections complètes, Exposé XXI in SGA 7, Lect. Notes Math. vol. **340**, 363–400, Berlin Heidelberg New York Springer 1973.

[6] Deligne, P.: La conjecture de Weil, II. Publ. Math. IHES **52** (1981), 137-252.

[7] Esnault, H.: Deligne's integrality theorem in unequal characteristic and rational points over finite fields, with an appendix with P. Deligne, Ann. of Math. **164** (2006), 715–730.

[8] Esnault, H.: Coniveau over $\mathfrak{p}$-adic fields and points over finite fields, preprint 2007, 5 pages, to appear in C. R. Acad. Sci. Paris.

[9] Fujiwara, K.: A Proof of the Absolute Purity Conjecture (after Gabber), in Algebraic Geometry 2000, Azumino, Advanced Studies in Pure Mathematics **36** (2002), Mathematical Society of Japan, 153–183.

[10] Grothendieck, A.: Éléments de Géométrie Algébrique (EGA): **III** (2): Étude cohomologique des faisceaux cohérents, Publ. Math. IHES **17** (1963)
IV (3): Études locales des schémas et des morphismes de schémas, Publ. Math. IHES **28** (1966).

[11] Keel, S.: Basepoint freeness for nef and big line bundles in positive characteristic, Ann. of Math. (2) **149** (1999), no.1, 253–286.

[12] Milne, J.: Lectures on Étale Cohomology, v2 02, August 9 (1998), http://www.jmilne.org/math/.

UNIVERSITÄT DUISBURG-ESSEN, MATHEMATIK, 45117 ESSEN, GERMANY
E-mail address: `esnault@uni-due.de`

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544, USA
E-mail address: `chenyang@math.princeton.edu`