

Quasi-projective quotients by compact equivalence relations

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1 Introduction

In [9, III], we tried to construct quasi-projective moduli schemes M_h for certain moduli functors \mathcal{M}_h of polarized compact complex manifolds X with Hilbert polynomial h . However, being unable to handle the compact part of the equivalence relation, we could achieve our aim only after changing the definition of the moduli functor [9, III, 1.10]. Instead of considering the functor of polarized manifolds up to \mathbb{Q} -numerical equivalence, as it was done in [6] and [7], we allowed only isomorphisms of the invertible sheaves giving the polarization. In this article we want to show that our “cheating” was unnecessary and we will prove:

Let \mathcal{N}_h be the functor of compact complex polarized manifolds X , as defined in [6] or [7], with ω_X semi-ample. Then there exists a coarse quasi-projective moduli scheme N_h for \mathcal{N}_h .

In fact, as in [9, III], we will replace “semi-ample” by some other condition which is certainly satisfied for X with ω_X numerically effective. As Kollár pointed out, the functor \mathcal{N}_h is not the right one for singular varieties. Since we want to allow X to have rational double points, as long as the functors considered stay separated and bounded, we will work with the functor \mathcal{P}_h , parametrizing pairs (X, \mathcal{H}) up to numerical equivalence (as in [4]). The reader finds the definitions and main results on moduli in Sect. 4.

To fix ideas, let us assume for the moment that M_h and P_h , the coarse moduli spaces of \mathcal{M}_h and \mathcal{P}_h , exist in the category of analytic spaces (which in fact is true, due to the work of Artin [1], as explained in [6, p. 172] or [4, 4.1.1]). M_h is a quotient of some Hilbert scheme by a reductive group and, in certain cases it is quasi-projective as shown in [9, III], using geometric invariant theory. We have a natural morphism $\pi: M_h \rightarrow P_h$ whose fibres are quotients of Pic^c of the corresponding varieties. In fact, some easy manipulations allow to replace Pic^c by Pic^0 .

If π were flat, then 2.6 would give a quite simple argument to descend quasi-projectivity from M_h to P_h . If the equivalence relation in $M_h \times M_h$ given by numerical equivalence were flat over M_h , one could use 2.5 to construct P_h as quasi-projective scheme. Even if we do not believe that either one of those flatness condition holds true in general, we include in Sect. 2 the arguments one could use in those cases.

If for (X, \mathcal{H}) and $(X', \mathcal{H}') \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ with $X \neq X'$ the abelian varieties $\text{Pic}^0(X)$ and $\text{Pic}^0(X')$ were different, then P_h should be a subvariety of the moduli space of polarized abelian varieties and hence quasi-projective.

Both remarks indicate possible ways to handle \mathcal{P}_h in general. The first one, not followed in this article, would be to consider moduli of polarized manifolds with some level structure and to hope that this enforces the flatness condition of 2.5. Since quasi-projectivity descends under quotients by finite groups, this should be enough. The second approach considers P_h as part of a “moduli space of polarized abelian varieties together with a morphism to M_h , finite over its image”. Instead of trying to define the corresponding moduli problem and to apply geometric invariant theory as in [9] or [10], we will construct P_h directly as a quotient of some subscheme of a Hilbert scheme in Sect. 3. As in [10] for the quotients by $\mathbb{P}\text{GL}(r, \mathbb{C})$, we will try to formulate the result on the quasi-projectivity of quotients by compact equivalence relations in 3.2 in a more general set-up (even if the statement obtained is quite monstrous).

After a first version of this paper was finished, H el ene Esnault and the author succeeded in [2] to obtain quite natural ample sheaves on M_h . Building up on those results (see 5.1 and 5.2) we are able now to make the construction in Sect. 3 effective and to give in Sect. 5 a description of a reasonably looking ample sheaf on P_h as well.

We keep the notations and conventions from [9]. Especially, all schemes and morphisms are supposed to be of finite type and separated and everything should be defined over \mathbb{C} . Points are always closed. An analytic space Z should be Zariski open in some compact analytic space Z' of finite type. We fix a bimeromorphic equivalence class of such compactifications Z' for each Z and we assume that all morphisms and coherent sheaves extend as such to some compactification in the equivalence classes chosen.

2 Compact flat equivalence relations

Definition 2.1. Let M be a quasi-projective scheme and $\tau: R \rightarrow M \times M$ a closed embedding. We call R a *compact connected equivalence relation* if

- i) R is an equivalence relation,
- ii) $p_1 = \text{pr}_1 \circ \tau$ is proper,
- iii) $p_{1*} \mathcal{O}_R = \mathcal{O}_M$.

We will call R a *flat compact connected equivalence relation* if in addition p_1 is flat.

Definition 2.2. Let $\tau: R \rightarrow M \times M$ be compact connected equivalence relation and P a scheme (or analytic space). We will say that $\pi: M \rightarrow P$ is a *quotient of M by R* if

- a) π is a surjective proper morphism,
- b) $\pi_* \mathcal{O}_M = \mathcal{O}_P$,

c) there is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{p_2 = \text{pr}_2 \circ \tau} & M \\ p_1 \downarrow & & \downarrow \pi \\ M & \xrightarrow{\pi} & P \end{array}$$

and the induced morphism $\delta: R \rightarrow M \times_P M$ is bijective.

Remarks 2.3. a) If R is flat we even obtain in 2.5 that δ is an isomorphism of schemes.

b) The assumption 2.1 iii is not really necessary if one is willing to weaken b in 2.2.

c) To prove the existence of quotients one may assume that M is connected. In fact, R contains the diagonal and the fibres of p_1 are connected.

Lemma 2.4. *Let $\tau: R \rightarrow M \times M$ be a compact connected equivalence relation and P a quotient of M by R . If $\varepsilon: M \rightarrow Z$ is a morphism of schemes (or analytic spaces) with $\varepsilon \circ p_1 = \varepsilon \circ p_2$, then there exists a unique morphism $\gamma: P \rightarrow Z$ such that $\varepsilon = \gamma \circ \pi$.*

Proof. For $x \in P$ the morphism ε maps $\pi^{-1}(x)$ to some point $\gamma(x) \in Z$. Hence there is a unique map of sets $\gamma: P \rightarrow Z$. If $U \subseteq Z$ is open then $\varepsilon^{-1}(U)$ is open in M . It is of the form, $\pi^{-1}(V)$ for some subset V of P and, since π is proper V is open. Hence γ is continuous. We have maps $\mathcal{O}_Z \rightarrow \varepsilon_* \mathcal{O}_M = \gamma_* \pi_* \mathcal{O}_M = \gamma_* \mathcal{O}_P$.

Unfortunately, the equivalence relations we are interested in do not seem to be flat. Nevertheless, we include the following criterion whose proof may serve as an introduction to the methods employed in Sect. 3.

Proposition 2.5. *Let $\tau: R \rightarrow M \times M$ be a flat compact connected equivalence relation. Then there exists a quasi-projective quotient $\pi: M \rightarrow P$ of M by R .*

Proof. Assume that M is connected and $i: M \rightarrow \mathbb{P}^k$ an embedding. Since $i \circ p_2$ embeds all fibres $p_1^{-1}(y)$, the sheaf $\mathcal{H} = p_2^* i^* \mathcal{O}_{\mathbb{P}^k}(1)$ is very ample with respect to p_1 .

By [3] $R \rightarrow M \times M \rightarrow M \times \mathbb{P}^k$ induces a morphism from M to some Hilbert scheme. If H' denotes the image and $g': \mathcal{H}' \rightarrow H'$ the restriction of the universal family we have a diagram

$$\begin{array}{ccccc} R & \xrightarrow{\Phi} & \mathcal{X}' & \xrightarrow{i} & H' \times \mathbb{P}^k & \xrightarrow{\text{pr}_2} & \mathbb{P}^k \\ p_1 \downarrow & & g' \downarrow & \text{pr}_1 \swarrow & & & \\ M & \xrightarrow{\varepsilon} & H' & & & & \end{array}$$

where the left square is a fibre product. Of course, $\text{pr}_2 \circ i \circ \Phi$ is nothing but

$$R \xrightarrow{p_2} M \xrightarrow{i} \mathbb{P}^k.$$

Hence Φ and ε are proper and, replacing H' by its Stein factorization, we may assume that $\varepsilon_* \mathcal{O}_M = \mathcal{O}_{H'}$. Since R is an equivalence relation we have a diagonal embedding

$$\Delta: M \rightarrow R$$

and $p_2 \circ \Delta$ as well as $p_1 \circ \Delta$ are isomorphisms. By flat base change we have $\Phi_* \mathcal{O}_R = \mathcal{O}_{\mathcal{X}'}$ and hence $\text{pr}_{2*} i_* \Phi_* \mathcal{O}_R = \text{pr}_{2*} i_* \mathcal{O}_{\mathcal{X}'} = i_* \mathcal{O}_M$. Since the fibres of g' are disjoint subschemes of M the morphism $\text{pr}_2 \circ i$ is injective and $\mathcal{X}' \rightarrow M$ will be an isomorphism.

Since $p_1 \circ \Delta$ and $\Phi \circ \Delta$ are isomorphisms we can identify g' and ε and we have constructed the quotient $H' = P$.

If $f: X \rightarrow Y$ is a morphism of schemes or analytic spaces and X quasi-projective, then Y may be non-quasi-projective and even not a scheme. However, we have

Proposition 2.6. *Let $f: X \rightarrow Y$ be a surjective proper morphism of schemes or analytic spaces. Assume that one of the following conditions holds:*

- i) X and Y are reduced and f is flat.
- ii) $f_* \mathcal{O}_X = \mathcal{O}_Y$ and f is flat.
- iii) Y is normal and f is finite.

Then Y is quasi-projective.

Proof. iii is well known. Replacing X by its Galois closure over Y we can assume that Y is a quotient of X by a finite group.

ii is a special case of 2.5. In fact, if $R = X \times_Y X \hookrightarrow X \times X$, then the assumptions of 2.5 hold true. However, the quotient is uniquely determined by 2.4 and Y must be the same as P . For i we follow the same line as in 2.5. We may assume Y to be connected and we choose an imbedding $i: X \rightarrow \mathbb{P}^k$. The graph of f induces embeddings over Y

$$X \rightarrow Y \times X \rightarrow Y \times \mathbb{P}^k$$

and hence again a morphism from Y to some Hilbert scheme. If H' is the reduced image one has

$$\begin{array}{ccccc} X & \xrightarrow{\Phi} & \mathcal{X}' & \xrightarrow{i} & H' \times \mathbb{P}^k & \xrightarrow{\text{pr}_2} & \mathbb{P}^k \\ f \downarrow & & g' \downarrow & \text{pr}_1 \swarrow & & & \\ Y & \xrightarrow{\varepsilon} & H' & & & & \end{array}$$

Again $\text{pr}_2 \circ i \circ \Phi$ is an open embedding and Φ an injection on points. Hence Φ must be an isomorphism. Since g' is flat and surjective and the square a fibre product, ε must be an isomorphism as well.

3 Compact equivalence relations lifting to smooth families

Definition 3.1. Let $\tau: R \rightarrow M \times M$ be a compact connected equivalence relation, see 2.1. We will call $f: Y \rightarrow H$ a *good smooth lifting of R* if we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi'} & R \\ f \downarrow & & p_1 \downarrow \\ H & \xrightarrow{\phi} & M \end{array}$$

such that $\phi(H) = M$, $\phi'(Y) = R$ and:

a) The fibres of ϕ are connected and, if $\zeta: H \rightarrow M'$ is a morphism contracting all fibres of ϕ to points, then there is a unique morphism $\varepsilon: M \rightarrow M'$ with $\zeta = \varepsilon \circ \phi$ (for example: M is the quotient of H by a connected group).

b) f is surjective, smooth and proper and the fibres of f are connected.

c) For all $x \in H$ the morphism $\phi'|_{f^{-1}(x)}$ is smooth onto its image and the fibres of the Stein factorization of

$$f^{-1}(x) \rightarrow \phi'(f^{-1}(x))$$

are all isomorphic.

- d) ϕ' has equidimensional fibres of constant dimension.
- e) $\omega_{Y/H}|_{f^{-1}(x)}$ is numerically effective for all $x \in H$.

Theorem 3.2. *If $\tau : R \rightarrow M \times M$ is a compact connected equivalence relation which has a good smooth lifting, then there exists a quasi-projective quotient $\pi : M \rightarrow P$ of M by R .*

Let us recall the following well-known lemma. It will allow in 3.4 to replace the good smooth lifting h by one, for which ϕ' is finite. In fact, since in the situation of Sect. 4 h will be a family of abelian varieties and the fibres of ϕ' subgroups, 3.3 and 3.4 are not really needed in the proof of our main result 4.6.

Lemma 3.3. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ & \searrow h & \downarrow f \\ & & S \end{array}$$

be a commutative diagram of surjective morphisms.

- i) *If g and h are flat then f is flat as well.*
- ii) *If h is smooth and f unramified then g is smooth and f étale.*
- iii) *If g is unramified, and if h and f are smooth with equidimensional fibres of the same dimension, then g is étale.*
- iv) *If h is smooth and if all fibres of f and g are smooth and equidimensional (of dimension m and r , respectively), then f and g are smooth.*

Proof (see also SGA 1). Since g and h are faithfully flat i is obvious. For ii consider the commutative diagram of fibre products

$$\begin{array}{ccccc} Y & \xrightarrow{\Gamma} & X \times_S Y & \xrightarrow{\text{pr}_2} & Y \\ g \downarrow & & \downarrow & & g \downarrow \\ X & \xrightarrow{\Delta} & X \times_S X & \xrightarrow{\text{pr}_2} & X \\ & & \downarrow & & f \downarrow \\ & & X & \xrightarrow{f} & S, \end{array}$$

where Δ is the diagonal and Γ the graph of the S -morphism g . Since f is unramified Δ is an open immersion and hence Γ as well. As pullback of h the middle vertical morphism $\text{pr}_1 : X \times_S Y \rightarrow X$ is smooth and hence $g = \text{pr}_1 \circ \Gamma : Y \rightarrow X$ as well. By i f will be flat and hence étale.

In iii we have to show that g is flat. Since h and f are smooth, this can be done for $h^{-1}(s) \rightarrow f^{-1}(s)$. However, an unramified morphism between smooth varieties of the same dimension is flat.

For iv we fix some $y \in Y$, $x = g(y)$, and $s = f(x)$ and, whenever it is necessary we may replace Y , X , and S by neighbourhoods of y , x , and s . By assumption we find a commutative diagram

$$\begin{array}{ccccc} h^{-1}(s) & \xrightarrow{g_s} & g^{-1}(s) & \longrightarrow & s \\ \mu'_s \downarrow & & \mu_s \downarrow & & \\ \mathbb{A}^r \times U_s & \longrightarrow & \mathbb{A}^m & & \end{array}$$

with μ_s and μ'_s étale and $U_s \subset \mathbb{A}^m$ open. In fact, since $g^{-1}(s)$ and $h^{-1}(s)$ are smooth and g_s a morphism with smooth fibres g_s will be smooth. This diagram extends to

$$\begin{array}{ccccc} Y & \xrightarrow{g} & X & \xrightarrow{f} & S \\ \mu' \downarrow & & \mu \downarrow & \nearrow p & \\ \mathbb{A}^r \times U & \xrightarrow{p_2} & \mathbb{A}_S^m & & \end{array}$$

for some open subscheme U of \mathbb{A}_S^m . Choosing U, S, X , and Y small enough μ and μ' will be unramified. Since $p \circ p_2$ and h are smooth iii tells us that μ' is étale. By ii g is smooth and μ étale and hence f is smooth as well.

Lemma 3.4. *Under the assumption made in 3.2 we can find some good smooth lifting $f: Y \rightarrow H$ such that the morphism*

$$\phi'' : Y \rightarrow H \times_M R$$

induced by ϕ' and f is finite.

Remark 3.5. We will obtain in the proof of 3.4 as well that for all $x \in M$ the reduced fibres $p_1^{-1}(x)_{\text{red}}$ and hence $p_2^{-1}(x)_{\text{red}}$ are non-singular.

Proof. Let $f: Y \rightarrow H$ be a good smooth lifting of R and $\phi'' : Y \rightarrow H \times_M R$. Since f is proper ϕ'' is proper as well. Let

$$Y \xrightarrow{g} Y' \xrightarrow{h} H \times_M R$$

be the Stein factorization of ϕ'' and $f' = \text{pr}_1 \circ h: Y' \rightarrow H$. We claim that f' is again a good smooth lifting.

For $x \in H$ we write $Y_x = f^{-1}(x)$, $Y'_x = f'^{-1}(x)$, and $R_x = (\{x\} \times_M R)_{\text{red}} = p_1^{-1}(\phi(x))_{\text{red}}$. We have morphisms

$$Y_x \xrightarrow{g_x} Y'_x \xrightarrow{h_x} R_x.$$

By condition c in 3.1 $h_x \circ g_x = \phi'|_{Y_x}$ is smooth. Since Y_x is non-singular g_x must be smooth and h_x étale. Hence the assumption c and d in 3.1 hold true for f' . Since all fibres of g_x are isomorphic and since ω_{Y_x} is numerically effective, g_x will be isomorphic to a product $Z \times F$ if we pull it back to some Z generically finite over Y'_x . We may choose Z in Y'_x and have $\omega_{Y'_x|Z} = g_x^* \omega_{Y_x|Z}$. Hence $\omega_{Y'_x}$ must be numerically effective and 3.1e holds true.

Next, we claim that the fibres of f' are reduced and hence $Y'_x = f'^{-1}(x)$. To this aim we may replace H by the normalization \tilde{H} of H_{red} and Y by the normal scheme $\tilde{Y} = Y \times_H \tilde{H}$ and Y' by $\tilde{Y}' = Y' \times_H \tilde{H}$ and assume thereby that H and Y are normal. Since $g: Y \rightarrow Y'$ has smooth connected fibres, g must be smooth and Y' normal under this assumption. Since f has reduced fibres f' must have reduced fibres.

Hence f' has smooth fibres as well as g . By 3.3ii we find f' to be smooth. The other assumptions in 3.1 obviously hold for f' and $\text{pr}_2 \circ h: Y' \rightarrow R$ if they hold true for f and ϕ' .

Remark 3.6. The arguments used in 3.4 do not seem to imply that R is flat over M or that $H \times_M R$ is flat over H . Since the fibres of ϕ'' are not necessarily connected it might happen that $R \rightarrow M$ has non-reduced fibres, in spite of the assumption 3.1c and 3.5.

Remark 3.7. In Sect. 5 we want to describe explicitly some ample sheaf $\mathcal{L}^{(p)}$ on H such that the morphism

$$H \xrightarrow{\phi} M \xrightarrow{\pi} P \hookrightarrow \mathbb{P}(V)$$

is given by a subspace V of $H^0(H, \mathcal{L}^{(p)})$ for some $p > 0$. To this aim we will include several “Addenda”, which are not needed in the proof of the quasi-projectivity of P and which will only be used in Sect. 5.

Proof of 3.2. Let us assume that M is connected and, using 3.4, that $f: Y \rightarrow H$ is a smooth good lifting of R with $\phi': Y \rightarrow H \times_M R$ finite. Let $M \rightarrow \mathbb{P}^k$ be an embedding and γ the composition of

$$Y \xrightarrow{\phi'} R \xrightarrow{\tau} M \times M \longrightarrow M \times \mathbb{P}^k \longrightarrow \mathbb{P}^k.$$

Since $\gamma|_{f^{-1}(x)}$ is finite over its image for all $x \in H$ the sheaf $\mathcal{G} = \gamma^* \mathcal{O}_{\mathbb{P}^k}(1)$ will be relatively ample for f . Using [9, III, 1.3], we can find some $\mu \geq 0$ such that $\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'}$ is very ample on the fibres of f and without higher cohomology for all $e' \geq 0$. Hence $\mathcal{E} = f_* (\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'})$ will be locally free.

$$c_1(\mathcal{G}^\mu|_{f^{-1}(x)})^{\dim(f^{-1}(x))} + 1$$

is independent of $x \in H$ and we choose e' to be larger than that number (see Remark 4.4i). We have an embedding over H

$$\iota: Y \rightarrow \mathbb{P}(\mathcal{E}) \times_H H \times_M M \times \mathbb{P}^k = \mathbb{P}(\mathcal{E}) \times \mathbb{P}^k.$$

Remark 3.8. If $\mathcal{O}(1, 1)$ denotes the sheaf $\text{pr}_1^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^k}(1)$ on $\mathbb{P}(\mathcal{E}) \times \mathbb{P}^k$ and $p: \mathbb{P}(\mathcal{E}) \times \mathbb{P}^k \rightarrow H$ the projection, then the map

$$p_* \mathcal{O}(1, 1) \rightarrow f_*(\iota^* \mathcal{O}(1, 1))$$

is nothing but the map

$$\bigoplus^{k+1} \mathcal{E} = \bigoplus^{k+1} f_*(\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'}) \rightarrow f_*(\mathcal{G}^{\mu+1} \otimes \omega_{Y/H}^{e'})$$

given by the pullback under γ of the linear forms on \mathbb{P}^k . Especially, knowing ι we can reconstruct \mathcal{G} .

Addendum 3.9. For $\delta \geq 0$ the morphism

$$p_* \mathcal{O}(\delta, \delta \cdot \mu) \rightarrow f_*(\iota^* \mathcal{O}(\delta, \delta \cdot \mu))$$

will be surjective and its kernel spans the ideal sheaf of $\iota(Y)$. Hence locally on H , if one considers the induced morphism to the Hilbert scheme of subschemes of $\mathbb{P}^{r-1} \times \mathbb{P}^k$ and the Plücker embedding, it is given by sections of

$$\mathcal{L} = \det(f_*(\mathcal{G}^{2 \cdot \mu \cdot \delta} \otimes \omega_{Y/H}^{e' \cdot \delta}))^{r_1} \otimes \det(f_*(\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'}))^{-r_2}$$

for $r_1 = \text{rank}(f_*(\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'}))$ and $r_2 = \delta \cdot \text{rank}(f_*(\mathcal{G}^{2 \cdot \mu \cdot \delta} \otimes \omega_{Y/H}^{e' \cdot \delta}))$. In fact,

$$p_* \mathcal{O}(\delta, \delta \cdot \mu) = S^\delta(f_*(\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'})) \otimes S^{\delta \cdot \mu}(\mathbb{C}^{k+1})$$

and

$$f_*(\iota^* \mathcal{O}(\delta, \delta \cdot \mu)) = f_*(\mathcal{G}^{2 \cdot \mu \cdot \delta} \otimes \omega_{Y/H}^{e' \cdot \delta}).$$

Let us return to the proof of 3.2: Let $r = \text{rk}(\mathcal{E})$ and $\Pi' = \mathbb{P}(\bigoplus^r \mathcal{E}^\vee) \xrightarrow{\sigma'} H$. As in [9, I, 2.5], the tautological map

$$\sigma'^* \bigoplus^r \mathcal{E}^\vee \rightarrow \mathcal{O}_{\Pi'}(1)$$

induces a “universal basis”

$$\underline{s} \oplus^r \mathcal{O}_\Pi(-1) \rightarrow \sigma'^* \mathcal{E}.$$

If we write $\Pi = \Pi' - V(\det(\underline{s}))$ and $\sigma = \sigma'|_\Pi$ then $\sigma^* \mathcal{E} \cong \oplus^r \mathcal{O}_\Pi(-1)$ and

$$\Pi \times_H \mathbb{P}(\mathcal{E}) \simeq \Pi \times \mathbb{P}^{r-1}.$$

Pulling back f and ι to Π we obtain

$$\begin{array}{ccc} Y' & \xrightarrow{\iota'} & \Pi \times \mathbb{P}^{r-1} \times \mathbb{P}^k \\ & \searrow f' & \downarrow p' \\ & & \Pi. \end{array}$$

By [3] we obtain a morphism Φ from Π to the Hilbert scheme of subschemes of $\mathbb{P}^{r-1} \times \mathbb{P}^k$. Let H' denote the image of Φ in the Hilbert scheme and

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{j} & H' \times \mathbb{P}^{r-1} \times \mathbb{P}^k \\ & \searrow g' & \downarrow \text{pr}_1 \\ & & H' \end{array}$$

the morphisms obtain by restricting the universal family to H' . By definition of the Hilbert scheme (f', ι') are the pullback of (g', j) under Φ .

In [9, III, 1.5] we considered the Hilbert scheme H'' parametrizing pairs

$$(g : X \rightarrow S, \mathcal{N})$$

together with an embedding

$$X \rightarrow \mathbb{P}(g_* (\mathcal{N}^n \otimes \omega_{X/S}^e)) \simeq \mathbb{P}^{r-1} \times S.$$

By 3.8 $(g' : \mathcal{Y}' \rightarrow H', (\text{pr}_3 \circ j)^* \mathcal{O}_{\mathbb{P}^k}(1))$ and $\text{pr}_{1,2} \circ j : \mathcal{Y}' \rightarrow H' \times \mathbb{P}^{r-1}$ give rise to a morphism $\Psi : H' \rightarrow H''$.

Let $U \subset H$ be an open subscheme such that $\mathcal{E}|_U = \oplus^r \mathcal{O}_U$. In a natural way the group $G = \text{PGL}(r, \mathbb{C})$ acts on $\Pi_U = \sigma^{-1}(U) \cong G \times U$. Moreover, G acts on the Hilbert scheme of subschemes of $\mathbb{P}^{r-1} \times \mathbb{P}^k$ by change of coordinates in the first factor. Since Φ is compatible with this action H' is G invariant. G also acts on H'' , on \mathcal{Y}' , and on $f'^{-1}(\Pi_U)$ and Ψ, g' , and $f'|_{f'^{-1}(\Pi_U)}$ are G -linear.

In [9, III, 1.7], we obtained some invertible sheaf \mathcal{L}''_β on H'' such that all points of H'' are stable with respect to \mathcal{L}''_β under the G -action. In our notation, since the stabilizers of points of H in G are finite, this means $H'' = H''(\mathcal{L}''_\beta)^G$. Proposition 2.18 of [6] tells us:

Claim 3.10. If G operates on the quasi-projective scheme Z , if Z has a G -linearized ample sheaf and if $\delta : Z \rightarrow H''$ is a G -linear morphism, then there exists some G -linearized sheaf \mathcal{L} on Z with $Z = Z(\mathcal{L})^G$. Especially, the geometric quotient Z/G exists as a quasi-projective scheme.

Proposition 2.18 in [6] is obtained as an application of the Hilbert-Mumford-Criterion. Since in [9] and [10] we replaced this criterion by some condition on the boundary behaviour of certain G -linearized sheaves on orbits it is not surprising that we can show 3.10 directly. In [9, III, 4.2 and 4.3] we showed the existence of some G -linearized invertible sheaf λ'' on H'' which satisfied

Assumption 3.11. Let \mathcal{L}''_0 be G -linearized and ample on H'' . Then

$$\mathcal{L}''_\beta = \mathcal{L}''_0 \otimes \lambda''^\beta$$

is ample for all $\beta \geq 0$. Moreover, let $x \in H''$ and H'_x be the G -orbit. Then there is some projective compactification $i: H'' \rightarrow \bar{H}$ and some $\beta_0 > 0$, such that for the closure \bar{H}_x of H'_x in \bar{H} one has:

- a) $\bar{H}_x - H'_x$ is the exact support of some effective Cartier divisor Γ_x .
- b) For all multiples β of β_0 we can find $\alpha > 0$ and a coherent subsheaf $\mathcal{L}_\beta^{(\alpha)}$ of $i_*(\mathcal{L}_\beta'')^\alpha$, generated by global sections, and a inclusion

$$\mathcal{O}_{\bar{H}_x}(\Gamma_x) \rightarrow \mathcal{L}_\beta^{(\alpha)}|_{\bar{H}_x},$$

isomorphic over H'_x .

Proof of 3.10. Obviously, for δ as in 3.10 and $\lambda' = \delta^*\lambda''$, Assumption 3.11 lifts to Z , since the G -orbits in Z are finite over those in H'' . On the other hand, we have shown in [9, I and III] (see also [10] for a more general point of view) that 3.11 implies that $H'' = H''(\mathcal{L}_\beta'')^s$ for some $\beta \gg 0$. Hence we obtain 3.10 again.

Addendum 3.12. By the choice of the sheaf \mathcal{L} in (3.9) $\sigma'^*(\mathcal{L})|_H$ is the pullback of the ample sheaf \mathcal{L}' on H' induced by the Plücker embedding. If \mathcal{N}'' and \mathcal{L}_0'' are G -invariant sheaves on H'' , \mathcal{L}_0'' ample and λ'' the sheaf satisfying Assumption 3.11, then for all $\alpha \in \mathbb{Z}$ one can find some $\gamma \gg \beta \gg 0$ such that for

$$\mathcal{L}'_{\alpha, \beta, \gamma} = \mathcal{L}' \otimes \psi^*(\mathcal{N}''^{\alpha} \otimes \mathcal{L}_0''^{\beta} \otimes \lambda''^{\gamma})$$

one has $H' = H'(\mathcal{L}'_{\alpha, \beta, \gamma})^s$. Since $\mathcal{L}'_{\alpha, \beta, \gamma}$ is G -invariant it is again coming from a sheaf on H which we will call $\mathcal{L}_{\alpha, \beta, \gamma}$ in the sequel.

Claim 3.13. Let G act on Z and Z' and let $\delta': Z \rightarrow Z'$ be a G -linear morphism. If there exist geometric quotients Z/G and Z'/G and if $\delta'_*\mathcal{O}_Z = \mathcal{O}_{Z'}$, then $\delta''_*\mathcal{O}_{Z/G} = \mathcal{O}_{Z'/G}$ for the induced morphism

$$\delta'': Z/G \rightarrow Z'/G.$$

Proof. $\delta''_*\mathcal{O}_{Z/G}$ consists of G -invariant functions on Z , coming from functions on Z' .

Let us return to the proof 3.2. We had obtained a commutative diagram

$$\begin{array}{ccccc} Y'_U = f'^{-1}(\Pi_U) & \xrightarrow{\phi'} & \mathcal{Y}' & \xrightarrow{\text{pr}_1, s \circ j} & H' \times \mathbb{P}^k \\ f' \downarrow & & \theta' \downarrow & \swarrow \psi & \\ \Pi_U & \xrightarrow{\phi} & H' & \xrightarrow{\quad} & H'' \end{array}$$

of G -invariant morphisms and 3.10 allows us to divide out G :

$$\begin{array}{ccccc} f^{-1}(U) & \xrightarrow{\phi'_U} & \mathcal{Y}'/G & \xrightarrow{j} & P \times \mathbb{P}^k \\ f \downarrow & & \pi' \downarrow & \swarrow \text{pr}_1 & \\ U & \xrightarrow{\zeta_U} & P = H'/G & \xrightarrow{\quad} & H''/G. \end{array}$$

The points of P parametrize the fibres of f with the polarization given by $\mathcal{G}^\mu \otimes \omega_{Y/H}^{\epsilon'}$ and with a fixed morphism to \mathbb{P}^k given by sections of \mathcal{G} . If for $x, x' \in H$ one has $\phi(x) = \phi(x')$ then $\phi'(f^{-1}(x)) = \phi'(f^{-1}(x'))$ in R . Since $f^{-1}(x) \rightarrow \phi'(f^{-1}(x))$ is étale and since $f^{-1}(x)$ and $f^{-1}(x')$ are fibres of the smooth family

$$f^{-1}(\phi^{-1}(\phi(x))) \rightarrow \phi^{-1}(\phi(x))$$

one finds an isomorphism

$$f^{-1}(x) \xrightarrow{\cong} f^{-1}(x')$$

respecting the polarization and the morphism to \mathbb{P}^k . Hence ζ_U and Φ''_U glue together and give morphisms $\Phi'' : Y \rightarrow \mathcal{Y}'/G$ and $\zeta : H \rightarrow P$. Moreover, since ζ is constant on the fibres of $\phi : H \rightarrow M$, we obtain by 3.1 a morphisms

$$H \xrightarrow{\phi} M \xrightarrow{\varepsilon} P$$

with $\zeta = \varepsilon \circ \phi$.

Addendum 3.14. If $\mathcal{L}_{\alpha, \beta, \gamma}$ is the sheaf on H constructed in 3.12, then for some $p > 0$ its p -th power descends to some ample sheaf $\mathcal{L}_{\alpha, \beta, \gamma}^{(p)}$ on P and we found the sheaf $\mathcal{L}^{(p)}$ asked for in 3.7.

To finish the proof of 3.2 we still have to show that P is really the quotient of M by R :

The morphism $j' \circ \Phi'' : Y \rightarrow P \times \mathbb{P}^k$ is obtained by glueing quotients of the G -invariant morphisms

$$Y'_U \rightarrow \Pi_U \times \mathbb{P}^{r-1} \times \mathbb{P}^k \rightarrow \Pi_U \times \mathbb{P}^k \rightarrow H' \times \mathbb{P}^k$$

for different $U \subset H$ and hence it factors over

$$Y \xrightarrow{\phi''} H \times_M R \rightarrow M \times \mathbb{P}^k,$$

i.e. over the image R of Y in $M \times \mathbb{P}^k$. Alltogether we obtain

$$\begin{array}{ccccc} Y & \xrightarrow{\phi'} & R & \xrightarrow{\varepsilon'} & P \times \mathbb{P}^k \xrightarrow{\text{pr}_2} \mathbb{P}^k \\ f \downarrow & & p_1 \downarrow & & \text{pr}_1 \downarrow \\ H & \xrightarrow{\phi} & M & \xrightarrow{\varepsilon} & P \end{array}$$

and

$$Y \xrightarrow{\Phi''} \mathcal{Y}'/G \xrightarrow{j'} P \times \mathbb{P}^k$$

with $\varepsilon' \circ \phi' = j' \circ \Phi''$.

By construction \mathcal{Y}'/G is the image of \mathcal{Y}' and hence of Y . Therefore, the image of R in $P \times \mathbb{P}^k$ is the same as that of Y and we denote it by M' .

Claim 3.15. $\text{pr}_2|_{M'} : M' \rightarrow \mathbb{P}^k$ is an isomorphism of M' and M .

Proof. The fibres of R for p_1 are embedded by $\text{pr}_2 \circ \varepsilon'$ into M and M is the union of those fibres. Therefore, M' and M are isomorphic as sets. We have

$$\text{pr}_{2*} \mathcal{O}_{M'} \hookrightarrow \text{pr}_{2*} \varepsilon'_* \mathcal{O}_R = \text{pr}_{2*} \mathcal{O}_R = \mathcal{O}_M.$$

On the other hand, M is the image of Y in \mathbb{P}^k , and hence of M' , and $\text{pr}_{2*} \mathcal{O}_{M'}$ contains \mathcal{O}_M .

From 3.15 we get a second morphism $\pi = \text{pr}_1 \circ (\text{pr}_2|_{M'})^{-1} : M \rightarrow P$ and a diagram

$$\begin{array}{ccc} R & \xrightarrow{p_2} & M \longleftarrow \mathcal{Y}'/G \\ p_1 \downarrow & & \pi \downarrow \swarrow \pi' \\ M & \xrightarrow{\varepsilon} & P. \end{array}$$

Since R contains the diagonal of $M \times M$ the morphism ε must be the same as π .

Claim 3.16. $\pi: M \rightarrow P$ is a quasi-projective quotient of M by R .

Proof. By construction π is surjective and proper. Since π' is the quotient of the smooth morphism $g': \mathcal{Y}' \rightarrow H'$ and $g'_* \mathcal{O}_{\mathcal{Y}'} = \mathcal{O}_{H'}$, we have $\pi'_* \mathcal{O}_{\mathcal{Y}'/G} = \mathcal{O}_P$ by 3.13. Since M is the image of \mathcal{Y}'/G we have

$$\mathcal{O}_P \subset \pi_* \mathcal{O}_M \subset \pi'_* \mathcal{O}_{\mathcal{Y}'/G} = \mathcal{O}_P$$

and $\pi_* \mathcal{O}_M = \mathcal{O}_P$. Let $\delta: R \rightarrow M \times_P M$ be the induced map. δ must be a closed embedding. If $x \in M$ is a closed point then $p_1^{-1}(x)$ consists of those points equivalent to x and p_2 maps $p_1^{-1}(x)$ onto $\pi^{-1}(\pi(x))$. Hence R contains $\pi^{-1}(\pi(x)) \times \pi^{-1}(\pi(x))$ and δ is bijective on sets.

Remark 3.17. Theorem 3.2 can be applied to quotients by the action of an abelian variety on a quasi-projective variety, as studied by Seshadri in [8].

4 Moduli of polarized manifolds

Let us return to the discussion of different moduli functors of polarized varieties, started in [9, III, Sect. 1]. For a complex scheme S we consider

$$\Gamma(S) = \{(f: X \rightarrow S, \mathcal{H}); f \text{ flat proper and surjective, } \mathcal{H} \text{ invertible and relatively ample for } f; \text{ all fibres of } f \text{ are normal irreducible varieties with rational Gorenstein singularities, not uniruled}\}.$$

There are several ways to define isomorphisms for objects in $\Gamma(S)$.

Definition 4.1. Let $(f: X \rightarrow S, \mathcal{H})$ and $(f': X' \rightarrow S, \mathcal{H}')$ be elements of $\Gamma(S)$.

a) $(f, \mathcal{H}) \simeq (f', \mathcal{H}')$ if there exists an invertible sheaf \mathcal{B} on S , an S -isomorphism $\tau: X \rightarrow X'$ and an isomorphism $\tau^* \mathcal{H}' \simeq \mathcal{H} \otimes f^* \mathcal{B}$.

b) $(f, \mathcal{H}) \equiv (f', \mathcal{H}')$ if there exists an S -isomorphism $\tau: X \rightarrow X'$ such that for all $s \in S$ the sheaves $\mathcal{H}|_{f^{-1}(s)}$ and $\tau^* \mathcal{H}'|_{f'^{-1}(s)}$ are numerically equivalent.

c) $(f, \mathcal{H}) \equiv_{\mathbb{Q}} (f', \mathcal{H}')$ if there exist $a, b \in \mathbb{N} - \{0\}$ with $(f, \mathcal{H}^a) \equiv (f', \mathcal{H}'^b)$.

Let $(f: X \rightarrow S, \mathcal{H})$ be an element of $\Gamma(S)$ and F a smooth fibre of f . $\mathcal{H}|_F$ defines a class $[\mathcal{H}|_F]$ in the Neron-Severi group $NS(F)$. Let us denote by $(\mathcal{H}|_F)_{in}$ an ample sheaf on F which generates the maximal free rank one \mathbb{Z} -submodule of $NS(F)$ which contains $[\mathcal{H}|_F]$. This construction behaves nice in families: If f is smooth we can find some étale neighbourhood S' of S such that on $f^{-1}(S')$ one has a sheaf \mathcal{H}_{in} with $(\mathcal{H}|_{f^{-1}(x)})_{in} = \mathcal{H}_{in}|_{f^{-1}(x)}$ for $x \in S'$.

Definition 4.2. We define functors \mathcal{M}_h' , \mathcal{P}_h' , and \mathcal{N}_h'' by

a) (see [9])

$$\mathcal{M}_h'(S) = \{(f: X \rightarrow S, \mathcal{H}) \in \Gamma(S); h(v) = X(\mathcal{H}|_F^v) \text{ for all } v \geq 0 \text{ and all fibres } F \text{ of } f\} / \simeq$$

b) (see [4])

$$\mathcal{P}_h'(S) = \mathcal{M}_h'(S) / \equiv$$

c) ([6, 7])

$$\mathcal{N}_h''(S) = \{(f: X \rightarrow S, \mathcal{H}) \in \Gamma'(S); f \text{ smooth, for all fibres } F \text{ of } f \text{ and all } v \geq 0 \ h(v) = X((\mathcal{H}|_F)_{\text{in}}^v)/\cong_{\mathbb{Q}}\}.$$

d) We have a natural transformation $\Sigma: \mathcal{M}_h' \rightarrow \mathcal{P}_h'$.

As in [9, III, Sect. 1] we consider a subfunctor \mathcal{M}_h'' of \mathcal{M}_h' which satisfies

Assumptions 4.3. a) \mathcal{M}_h'' is bounded and separated. For $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h''(S)$ the subset

$$S_0 = \{s \in S; (f^{-1}(s), \mathcal{H}|_{f^{-1}(s)}) \in \mathcal{M}_h''(\text{Spec}(\mathbb{C}))\}$$

is locally closed in S .

b) For example, we can take

$$\mathcal{M}_h''(S) = \{(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h'(S); f \text{ smooth}\},$$

but [4] gives some results allowing to consider larger \mathcal{M}_h'' if $n \leq 3$. We do not know what is the best possible choice for \mathcal{M}_h'' . In any case, the assumptions made in 4.3c are enough for our construction.

c) Let $\mu > 0$ be an integer such that \mathcal{H}^μ is very ample for all $(X, \mathcal{H}) \in \mathcal{M}_h''(\text{Spec}(\mathbb{C}))$. In fact, as we pointed out in [2, 2.12], the property 2.2a in [9, III] only holds true for manifolds. Otherwise, in order to have the bound given in [9, III, 2.2a], for $e(\mathcal{H}^\mu)$ one even has to choose $\mu > 0$ such that:

For all $(X, \mathcal{H}) \in \mathcal{M}_h''(\text{Spec}(\mathbb{C}))$ there is a desingularization $\tau: X' \rightarrow X$ and an effective exceptional divisor E on X' such that $\tau^ \mathcal{H}^\mu \otimes \mathcal{O}_{X'}(-E)$ is very ample.*

By the boundedness of \mathcal{M}_h'' such a μ exists.

d) Let c be the highest coefficient of h and μ as in c. For $v = \mu(n+2)$ and $e \geq (n!) \cdot c \cdot v^n + 1$ we consider the subfunctor \mathcal{M}_h of \mathcal{M}_h'' given by

$$\mathcal{M}_h(\text{Spec}(\mathbb{C})) = \{(X, \mathcal{H}) \in \mathcal{M}_h''(\text{Spec}(\mathbb{C})); \mathcal{H}^v \otimes \omega_X^e \text{ very ample}\}.$$

Correspondingly, we have the subfunctor \mathcal{P}_h of \mathcal{P}_h' with $\mathcal{P}_h(S) = \Sigma(\mathcal{M}_h(S))$ and, if we choose μ such that $\mathcal{H}_{\text{in}}^\mu$ is very ample for all $(X, \mathcal{H}) \in \mathcal{N}_h''(\text{Spec}(\mathbb{C}))$, the subfunctor \mathcal{N}_h of \mathcal{N}_h'' with

$$\mathcal{N}_h(\text{Spec}(\mathbb{C})) = \{(X, \mathcal{H}) \in \mathcal{N}_h''(\text{Spec}(\mathbb{C})); \mathcal{H}_{\text{in}}^v \otimes \omega_X^e \text{ very ample}\}.$$

Remarks 4.4. i) In [9, III], we had chosen $e = (n!) \cdot c \cdot v^n + 1$, but we only used $e \geq (n!) \cdot c \cdot v^n + 1$. In fact, this freedom to choose e slightly larger than in [9, III] is necessary for the description of an ample sheaf in Sect. 5. Moreover, we added in [9, III], in the definition of \mathcal{M}_h'' the condition that $\mathcal{H}^v \otimes \omega_X^e$ has no higher cohomology. Obviously, this is always true by Kodaira's vanishing theorem.

ii) By [9, III, 1.3], the condition in the definition of \mathcal{N}_h is independent of the sheaf \mathcal{H}_{in} chosen. Moreover, it follows that $\mathcal{N}_h(\text{Spec}(\mathbb{C}))$ contains all

$$(X, \mathcal{H}) \in \mathcal{N}_h''(\text{Spec}(\mathbb{C}))$$

with ω_X numerically effective. The same holds for \mathcal{P}_h and \mathcal{M}_h and we have inclusions

$$\mathcal{M}_h^{\text{sa}}(S) \subset \mathcal{M}_h^{\text{nef}}(S) \subset \mathcal{M}_h(S),$$

$$\mathcal{P}_h^{\text{sa}}(S) \subset \mathcal{P}_h^{\text{nef}}(S) \subset \mathcal{P}_h(S),$$

$$\mathcal{N}_h^{\text{sa}}(S) \subset \mathcal{N}_h^{\text{nef}}(S) \subset \mathcal{N}_h(S).$$

where “sa” stands for the condition “ $\omega_{X/S|F}$ semi-ample for all fibres F of f ”, and “nef” for the condition “ $\omega_{X/S|F}$ numerically effective for all fibres F of f ”.

By [5] we know that the condition “semi-ample” is Zariski open. One hopes that $\mathcal{N}_h^{\text{sa}} = \mathcal{N}_h^{\text{nef}}$, but even for $n=3$ this is unknown. The main result of [9, III] gives:

Theorem 4.5. *For the functors \mathcal{M}_h and $\mathcal{M}_h^{\text{sa}}$ considered above there exist coarse quasi-projective moduli schemes M_h and M_h^{sa} .*

The additional information obtained in this article is:

Theorem 4.6. a) *We have coarse quasi-projective moduli schemes P_h and P_h^{sa} for the functors \mathcal{P}_h and $\mathcal{P}_h^{\text{sa}}$.*

b) *We have coarse quasi-projective moduli schemes N_h and N_h^{sa} for the functors \mathcal{N}_h and $\mathcal{N}_h^{\text{sa}}$.*

Remark 4.7. We have natural morphisms

$$\begin{array}{ccc} M_h^{\text{sa}} & \xrightarrow{i} & M_h \\ \Sigma^{\text{sa}} \downarrow & & \Sigma \downarrow \\ P_h^{\text{sa}} & \xrightarrow{i} & P_h \end{array}$$

and $N_h^{\text{sa}} \xrightarrow{j} N_h$. The square is a fibre product, i, ι , and j are open embeddings and Σ is proper an surjective with $\Sigma_* \mathcal{O}_{M_h} = \mathcal{O}_{P_h}$, as we will see in the proof of 4.6. For $\mathcal{N}_h^{\text{nef}}$ (and in a similar way for $\mathcal{P}_h^{\text{nef}}, \mathcal{M}_h^{\text{nef}}$) we only know that $\mathcal{N}_h^{\text{nef}}(\text{Spec}(\mathbb{C}))$ is parametrized by some subset N_h^{nef} with

$$N_h^{\text{sa}} \subset N_h^{\text{nef}} \subset N_h,$$

but we do not know whether N_h^{nef} is a subscheme.

Proof of 4.6 a. \mathcal{M}_h is bounded and hence we may choose some $\eta > 0$ such that for all $(X, \mathcal{H}) \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ we have

$$\eta \cdot \text{Pic}^c(X) = \text{Pic}^0(X).$$

If $h'(t) = h(\eta \cdot t)$ replacing \mathcal{H} by \mathcal{H}^η gives natural transformations

$$\begin{array}{ccc} \mathcal{M}_h & \xrightarrow{\Phi} & \mathcal{M}_{h'} \\ \Sigma \downarrow & & \Sigma' \downarrow \\ \mathcal{P}_h & \xrightarrow{\Phi'} & \mathcal{P}_{h'}, \end{array}$$

where $\Phi'(S)$ is an isomorphism. The image of $\Phi(S)$ is

$$\{(f: X \rightarrow S, \mathcal{H}') \in \mathcal{M}_{h'}(S); \mathcal{H}' = \mathcal{H}^\eta \text{ for some invertible sheaf } \mathcal{H} \text{ on } X\}.$$

By the choice of η the fibres of $\Sigma'(\text{Spec} \mathbb{C})$ restricted to this image are quotients of the connected scheme Pic^0 . Since $\text{Im}(\Phi(S)) \subset \mathcal{M}_{h'}(S)$ is given by a constructible condition we may replace \mathcal{M}_h by this image and assume that the same connectedness holds for the fibres of

$$\Sigma(\text{Spec}(\mathbb{C})) : \mathcal{M}_h(\text{Spec}(\mathbb{C})) \rightarrow \mathcal{P}_h(\text{Spec}(\mathbb{C})).$$

As in [9, III, 1.5], one has a quasi-projective Hilbert scheme H and a universal family

$$(g: \mathcal{X} \rightarrow H, \mathcal{L}) \in \mathcal{M}_h(H)$$

together with an isomorphism

$$\mathbb{P}(g_*(\mathcal{L}^\vee \otimes \omega_{\mathcal{X}/H}^e)) \rightarrow \mathbb{P}^{r-1} \times H.$$

On $H \times H$ we have two equivalence relations, the one given by the $G = \mathbb{P}GL(r, \mathbb{C})$ action

$$G \times H \rightarrow H \times H,$$

and the one given by numerical equivalence

$$\hat{Y} \xrightarrow{\delta} H \times H$$

with

$$\hat{Y} = \{(h_1, h_2, \sigma); \sigma: g^{-1}(h_1) \rightarrow g^{-1}(h_2) \text{ an isomorphism and } \sigma^* \mathcal{L}|_{g^{-1}(h_2)} \cong \mathcal{L}|_{g^{-1}(h_1)}\}.$$

By [9, III, 1.8], there is a quasi-projective quotient $\phi: H \rightarrow M$ with respect to the G -action. ϕ satisfies the condition asked for in 3.1 a. Using the description of \hat{Y} given above we have a morphism from \hat{Y} to $\text{Pic}^r(\mathcal{X}/H)$ given by

$$(h_1, h_2, \sigma) \mapsto \sigma^* \mathcal{L}|_{g^{-1}(h_2)} \otimes \mathcal{L}^{-1}|_{g^{-1}(h_1)}.$$

Since we assume that the kernel of $\Sigma(\text{Spec}(\mathbb{C}))$ is connected the image of this morphism is $\text{Pic}^0(\mathcal{X}/H)$. Let us write

$$\begin{array}{ccccc} \hat{Y} & \xrightarrow{\gamma} & Y = \text{Pic}^0(\mathcal{X}/H) & \xrightarrow{\phi'} & M \times M \\ \downarrow & & f \downarrow & & \text{pr}_1 \downarrow \\ H \times H & \xrightarrow{\text{pr}_1} & H & \xrightarrow{\phi} & M \end{array}$$

for the corresponding morphisms and R for the image of ϕ' in $M \times M$. Obviously, R is an equivalence relation and f satisfies the assumptions made in 3.1 b and e.

As shown in [4, 4.1.4], \hat{Y} has equidimensional fibres of constant dimension over its image in $H \times H$. Since this image is $G \times G$ invariant and $H \times H \rightarrow M \times M$ the quotient by $G \times G$, \hat{Y} must have equidimensional fibres of constant dimension over R . Since the fibres of γ are just given by different choices of coordinates in \mathbb{P}^{r-1} the same holds for Y over R .

The image of Y in $H \times M$ is again G -invariant and R is the quotient of this image by the G -action on the first factor. Hence R will be proper over M and, by 3.13, $(\text{pr}_1|_R)_* \mathcal{O}_R = \mathcal{O}_M$. Hence R is a compact connected equivalence relation as defined in 2.1.

It remains to verify 3.1 c. For $x \in H$ the scheme $f^{-1}(x)$ is an abelian variety and the morphism $f^{-1}(x) \rightarrow R$ is just obtained by dividing $f^{-1}(x)$ by some subgroup and 3.1 c is obvious.

By 3.2 we obtain a quasi-projective quotient $\pi: M \rightarrow P$ of M by R . Since M is a geometric quotient of H by G and since we have the universal property 2.4 for $\pi: M \rightarrow P$, P is a coarse moduli scheme for \mathcal{P}_h .

As already mentioned the condition " $\omega_{\mathcal{X}}$ semi-ample" is an open condition by [5] and we obtain both parts of 4.6a.

It is well-known, that **b** follows from **a**: Let us consider the subfunctor $\mathcal{P}_h^{\text{in}}$ of \mathcal{P}_h given by $\mathcal{P}_h^{\text{in}}(\text{Spec}(\mathbb{C})) = \{(X, \mathcal{H}) \in \mathcal{P}_h(\text{Spec}(\mathbb{C})); X \text{ smooth and } \mathcal{H} = \mathcal{H}_{\text{in}}\}$. Since both conditions are constructible we obtain a coarse moduli scheme as subscheme of P_h . On the other hand, sheafifying \mathcal{N}_h and \mathcal{P}_h with respect to the étale topology [7] we obtain an isomorphism

$$\Delta: \mathcal{N}_h \rightarrow \mathcal{P}_h^{\text{in}}.$$

Since sheafification does not effect coarse moduli schemes we finished the proof of 4.6.

5 Ample sheaves on P_h

The aim of this section is to find a reasonably natural ample sheaf on P_h , or, in the notations of 3.14 a natural candidate $\mathcal{L}_{\alpha, \beta, \gamma}^{(p)}$. Since we have to use the results from [2], as recalled below, we have to choose $e \geq (n!) \cdot c \cdot v^n + 2$ (see 4.4i).

Theorem 5.1 [2, 5.6]. *Let \mathcal{M}_h be the moduli functor considered in Sect. 4 and M_h the coarse quasi-projective moduli scheme for \mathcal{M}_h . Then, for v as in 4.3d, $e \geq (n!) \cdot c \cdot v^n + 2$, and for some $p > 0$ there is an ample invertible sheaf $\lambda^{(p)}$ on M_h with the following property:*

For $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ let $\varphi: S \rightarrow M_h$ be the induced morphism, $r' = \text{rank}(f_(\mathcal{H}^v))$ and $r = \text{rank}(f_*(\mathcal{H}^v \otimes \omega_{X/S}^e))$. Then*

$$\varphi^* \lambda^{(p)} = (\det(f_*(\mathcal{H}^v \otimes \omega_{X/S}^e))^{r'} \otimes \det(f_*(\mathcal{H}^v))^{-r})^p.$$

If for some $\delta > 0$ and all $(F, \mathcal{H}) \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ one knows that $\omega_F^\delta = \mathcal{O}_F$, then for all $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ one finds some sheaf $\theta_{X/S}^{(\delta)}$ with $f^* \theta_{X/S}^{(\delta)} = \omega_{X/S}^\delta$. Hence, using the notations from 5.1, $\varphi^* \lambda^{(p)} = (\theta_{X/S}^{(\delta)})^{e \cdot \delta^{-1} \cdot r' \cdot r^{-p}}$ and by (5.1) we have:

Corollary 5.2 (see [2, 5.10]). *Keeping the assumptions from 5.1, assume that for some $\delta > 0$ and all $(F, \mathcal{H}) \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ one has $\omega_F^\delta = \mathcal{O}_F$. Assume, moreover, that e is divisible by δ . Then, for some $p > 0$, there is an ample invertible sheaf $\theta^{(p)}$ on M_h with: For $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ and the induced morphism $\varphi: S \rightarrow M_h$*

$$f^* \varphi^* \theta^{(p)} = \omega_{X/S}^p.$$

5.2 implies that M_h cannot contain a projective curve on C such that all $c \in C$ correspond to $(F, \mathcal{H}_c) \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ for the same F . In other terms, the assumptions of 5.2 imply that the fibres of the proper morphism $\sigma: M_h \rightarrow P_h$ induced by $\Sigma: \mathcal{M}_h \rightarrow \mathcal{P}_h$, see 4.2d, are zero-dimensional. Since, as in the proof of 4.6a, we can even assume σ to be an isomorphism, and since, as in the proof of 4.6b, we can assume that N_h is a subscheme of P_h , we obtain

Corollary 5.3. *5.2 remains true, if we replace (\mathcal{M}_h, M_h) by (\mathcal{P}_h, P_h) or (\mathcal{N}_h, N_h) .*

Let us return to the general case, i.e. to the functor \mathcal{P}_h defined in 4.3d. As in [2] and as in the proof of 4.6a let us assume:

Assumption 5.4. Let $v \geq 0$ satisfy the assumptions made in 4.3 and assume, moreover, that for all $(F, \mathcal{H}) \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ one has $v \cdot \text{Pic}^r(F) = \text{Pic}^0(F)$. Let us assume that in 4.3d we have chosen $e \geq (n!) \cdot c \cdot v^n + 2$.

In order to find a sheaf on P_h which is a candidate for an ample sheaf, we have first to define natural sheaves on S for $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{P}_h(S)$, which are independent of the sheaf \mathcal{H} chosen in the numerical equivalence class.

Constructions 5.5 (of $\theta_{X/S}$). For $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ let $\pi: Y = \text{Pic}^0(X/S) \rightarrow S$ be the connected component of the relative Picard scheme of Y over S , which contains the zero-section. Let $A = \text{Aut}^0(X/S)$ be the connected component of the zero section in the scheme of automorphisms of X over S .

For $\sigma \in \text{Aut}_{X/S}(T)$ we have $\mathcal{H}^\nu \otimes \sigma^* \mathcal{H}^{-\nu} \in \text{Pic}_{X/S}(T)$ and we obtain thereby a S -morphism $A \xrightarrow{q} Y$.

The image of q is independent of ν . Let $\pi': Y' \rightarrow S$ be the quotient of Y by $q(A)$. Since $\pi': Y' \rightarrow S$ is again a family of abelian varieties, we can find an invertible sheaf $\theta_{X/S}$ on S with $\pi'^* \theta_{X/S} = \omega_{Y'/S}$.

Constructions 5.6 (of $\chi_{X/S, \mathcal{H}}^\eta$). Let us keep the notations from 5.5.

If $f: X \rightarrow S$ has a section $s: S \rightarrow X$, then Grothendieck's theorem on $\text{Pic}_{X/S}$, see [6, 0, Sect. 5], shows that $\text{Pic}(X/S)$ represents the functor $\text{Pic}_{X/S}$. Especially, we have a universal bundle \mathcal{L} on $Y \times_S X$.

Let us write $X' = Y \times_S X$, $\pi' = \text{pr}_2: X' \rightarrow X$, $f' = \text{pr}_1: X' \rightarrow Y$ and $\mathcal{H}' = \mathcal{L} \otimes \pi'^* \mathcal{H}$. For $r = \text{rank}(f'_*(\mathcal{H}'^\nu \otimes \omega_{X'/Y}^\nu))$ and $r' = \text{rank}(f'_*(\mathcal{H}'^\nu))$ we define

$$\mathcal{A} = \det(f'_*(\mathcal{H}'^\nu \otimes \omega_{X'/Y}^\nu))^{r'} \otimes \det(f'_*(\mathcal{H}'^\nu))^{-r}.$$

\mathcal{A} remains the same, if we replace \mathcal{H}' by $\mathcal{H}' \otimes f'^* \mathcal{B}'$ for some \mathcal{B}' on Y . Especially, if $s_0: S \rightarrow X$ is a second section of X and if \mathcal{L}_0 is the universal bundle with $(\text{id}_Y \times s_0)^* \mathcal{L}_0 = \mathcal{O}_Y$, then the sheaf $\mathcal{H}'_0 = \mathcal{L}_0 \otimes \pi'^* \mathcal{H}$ and \mathcal{H}' give rise to the same sheaf \mathcal{A} . Moreover, \mathcal{A} does not depend on the choice of \mathcal{H} in the equivalence class with respect to \simeq (see 4.1 a).

For $\eta > 0$ let us write $\chi_{X/S, \mathcal{H}}^{(\eta)} = \det(\pi_* \mathcal{A}^\eta)$. If $f: X \rightarrow S$ does not have a section we consider

$$(\text{pr}_2: X \times_S X \rightarrow X, \text{pr}_1^* \mathcal{H}) \in \mathcal{P}_h(X).$$

The diagonal $\Delta: X \rightarrow X \times_S X$ is a section and

$$\chi^{(\eta)} = \chi_{X \times_S X/X, \text{pr}_1^* \mathcal{H}}^{(\eta)}$$

is defined. Since our construction of $\chi^{(\eta)}$ is independent of the section, $\chi^{(\eta)}$ must be constant on the fibres of $f: X \times S$ and therefore, we find some invertible sheaf on S , called $\chi_{X/S, \mathcal{H}}^{(\eta)}$ again, with

$$f^* \chi_{X/S, \mathcal{H}}^{(\eta)} = \chi^{(\eta)}.$$

Lemma 5.7. *If for $i=1,2$ we have $(f: X \rightarrow S, \mathcal{H}_i) \in \mathcal{M}_h(S)$ with*

$$(f: X \rightarrow S, \mathcal{H}_1) \equiv (f: X \rightarrow S, \mathcal{H}_2),$$

then $\chi_{X/S, \mathcal{H}_1}^{(\eta)} \simeq \chi_{X/S, \mathcal{H}_2}^{(\eta)}$.

Proof. As in 5.6 we may assume that f has a section s . Let us write \mathcal{H}'_i and \mathcal{A}_i for the sheaves constructed in 5.6 starting from \mathcal{H}_i . We can assume that $s^* \mathcal{H}_1 = s^* \mathcal{H}_2$ and hence $\mathcal{H}'_1 \otimes \mathcal{H}'_2^{-\nu}$ lies in $\text{Pic}_{X/S}^0(S)$. If $\sigma: S \rightarrow Y$ is the section corresponding to $\mathcal{H}'_1 \otimes \mathcal{H}'_2^{-\nu}$ and $T_\sigma: Y \rightarrow Y$ the S -morphism "translation by σ ", then

$$(T_\sigma \times \text{id}_X)^* \mathcal{H}'_2^\nu = (T_\sigma \times \text{id}_X)^* \mathcal{L} \otimes \pi'^* \mathcal{H}'_2^\nu = \mathcal{L} \otimes \pi'^* \mathcal{H}'_1^\nu = \mathcal{H}'_1^\nu$$

and $T_\sigma^* \mathcal{A}_2 = \mathcal{A}_1$. This implies 5.7 since

$$\pi_* \mathcal{A}_1^\eta = (\pi \circ T_\sigma)_* \mathcal{A}_1^\eta \simeq \pi_* \mathcal{A}_2^\eta.$$

Theorem 5.8. *Under the assumptions made in 5.4 and using the notations of 5.5 and 5.6, let \mathcal{P}_h be the moduli functor of polarized varieties considered in Sect. 4. Let P_h be*

the coarse quasi-projective moduli scheme for \mathcal{P}_h constructed in 4.6a. Then for some integers $p \gg q \gg \eta \gg 0$ there exists an ample invertible sheaf $\mathcal{L} = \mathcal{L}^{(\eta, q, p)}$ on P_h satisfying:

For $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{P}_h(S)$ and the induced morphism $\phi: S \rightarrow P_h$ one has

$$\phi^* \mathcal{L}^{(\eta, q, p)} = (\theta_{X/S}^q \otimes \chi_{X/S, \mathcal{H}}^{(\eta)})^p.$$

Repeating the argument used to prove 4.6b we get as well:

Corollary 5.9. 5.8 remains true if one replaces (\mathcal{P}_h, P_h) by (\mathcal{N}_h, N_h) .

Proof of 5.8. In order to prove 4.6a we considered the equivalence relation R together with the good smooth lifting f given by

$$\begin{array}{ccc} Y = \text{Pic}^0(\mathcal{X}/H) & \xrightarrow{\phi'} & M_h \times M_h \\ f \downarrow & & \downarrow \text{pr}_1 \\ H & \xrightarrow{\phi} & M_h \end{array}$$

and $R = \phi'(Y)$. For $\eta \gg 0$ let $\lambda^{(\eta)}$ be the ample sheaf on M_h considered in 5.5. We may assume $\lambda^{(\eta)}$ to be very ample.

Obviously, $\phi'^* \text{pr}_2^* \lambda^{(\eta)}$ is nothing but the sheaf \mathcal{A}^n considered in 5.6 and, if $(g: \mathcal{X} \rightarrow H, \mathcal{H}) \in \mathcal{M}_h(H)$ denotes the universal family we have

$$\chi_{\mathcal{X}/H, \mathcal{H}}^{(\eta)} = \det(f_* \phi'^* \text{pr}_2^* \lambda^{(\eta)}).$$

Since all $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ are locally obtained by pullback from $(g: \mathcal{X} \rightarrow H, \mathcal{H})$ it is enough to prove that for $p \gg g \gg \eta \gg 0$ the sheaf

$$(\theta_{\mathcal{X}/H}^q \otimes \chi_{\mathcal{X}/H, \mathcal{H}}^{(\eta)})^p$$

descends to an ample sheaf on P_h , or, in other terms, that we can take the sheaf $\mathcal{L}_{\alpha, \beta, \gamma}^p$ in (3.14) to be of that form.

In 3.4 we replaced Y by some quotient Y' , such that ϕ' factors like

$$Y \xrightarrow{\pi'} Y' \xrightarrow{\phi''} R$$

and such that ϕ'' is finite and $\pi'_* \mathcal{O}_{Y'} = \mathcal{O}_Y$.

Let $h \in H$ be a point and let $(F, \mathcal{H}) \in \mathcal{M}_h(\text{Spec}(\mathbb{C}))$ be the pair corresponding to $\phi(h)$. If $y \in f^{-1}(h) = \text{Pic}^0(g^{-1}(h))$ corresponds to the sheaf \mathcal{L} , then $\phi'(y) = (\phi(h), m')$ where m corresponds to $(F, \mathcal{H} \otimes \mathcal{L}) \in \mathcal{M}(\text{Spec}(\mathbb{C}))$.

Hence $\phi'(y) = \phi'(y')$ if and only if there is an isomorphism $\tau: g^{-1}(h) \rightarrow g^{-1}(h)$ such that $\tau^* \mathcal{L}$ is the sheaf corresponding to y' . In other terms, Y' is just the quotient $Y/\delta(A)$ considered in 5.5. Obviously, if f' is the induced morphism from Y' to H , then $\det(f'_* \phi''^* \text{pr}_2^* \lambda^{(\eta)}) = \chi_{\mathcal{X}/H, \mathcal{H}}^{(\eta)}$ again. Hence, if we write $\mathcal{G} = \phi''^* \text{pr}_2^* \lambda^{(\eta)}$ and $\theta_{Y'/H} = \theta_{\mathcal{X}/H}$, we have $f'^* \theta_{Y'/H} = \omega_{Y'/H}$ and 5.8 is just saying

Claim 5.10. We can choose the sheaf $\mathcal{L}_{\alpha, \beta, \gamma}$ in 3.14 to be of the form

$$(\theta_{Y'/H} \otimes \det(f'_* \mathcal{G}^\mu))^p.$$

Proof. In order to prove 5.10 we can assume, as in the proof of 3.2 that Y is already the good smooth lifting of R with ϕ' finite. Since f is a family of abelian varieties, the sheaf \mathcal{L} of 3.9 is nothing but

$$\mathcal{L} = \det(f_* (\mathcal{G}^{2 \cdot \mu \cdot \delta}))^{r_1} \otimes \det(f_* (\mathcal{G}^\mu))^{-r_2}$$

for $r_1 = \text{rank}(f_*(\mathcal{G}^\mu))$ and $r_2 = \delta \cdot \text{rank}(f_*(\mathcal{G}^{2 \cdot \mu \cdot \delta}))$. In the same way, we have the equality

$$\mathcal{N} = \mathcal{L} \otimes \det(f_*(\mathcal{G}^\mu))^{-r_2} = \det(f_*(\mathcal{G}^{2 \cdot \mu \cdot \delta} \otimes \omega_{Y/H}^{2 \cdot e' \cdot \delta}))^{r_1} \otimes \det(f_*(\mathcal{G}^\mu \otimes \omega_{Y/H}^{e'}))^{-2 \cdot r_2}.$$

If one considers the embedding

$$\text{pr}_1 \circ \iota: Y \rightarrow \mathbb{P}(\mathcal{E}),$$

then \mathcal{N} locally corresponds to the pullback of the ample sheaf on the Hilbertscheme H'' of subschemes of \mathbb{P}^{r-1} , which is given by the Plücker coordinates. Hence, in the notations of 3.12 $\sigma^*(\mathcal{N})|_H$ is the pullback of some sheaf \mathcal{N}'' on H'' .

Let $(g'': \mathcal{Y}'' \rightarrow H'', \mathcal{G}'')$ and $(g': \mathcal{Y}' \rightarrow H', \mathcal{G}')$ be the universal families. Then, as shown in [9, III, 4.3], we can choose in 3.11

$$\lambda'' = \det(g''_*(\mathcal{G}''^\mu \otimes \omega_{\mathcal{Y}''/H''}^{e''}))^{r_1} \otimes \det(g''_*(\mathcal{G}''^\mu))^{-r_3}$$

for

$$r_3 = \text{rank}(g''_*(\mathcal{G}''^\mu \otimes \omega_{\mathcal{Y}''/H''}^{e''})).$$

Again, since

$$\omega_{\mathcal{Y}''/H''} = g''^* \theta_{\mathcal{Y}''/H''},$$

we have $r_3 = r_1$ and

$$\lambda'' = \theta_{\mathcal{Y}''/H''}^{e'' \cdot r_1^2}.$$

Moreover, by [2, 5.5], λ'' is ample on H'' and we can choose $\mathcal{L}_0'' = \lambda''$ in 3.12.

Finally, for $\alpha = -1$, the sheaf considered in 3.12 is

$$\mathcal{L}_{-1, \beta, \gamma}'' = \det(g'_*(\mathcal{G}'^\mu))^{r_2} \otimes \psi^* \lambda''^{\beta + \gamma} = \det(g'_*(\mathcal{G}'^\mu))^{r_2} \otimes \theta_{\mathcal{Y}'/H'}^{\beta + \gamma} \cdot e' \cdot r_1^2$$

and the sheaf $\mathcal{L}_{-1, \beta, \gamma}$ on H , considered in 3.12 and 3.14 is

$$\det(f_*(\mathcal{G}^\mu))^{r_2} \otimes \theta_{Y/H}^q$$

for some $q \geq 0$.

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