5. Some first examples for $S = \emptyset$

In the proof of 0.2 the assumption $S \neq 0$ was only used to obtain the splitting (3.4.1). If one assumes the existence of such a splitting, and the smoothness of $f : X \to Y$, one obtains a first very special case of Theorem 0.5.

Corollary 5.1. *Let $f : X \to Y$ be a smooth non-isotrivial family of abelian varieties reaching the Arakelov bound. Assume the decompositions $R^1 f_*(\mathbb{C}_X) = \mathbb{V} \oplus \mathbb{U}_1$ and $\mathrm{End}(\mathbb{V}) = \mathbb{W} \oplus \mathbb{U}$ in 2.7 are both defined over $\mathbb{Q}$. Then there exists an étale cover $Y' \to Y$, a family of false elliptic curves $h : A \to Y'$ (see Example 0.3, a), and a $g - g_0$ dimensional abelian variety B such that $f' : X' = X \times_Y Y' \to Y'$ is isogenous to*

$$B \otimes A \otimes_{Y'} \cdots \otimes_{Y'} A \longrightarrow Y'.$$

To illustrate the methods of proof for Theorems 0.5 and 0.7 let us consider a non-isotrivial smooth family $f : X \to Y$ of abelian surfaces reaching the Arakelov bound. Let us assume first that the unitary part $\mathbb{U}_1$ of $R^1 f_* \mathbb{C}_X$ is trivial. By Proposition 1.4

$$R^1 f_* \mathbb{C}_X = \mathbb{V} \simeq \mathbb{L} \otimes \mathbb{T}$$

where $\mathbb{T}$ is a unitary rank two local system.

By Remark 1.10 $\mathrm{End}(\mathbb{V}) \simeq \mathrm{End}_0(\mathbb{T}) \oplus \mathbb{C} \oplus \mathbb{W}$, where

$$\mathbb{W} = \mathrm{End}_0(\mathbb{T}) \otimes \mathrm{End}_0(\mathbb{L}) \oplus \mathrm{End}_0(\mathbb{L})$$

has a maximal Higgs field. By Corollary 2.7, ii), $\mathrm{End}_0(\mathbb{T}) \oplus \mathbb{C}$ and $\mathrm{End}_0(\mathbb{L})$ are both defined over a number field K. Let σ be an automorphism of $\bar{\mathbb{Q}}$. By 2.5 the local systems $\mathbb{L}^\sigma$ and $\mathbb{T}^\sigma$ are variations of Hodge structures. Since $\mathbb{L}^\sigma \otimes \mathbb{T}^\sigma$ is a variation of Hodge structures of weight one and width one, by 1.9, b), one of the two factors has a trivial Higgs field, and for the other factor the Higgs field is an isomorphism.

Identifying $\mathrm{End}(\mathbb{V}) \simeq \mathbb{V} \otimes \mathbb{V}$ the local system $\mathrm{End}_0(\mathbb{T}) \oplus \mathrm{End}_0(\mathbb{L})$ corresponds to the second wedge product

$$\bigwedge^2 (\mathbb{V}) \simeq S^2(\mathbb{L}) \oplus S^2(\mathbb{T})$$

(see 4.2). In particular, if the Higgs field of $\mathbb{T}^\sigma$ is an isomorphism, by 1.9, b), $\bigwedge^2(\mathbb{V})$ can not have a global section. On the other hand, the polarization of X induces a global section of $\bigwedge^2(\mathbb{V}) = R^2 f_* \mathbb{C}$.

So $\mathbb{L}^\sigma$ must have a maximal Higgs field and $\mathbb{T}^\sigma$ a trivial Higgs field. As in the proof of 2.3 this implies that $\mathrm{End}_0(\mathbb{T})$ remains invariant under the action of the Galois group of K over $\mathbb{Q}$, hence it is defined over $\mathbb{Q}$. Replacing $\mathbb{Y}$ by some étale covering, we again obtain the decomposition (3.4.1) and 5.1 implies:

Corollary 5.2. *Let $f : X \to Y$ be a smooth non-isotrivial family of abelian surfaces with a maximal Higgs field, and assume that $\mathbb{R}^1 f_* \mathbb{C}_X$ does not contain a unitary local sub system. Then there exists an étale covering $Y' \to Y$ such that Y' is a Shimura curve, parameterizing false elliptic curves, and*

$$f' : X' = X \times_Y Y' \to Y'$$

is the corresponding universal family.

For smooth non-isotrivial families of abelian surfaces over a projective curve Y, the condition on the unitary local sub system always holds true. In fact, assume one has a non trivial decomposition $R^1 f_* \mathbb{C}_X = \mathbb{V} \oplus \mathbb{U}_1$, necessarily with $\mathbb{V} = \mathbb{L}$ of rank two with a maximal Higgs field.

Since Y is compact, the general fibre of f can not be isogenous to the product of two elliptic curves, hence $\mathbb{U}_1$ can not be defined over $\mathbb{Q}$.

Then there exists some $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with $\mathbb{L}^\sigma$ unitary. As we will see in 6.3, the rank two local system $\mathbb{L}$ is given by a quaternion algebra A, defined over a real number field F, and such that A is ramified at all but one place of F at infinity. Since for some $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ $\mathbb{V}^\sigma$ is unitary, $F \neq \mathbb{Q}$. However, then there is no non trivial map from $\mathrm{Cor}_{F/\mathbb{Q}} A$ to $M(4, \mathbb{Q})$.