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Vanishing Theorems and Positivity in Algebraic Fibre Spaces

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This note tries to explain some methods used in and initiated by the birational classification theory of complex projective manifolds. We want to draw attention to improvements of vanishing theorems and to properties of direct images of certain sheaves which should be of some interest outside of the birational geometry as well. First, however, we have to recall briefly the open problems and the state of the art in classification theory. (See the survey articles [**1–7**] for a more complete description.)

1. Minimal models. S. Mori's theory of the cone of curves and its extremal rays, as well as his contraction theorem [**22**], gave some hope that any complex projective manifold X might have a (singular) minimal model.

As indicated by M. Reid's structure theorems [**5**] on the canonical model of threefolds, one has to allow at least "canonical singularities" (Conditions 1 and 2):

DEFINITION 1. A normal projective variety V is called a minimal variety if there exists some $r > 0$ such that:

(1) The reflexive hull $\omega_V^{[r]}$ of the rth power of the canonical sheaf is invertible.

(2) For all (or one) desingularization $\delta\colon X \to V$, one has $\omega_X^r \supset \delta^*\omega_V^{[r]}$.

(3) $\omega_V^{[r]}$ is numerically effective; i.e.,

$$\deg(\omega_V^{[r]}|_C) \geq 0$$

for all curves C in V.

(The minimal number r satisfying (1) is called the *index* of V.)

THE MINIMAL MODEL CONJECTURE. *Any non-uniruled complex projective manifold X is birationally equivalent to a minimal variety V (called the minimal model).*

Added in proof. Shigefumi Mori recently gave a proof of the minimal model conjecture for threefolds.

Even though S. Mori does not like the word "conjecture" in this connection, the quite encouraging results of Y. Kawamata and himself (for example, [**22**, **18**]) seem to justify this expression. In fact, most results mentioned in the sequel assume that this conjecture holds.

2. Semi-ampleness.

DEFINITION 2. Let V be a normal projective variety and $\mathcal{L}$ an invertible sheaf.

(a) $\mathcal{L}$ is called semi-ample if for some $\mu > 0$, $\mathcal{L}^\mu$ is generated by $H^0(V, \mathcal{L}^\mu)$.

(b) The Iitaka dimension of $\mathcal{L}$ is

$$\kappa(V, \mathcal{L}) := \begin{cases} -\infty & \text{if } R(V, \mathcal{L}) = \mathbf{C}, \\ \operatorname{tr} \deg_{\mathbf{C}} R(V, \mathcal{L}) - 1 & \text{otherwise,} \end{cases}$$

where $R(V, \mathcal{L}) := \bigoplus_{i \geq 0} H^0(V, \mathcal{L}^i)$.

DEFINITION 3. If V is nonsingular (or a minimal model of index r), then the *Kodaira dimension* of V is

$$\kappa(V) := \kappa(V, \omega_V) \quad (\text{or } \kappa(V) := \kappa(V, \omega_V^{[r]})).$$

SEMI-AMPLENESS CONJECTURE. *If V is a minimal variety of index r, then $\omega_V^{[r]}$ is semi-ample.*

Using the vanishing theorems of §4, this conjecture has been verified by X. Benveniste, Y. Kawamata, and V. Shokurov if $\kappa(V) = \dim V$ and $\kappa(V) = \dim V - 1$ (see [**16**, **18**] for details). Of course, the conjecture implies that $\kappa(V) \geq 0$ for all minimal varieties V. By studying deformations of subvarieties along foliations, Y. Miyaoka [**21**] recently verified this implication directly in the 3-dimensional case.

3. The Iitaka-Ueno program ([13, 24]).

The starting point of the recent development in classification theory was S. Iitaka's fibration theorem, saying roughly that each manifold with $\kappa(X) \geq 0$ carries a fibration by subvarieties of dimension $\dim X - \kappa(X)$ and of Kodaira dimension zero. In a similar way, one would like to use the Albanese map to reduce the study of manifolds X with $\kappa(X) \leq 0$ and $q(X) = \dim H^0(X, \Omega_X^1) > 0$ to families of lower-dimensional manifolds. In order to do so, one needs an affirmative answer to the

GENERALIZED IITAKA CONJECTURE. *For any fibre space $f\colon X \to Y$, with $\kappa(Y) \geq 0$, one has $\kappa(X) \geq \operatorname{Max}\{\operatorname{Var}(f), \kappa(Y)\} + \kappa(X_y)$.*

We use

DEFINITION 4. (a) A surjective morphism $f\colon X \to Y$ between projective manifolds is called a *fibre space* if the general fibre X_y is connected.

(b) $\operatorname{Var}(f)$ is the minimal transcendental degree over $\mathbf{C}$ of a field of definition for the birational equivalence class of X_y.

As explained in §7, this conjecture is now solved by J. Kollár, if $\kappa(X_y) = \dim X_y$ [**19**], and by Kawamata if X_y has a minimal model satisfying the Semi-ampleness Conjecture [**15**]. However, it can be applied successfully to the Albanese map only, if the structure of X_y is sufficiently well known. (See [**15**, §8] and [**25**, §9] for typical applications.)

The Iitaka Conjecture, the Semi-ampleness Conjecture, and even some of the methods used to attack the Minimal Model Conjecture are dealing with the general problem: "How to get lots of sections of powers of canonical sheaves." It is not too surprising that the same type of methods appears in most results obtained in one of the three directions.

4. Vanishing theorems. One method of studying sections (or cohomology) of an invertible sheaf $\mathcal{M}$ on a projective manifold X consists of restricting it to an effective divisor B and hoping that the surjectivity of the restriction map allows induction on the dimension. Hence, if we write $\mathcal{M} = \mathcal{L} \otimes \omega_X(B) := \mathcal{L} \otimes \omega_X \otimes \mathcal{O}_X(B)$, we ask for criteria implying that the adjunction map

$$\phi_{q,B} \colon H^q(X, \mathcal{L} \otimes \omega_X(B)) \to H^q(B, \mathcal{L} \otimes \omega_B)$$

is surjective. As a generalization of Kodaira's vanishing theorem, Kawamata and I obtained (see [**18**, 1-2] or [**8**, 2.13])

THEOREM 5. *Let $C = \sum r_i C_i$ be an effective normal crossing divisor such that for some $N > \operatorname{Max}\{r_i\}$, $\mathcal{L}^N(-C)$ is numerically effective. Then*

$$H^q(\mathcal{L} \otimes \omega_X) = 0$$

(and hence $\phi_{q-1,B}$ is surjective for all B) for $q > \dim X - \kappa(X, \mathcal{L})$.

Usually Theorem 5 is called "the vanishing theorem for integral parts of **Q**-divisors" to show that the "positivity assumption" is just made for $\mathcal{L}(-\frac{1}{N} \cdot C)$. This tiny improvement turns out to be quite important for applications. A more direct approach to study the adjunction map was given by Kollár [**20**]:

THEOREM 6. *If $\mathcal{L}^N$ is generated by its global sections and $\mathcal{O}_X(B) \subset \mathcal{L}^N$ for some $N \gg 0$, then $\phi_{q,B}$ is surjective for all q.*

In fact, both Theorems 5 and 6 can be obtained using the degeneration of the Hodge-Deligne spectral sequence. For example, if B is smooth and $\mathcal{O}_X(B) = \mathcal{L}^N$, the proof of (6) given in [**8**] reads:

Let $\pi \colon X' \to X$ be the cyclic cover obtained by taking the Nth root out of a section of $\mathcal{L}^N$ defining B. Then $\mathcal{L}^{-1}$ is a direct summand of $\pi_* \mathcal{O}_X$, and the differential $d \colon \mathcal{O}_{X'} \to \Omega^1_{X'}\langle B \rangle$ induces an integrable connection $\nabla \colon \mathcal{L}^{-1} \to \mathcal{L}^{-1} \otimes \Omega^1_{X'}\langle B \rangle$. For the Poincaré residue $r \colon \Omega^1_X\langle B \rangle \to \mathcal{O}_B$, $(\mathrm{id} \otimes r) \circ \nabla$ is nothing but the restriction map $\mathcal{L}^{-1} \to \mathcal{L}^{-1}|_B$.

By Deligne's mixed Hodge theory, $H^q(d) = 0$ and hence

$$0 = H^q((\mathrm{id} \otimes r) \circ \nabla) \colon H^q(\mathcal{L}^{-1}) \to H^q(\mathcal{L}^{-1}|_B).$$

A similar proof gives a slight generalization of Theorem 6:

THEOREM 7. *If $\mathcal{L}^N(-C) = \mathcal{O}_X$ for C and N as in Theorem 5, then $\phi_{q,B}$ is surjective for all divisors B with $B_{\text{red}} \leq C_{\text{red}}$.*

Besides the applications of Theorems 5 and 6 to the Semi-ampleness Conjecture mentioned already, and the simple proof of the "Positivity Theorem" [**20**], described in §5, the two statements were successfully applied

– to the deformation behavior of plurigenera [**23**],

– in deformation theory of singularities [**14** and **12**],

– for Nullstellen Lemmata for homogeneous polynomials [**10**],

– and even, together with methods described in §5, to reprove the theorem of Roth on diophantine approximations of algebraic numbers [**11**].

5. Weak positivity. Let $f\colon X \to Y$ be a fibre space. In order to construct sections of powers of $\mathcal{L} \otimes \omega_X$, one can try to study positivity properties of $f_*(\mathcal{L} \otimes \omega_{X/Y})$, where $\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{-1}$. In order to have a notation of positivity compatible with tensor products, symmetric products, finite covers or blowing ups, one defines

DEFINITION 7. Let $\mathcal{F}$ be a sheaf on Y.

(a) $\mathcal{F}$ is called *generically generated* if $H^0(Y, \mathcal{F}) \otimes_{\mathbf{C}} \mathcal{O}_Y \to \mathcal{F}$ is surjective at the general point of Y.

(b) $\mathcal{F}$ is called *weakly positive* if for every ample invertible sheaf $\mathcal{H}$ on Y and every $a > 0$ there exists some $b > 0$ such that $\hat{S}^{a \cdot b}(\mathcal{F}) \otimes \mathcal{H}^b$ is generically generated (where $\hat{S}$ denotes the reflexive hull of the symmetric product).

(c) $\mathcal{F}$ is called *big* if for all ample invertible sheaves $\mathcal{H}$ there is some $c > 0$ such that $\mathcal{H}^{-1} \otimes S^c(\mathcal{F})$ is weakly positive.

For example, if $\mathcal{F}$ is invertible, $\mathcal{F}$ is weakly positive if and only if the corresponding divisor is in the closure of the cone of effective divisors in $\operatorname{Pic}(Y)$; and $\mathcal{F}$ is big if and only if $\kappa(Y, \mathcal{F}) = \dim Y$. A slight modification of the positivity theorem of Fujita-Kawamata [**17, 25**] says:

THEOREM 8. *If $f\colon X \to Y$ is a fibre space, then $f_*\omega_{X/Y}$ is weakly positive.*

Kollár [**20**] realized that the quite difficult study on variations of Hodge structures used by Kawamata to prove Theorem 8 can be replaced by his vanishing theorem (6), and it is even sufficient to consider a smooth divisor B with $\mathcal{O}_X(B) \subset \mathcal{L}^N$. Moreover, his approach gives an effective bound of the number b in the definition of weakly positive -independent of a:

THEOREM 8′. *Let $\mathcal{H}$ be a very ample invertible sheaf on Y. Then for $m > \dim Y$ and all $s > 0$ the sheaf $\mathcal{H}^m \otimes \omega_Y \otimes \hat{S}^s(f_*\omega_{X/Y})$ is generically generated.*

SKETCH OF THE PROOF. If $f^{(s)}\colon X^{(s)} \to Y$ is a desingularization of the s-fold product $X \times_Y X \times_Y \cdots \times_Y X$, there is a map $f_*^{(s)}\omega_{X^{(s)}/Y} \to (f_*\omega_{X/Y})^{\otimes s}$ which is generically an isomorphism. Hence it is enough to consider $s = 1$. If H is a general divisor of $\mathcal{H}$, $\mathcal{L} = f^*\mathcal{H}$ and $B = f^{-1}(H)$, Theorem 6 gives a surjection

$$H^0(\mathcal{H}^m \otimes \omega_Y \otimes f_*\omega_{X/Y}) = H^0(\mathcal{L}^{m-1} \otimes \omega_X(B)) \to H^0(\mathcal{L}^{m-1} \otimes \omega_B)$$
$$= H^0(\mathcal{H}^{m-1} \otimes \omega_H \otimes f_*\omega_{B/H}),$$

and by induction on $\dim Y$ we are done.

Applying Theorem 8 to a cyclic cover of X and playing around with the properties of weakly positive sheaves, one obtains [**25**]

THEOREM 9. (a) *Let $C = \sum r_i C_i$ be an effective normal crossing divisor, $\mathcal{L}$ an invertible sheaf on X, and $N > \mathrm{Max}\{r_i\}$ such that $\mathcal{L}^N(-C)$ is generated by its global sections. Then $f_*(\mathcal{L} \otimes \omega_{X/Y})$ is weakly positive.*

(b) *For all $\nu > 0$ the sheaf $f_*\omega^{\nu}_{X/Y}$ is weakly positive.*

If Y is of general type, Theorem 9(b) implies the Iitaka Conjecture ($\kappa(X) = \kappa(Y) + \kappa(X_y)$).

The effective bound of Kollár carries over, and if $\eta(Z)$ denotes the smallest number for which the ηth canonical map of a manifold Z is generically finite, one obtains [**9**]

$$\eta(X) \leq \eta(X_y) \cdot \left\{ \frac{(m+2) \cdot \eta(Y) + 2}{\eta(X_y)} \right\}$$

where $\{\ \}$ denotes the "round up" of a real number.

6. Big sheaves. The sheaves $f_*\omega^{\nu}_{X/Y}$ satisfy some kind of generic stability condition if $\nu > 1$ (see [**25** and **26**]). For example, if Y is a curve and if $f_*\omega^{\nu}_{X/Y}$ contains an ample invertible sheaf, then $f_*\omega^{\nu}_{X/Y}$ is ample. Obviously a similar statement is wrong for $\nu = 1$.

If the degenerated fibres of f are not too bad one obtains

THEOREM 10. *If $f\colon X \to Y$ is a fibre space and if the fibres of f are semistable outside a subvariety of Y of codimension two, then for all $\eta > 1$ one has*

$$\kappa(\det(f_*\omega^{\eta}_{X/Y})) = \dim Y \quad \textit{if and only if } f_*\omega^{\eta}_{X/Y} \textit{ is big.}$$

Of course, in order to have the bigness of $f_*\omega^{\eta}_{X/Y}$, the fibre space f should have the maximal number of moduli. Using the notations introduced in Definition 4(b) one hopes that:

BIGNESS PROBLEM. *Does $\kappa(X_y) \geq 0$ and $\mathrm{Var}(f) = \dim Y$ imply that $f_*\omega^{\eta}_{X/Y}$* is big for some $\eta \gg 0$?

Similar to the Minimal Model Conjecture and the Semi-ampleness Conjecture one might even be tempted to ask, in addition, whether there is a birational model for the fibre space such that $f_*\omega^{\eta}_{X/Y}$ is generated by its global sections. As explained in [**25**] and [**26**], an affirmative answer to the Bigness Problem gives a solution of the Generalized Iitaka Conjecture (§3).

Theorem 10 can be used together with a quite technical covering construction to reduce the Bigness Problem for $f\colon X \to Y$ to that for a fibre space

$f'\colon X' \to Y'$ whose general fibre X'_y is obtained as a cyclic cover of X_y, ramified along a multicanonical divisor. (See [**26** and **19**] for an extended version.)

7. Moduli. If $f\colon X \to Y$ is a family of curves or surfaces of general type, it follows from D. Mumford's and D. Gieseker's construction of a projective moduli-variety that for $k, s \gg 0$ and $r(k) = \operatorname{rank}(f_*\omega^k_{X/Y})$ one has

$$\kappa(Y, \det(f_*\omega^{k\cdot s}_{X/Y})^{\cdot r(k)} \otimes \det(f_*\omega^k_{X/Y})^{-s\cdot r(s\cdot k)}) \geq \operatorname{Var}(f).$$

Together with Theorem 9(b) this implies an affirmative answer to the Bigness Problem [**25**].

Instead of global moduli one can as well use local Torelli theorems for the general fibre X_y. By the remark at the end of the last section one has to have Torelli-type theorems only for certain coverings of X_y. Those are quite easily verified if $\kappa(X_y) = \dim X_y$ and if ω_{X_y} is semi-ample [**27**]. Kawamata considered this approach under much weaker assumptions and obtained, in [**15**],

THEOREM 11. *The bigness problem has an affirmative answer if X_y has a minimal model satisfying the semi-ampleness conjecture.*

Kollár used in [**19**] properties of variations of Hodge structures to get hold of the kernel of the multiplication map $\gamma_m\colon S^m(f_*\omega_{X/Y}) \to f_*\omega^m_{X/Y}$. If $\operatorname{Var}(f) = \dim Y$ and if the canonical map of X_y is birational, he obtained that the image of γ_m is "positive." Together with the usual covering construction this implies

THEOREM 12. *The bigness problem has an affirmative answer if $\kappa(X_y) = \dim X_y$.*

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