Classification Theory of Algebraic Threefolds *)

Eckart Viehweg (Mannheim)

We assume that all varieties are normal, complete, irreducible and defined over $\mathbb{C}$. For an invertible sheaf L on a variety V let:

$$\kappa(V,L) = \begin{cases} \operatorname{tr.deg}\left(\bigoplus_{n \geq 0} H^0(V,L^n)\right) - 1 & \text{if } L^n \text{ has non trivial sections for some } n > 0 \\ -\infty & \text{otherwise} \end{cases}$$

For a regular variety V the Kodaira-dimension is $\kappa(V) = \kappa(V,\omega_V)$ where ω_V denotes the canonical sheaf of V. The irregularity of V is $q = q(V) = \dim_{\mathbb{C}}(H^1(V,O_V)) = \dim(A(V))$ where

$\alpha: V \longrightarrow A(V)$ is the Albanese map.

<u>Theorem (1):</u> Let $\hat{V}$ be an algebraic threefold. Then $\hat{V}$ is birationally equivalent to one of the following threefolds V :

$\kappa(V)$	$q(V)$	$\dim(\alpha(V))$	Structure of V	
3			Threefold of general type	
2			There exists a surjective morphism $f: V \longrightarrow W$ with general fibre F ,	$\dim(F)=1$, $\kappa(F)=0$
1				$\dim(F)=2$, $\kappa(F)=0$
0	0	0	????	
	1	1	$\alpha: V \longrightarrow A(V)$ is smooth surjective, all fibres are isomorphic to some connected variety F	$\dim(F)=2$, $\kappa(F)=0$
	2	2		$\dim(F)=1$, $\kappa(F)=0$
	3	3	Abelian threefold	

---

*) This note contains a résumé of two talks the author gave at the ISTITUTO DI GEOMETRIA "C.SEGRE" , Torino , november 1978. The complete proofs will appear elsewhere. The author wishes to express his gratitude for the hospitality extended to him at Torino.

$-\infty$	0	0	?????	
	$q \geq 1$	1	The Stein factorisation $f: V \longrightarrow W$ of $\alpha: V \longrightarrow \alpha(V)$ has the general fibre F and a desingularisation W' of W with $q(W')=q(V)$	$\dim(F)=2$, $\kappa(F)= -\infty$
		2		$F \cong \mathbb{P}^1$

The first three lines of this table are just a special case of Iitaka's fundamental theorem on the pluricanonical fibrations [U_1 , p. 73]. In his Lecture Notes [U_1] Ueno has written down the kind of information required in order to prove theorem (1) :

I) Let W' be a desingularisation of $\alpha(V)$. Then $\kappa(W') \geq 0$ and $\kappa(W') = 0$, if and only if α is surjective.

II) Let V be a threefold with $\kappa(V) = 0$. Assume there exists a surjective morphism from V to an abelian threefold. Then V is an abelian threefold itself.

III) If $g: V' \longrightarrow V$ is a generically surjective rational map and $\dim(V') = \dim(V)$, then $\kappa(V') \geq \kappa(V)$.

IV) If $f: V \longrightarrow W$ is a surjective morphism of regular varieties, and if the general fibre V_w of f is connected and regular, then $\kappa(V) \leq \kappa(V_w) + \dim(W)$.

V) $C_{3,2}$ and $C_{3,1}$ are true, where $C_{n,m}$ denotes the following conjecture of Iitaka:

Conjecture $C_{n,m}$: Let $f: V \longrightarrow W$ be a surjective morphism of regular complete connected varieties, having a regular and connected general fibre V_w , where $n = \dim(V)$ and $m = \dim(W)$. Then (conjecturally) $\kappa(V) \geq \kappa(V_w) + \kappa(W)$. Moreover, if W is an abelian variety and $\kappa(V) = 0$, then (up to birational equivalence) f is smooth and all fibres are isomorphic.

Applying all these facts to the Stein factorisation of $\alpha: V \longrightarrow \alpha(V)$ one obtains theorem(1).

I), III) and IV) is proven in [U_1, p. 119, 73 and 74]. II) is proven in [U_2, §3]. Hence only V) remains to be proven.

What is known about $C_{n,m}$?

First Ueno gave a proof of $C_{2,1}$, using classification theory for surfaces; later (1975) he found another proof. In 1976 the author gave a different proof of $C_{2,1}$ and a proof of $C_{n,n-1}$ [V_1]. Moreover Ueno proved $C_{n,m}$, if the general fibre of f is an abelian variety [U_3], and Fujita and Ueno proved $C_{n,1}$, if $\kappa(V) \geqslant 0$ and W is a curve of genus larger or equal two.

Remark: Using the last special case of $C_{3,1}$ and $C_{3,2}$, Ueno obtained in [U_2] the "$\kappa(V) = 0$" case of theorem (1), except when $q(V) = 1$.

The second half of this note consists of an outline of a proof of $C_{n,n-1}$ and $C_{3,1}$:

Conjecture $C'_{n,m}$: Let $f: V \longrightarrow W$ be as in $C_{n,m}$. Then (after replacing V and W by birational equivalent models) $\kappa(V, \omega_V \otimes f^*\omega_W^{-1}) \geqslant \kappa(V_w)$. Moreover, if for some $\nu > 0$ $(\omega_V \otimes f^*\omega_W^{-1})^{\otimes \nu} \cong O_V$, then (up to birational equivalence) f is smooth and all fibres are isomorphic.

Methods, similar to [V_1, proof of 1.7] give:

Theorem (2):

If $C'_{k,k-r}$ is true for $k = r+1,\dots,n$, then $C_{n,n-r}$ is true.

<u>Definition:</u> A surjective flat morphism $g: T \longrightarrow S$ of normal complete varieties is called a <u>semistable family</u> (of (dim(T) - dim(S)) dimensional varieties),if

i) The general fibre T_s is connected and regular.

ii) Every fibre is reduced with regular components, intersecting transversally.

<u>Lemma (3):</u> If $g: T \longrightarrow S$ is a semistable family, then g is a Gorenstein morphism. If S has only rational singularities $[V_2]$, then T has only rational singularities.

Of course, not every $f: V \longrightarrow W$ is birationally equivalent to a semistable family. However, it is possible to replace W by a (generically) galois covering $W' \longrightarrow W$ such that the desingularisation V' of $V \times_W W'$ (viewed as a family over W') is birationally equivalent to a semistable family $g: T \longrightarrow S$, whenever $\dim(V) = \dim(W) + 1$ $[V_1, \S 5]$ or $\dim(W) = 1$ [TE, p. 53].

By keeping track of what happens to the canonical sheafs when you replace $f: V \longrightarrow W$ by some semistable family, it is possible to deduce $C'_{n,n-1}$ and $C'_{3,1}$ from:

<u>Theorem (4):</u> Let $g: T \longrightarrow S$ be

either (a) a semistable family of curves ,

or (b) a semistable family of surfaces over a curve S such that the general fibre T_s is a minimal model

and let $\omega_{T/S}$ be the relative dualising sheaf $[K_1]$.

Then $\kappa(T, \omega_{T/S}) \geqslant \kappa(T_s)$.

<u>Remark:</u> Lemma (3) tells us, that in order to prove theorem (4) we are allowed to replace S by any S' , surjective over S ,

and to prove the inequality for the pullback-family. Moreover, for regular S and any desingularisation $h: T' \longrightarrow T$, we have: $\kappa(T, \omega_{T/S}) = \kappa(T', \omega_{T'} \otimes (g\cdot h)^* \omega_S^{-1})$.

Notation: In the situation of our theorem (4) , we know, that $g_* \omega_{T/S}^{\otimes\nu}$ is a locally free sheaf of rank $r(\nu)$. Write

$$\lambda_\nu = \bigwedge^{r(\nu)} g_* \omega_{T/S}^{\nu} \ .$$

There are several possibilities to obtain some information about the relative dualising sheaf of a semistable family:

Ingrediants:

A) Moduli theory : In [M_1] and [G_1] it is proven that the "Hilbert point" of a curve or surface of general type, with respect to a n-canonical embedding, fulfills the "Stability-Criterium" of Mumford [M_1]. Hence the corresponding coarse moduli scheme exists [M_1], together with a natural embedding $i: M \longrightarrow \mathbb{P}^\mu$. Let $\varphi: S \longrightarrow M$ be the rational map, induced by $g: T \longrightarrow S$. We may assume, that $i\cdot\varphi$ is a morphism. Let $L = (i\cdot\varphi)^* O_{\mathbb{P}^\mu}(1)$. Going over the constructions of [M_1] and [G_1] carefully, one finds (without knowing anything about the compactification of M), that for $n,m \gg 0$ some power of L is a subsheaf of $\lambda_{n\cdot m}^{r(n)} \otimes \lambda_n^{-m\cdot r(n\cdot m)}$. Hence we obtain:

(A) $$\kappa(S, \lambda_{n\cdot m}^{r(n)} \otimes \lambda_n^{-m\cdot r(n\cdot m)}) \geqslant \dim(\varphi(S)) \quad \text{for} \quad n,m \gg 0 \ .$$

B) Period maps and variation of Hodge structures:

Fujita proves in [F_1] (using results of Griffiths) the following

Theorem : Let $f: V \longrightarrow W$ be as in $C_{n,1}$ and f be projective, then for every invertible quotientsheaf L of $f_* \omega_{V/W}$ we have $\deg(L) \geqslant 0$.

The same arguments can be used to prove $[F_2]$:

(B_1) Assume in addition, that $\omega_{V_w} = O_{V_w}$ and that not all regular fibres of f are isomorphic. Then $\deg(f_*\omega_{V/W}) > 0$.

Of course in general we are in trouble, because (for example) $f_*\omega_{V/W}$ may be trivial. However, in the situation of theorem (4), if S is a curve and T_s a surface of general type or a curve, it is possible to obtain more:

Apply Fujitas theorem to some family $g': T' \longrightarrow S$, where the general fibre is a covering $\varrho: T_s' \longrightarrow T_s$ with ramification sheaf $\varrho^*\omega_{T_s}^m$ for $m \gg 0$. After playing around with Chern classes of vector bundles on curves, you get:

(B_2) $\deg(g_*\omega_{T/S}^r) = \deg(\lambda_r) \geqslant 0$ for $r \gg 0$.

C) Relative Riemann-Roch theorem:

Although in general $g_*\omega_{T/S}^n$ is not the same as $g_!\omega_{T/S}^n$ (if it were, the proof of $C_{3,1}'$ would be somewhat easier), this canbe assumed (without less of generality) in case (a) of theorem (4). Copying the proof of Mumford of $[M_2$, Theorem 5.10$]$, one obtains:

(C) If T_s is a curve of genus $g \geqslant 2$, then (up to torsion) $\lambda_n = \mu^{\binom{n}{2}} \otimes \lambda_1$, where $\mu = \lambda_1^{12} \otimes \delta$ for some ideal-sheaf δ .

D) Weierstraß sections:

If T_s is a curve of genus $g \geqslant 1$, there is a natural injection of $g^*\lambda_1$ into some power of $\omega_{T/S}$ $[V_1$, § 2 $]$.

E) Riemann-Roch theorem for vector bundles on curves :

From this follows:

(E) If $\deg(\lambda_n)$ increases like $a \cdot n \cdot r(n)$ for $a > 0$, then

$$\kappa(T,\omega_{T/S}) = \kappa(T_s) + 1 .$$

In order to prove theorem (4) you just have to combine all these ingrediants.

Leitfaden:

dim(T)	dim(S)	Structure of T_s	Use ingrediant
2	1	elliptic curve	B_1
n	n-1	elliptic curve	$C'_{2,1}$ for elliptic curves
2	1	curve of genus 2	A & B_2 & E or A & B_2& E (also if you are working in characteristic p) or C & D & E ....
n	n-1	curve of genus 2	A & C & D or B & D [V_1]
3	1	abelian variety or K 3 surface	B_1
3	1	surface of general type	A & B_2
3	1	elliptic surface	$C'_{n,n-1}$ for elliptic curves

All these steps are quite simple, except the last one.
In order to get $C'_{3,1}$ if the fibre is an elliptic surface, it is necessary to study "the behavior of the dualising sheafs under basechange and normalisation" (i.e. the step from theorem (4) to $C'_{3,2}$) even more carefully.

<u>Remarks:</u> i) It is possible to prove $C_{2,1}$ also in characteristik p . However, this is not a great help, if you want to prove the classification of surfaces in characteristic p , since in general the Albanese map may have a non smooth general fibre.
ii) The reader may ask why these ideas do not work in the

cases $C_{4,2}$ or $C_{4,1}$. I think the most important gap is that at present it is unknown, whether the canonical ring of a threefold is finitly generated or not. A positive answer would give some good compactification of the moduli scheme of surfaces of general type; it would give a result like B_2 for families of threefolds and it would allow to use the relative Riemann-Roch theorem. I have no feeling as to what one should expect to be true about the canonical ring. Perhaps just the following statement is true (and would be of some help): Let V be a regular variety of general type. Is it true that there exists a polynominal P of degree $\dim(V) - 1$ and a positive real number d , such that for $n \gg 0$ we have $|\dim(H^0(V,\omega_V^n)) - d\cdot n^{\dim(V)}| \leqslant P(n)$?

BIBLIOGRAPHY :

[F_1] Fujita, T.: Kähler fibre spaces over curves; to appear in J. Math. Soc. Japan

[F_2] Fujita, T.: The sheaf of relative canonical forms of a Kähler fibre space over a curve; Proc. Japan Academy, 54 Ser. A (1978), 183-184

[G_1] Gieseker, D.: Global moduli for surfaces of general type; Inv. math., 43 (1977), 233-282

[K_1] Kleiman, S. L.: Relative duality for quasi-coherent sheaves; Preprint, see also: Hartshorne, R.: Residue and Duality; Lecture Notes in Math. 20 , Springer Verlag

[M_1] Mumford, D.: Geometric Invariant Theory; Ergebnisse, Springer Verlag

[M_2] Mumford, D.: Stability of projective varieties; l'Enseignement Math., 23 (1977)

[TE] Kempf, G.; Knudsen, F.; Mumford, D.; Saint-Donat, B.: Toroidal Embeddings I ; Lecture Notes in Math. 339, Springer Verlag

[U_1] Ueno, K.: Classification Theory of Algebraic Varieties and Compact Complex Spaces; Lecture Notes in Math. 439, Springer Verlag,

[U_2] Ueno, K.: Classification of algebraic varieties II - Algebraic threefolds of parabolic type - ; Preprint

[U_3] Ueno, K.: On algebraic fibre spaces of abelian varieties, Preprint,

[V_1] Viehweg, E.: Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension one; Comp. Math., 35 (1977), 197-223

[V_2] Viehweg, E.: Rational singularities of higher dimensional schemes; Proc. of the AMS, 63 (1977), 6-8