

Weak positivity and the stability of certain Hilbert points. III

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Whereas in the first and second part of this paper ([13] and [14]) we studied moduli of canonically polarized manifolds, we will regard here complex compact manifolds with an arbitrary polarization. Roughly speaking, we will show that, replacing the given polarization by another one, the Hilbert points obtained are stable for the action of the projective linear group, provided that the canonical sheaves of the manifolds considered are numerically effective.

By [7], we obtain a quasi-projective coarse moduli scheme M parametrizing pairs, consisting of a manifold F with a numerically effective canonical sheaf, together with some ample invertible sheaf \mathcal{H} with given polynomial, up to isomorphisms respecting the ample sheaf.

This however is not the moduli functor of polarized manifolds considered in [7] or [9]. There one allows isomorphisms respecting the maximal rank one \mathbb{Z} -submodule of $NS(F)$ containing the polarization. To get a moduli scheme for this functor one still has to divide M by the Pic^c -part of the equivalence relation, or replacing the polarization by some power, by the Pic^0 -part. Unfortunately we have not been able to construct this quotient as a quasi-projective scheme.

Therefore we can only state the existence of coarse quasi-projective moduli, in the sense of [7], if in addition the irregularity of the manifolds considered is zero. Especially we reprove the quasi-projectivity of the moduli scheme of polarized $K3$ -surfaces, established by I.I. Pjatetskij-Šapiro and I.R. Šafarevich in [8].

Of course the results of this paper apply again the quasi-projectivity of moduli schemes for canonically polarized manifolds. However the proof is more complicated than the one given in [14] and the ample sheaf constructed even worse than the one obtained there.

As in [13] and [14] the same results hold true for normal Gorenstein varieties F with at most rational singularities, provided that the moduli functor is bounded and separated. By [3], this holds true for surfaces, for threefolds with p_g larger than zero or for manifolds in any dimension.

The reader finds the exact assumptions made for the moduli functors considered and the main results on stability and the existence of coarse quasi-projective moduli schemes in §1.

In § 2 we recall some notations introduced in [1] and [14] and we formulate the positivity Theorem 2.6 needed in the sequel. This positivity theorem together with estimates on the singularities of certain divisors, taken from [1], allows to use again the mysterious minimizing principle (in 2.11) to prove the weak positivity of certain polarizations (see 2.7).

The proof of 2.6 is given in § 3. It is based again on our version of O. Gabber's extension theorem (§ 1 of [14]) and on the nilpotent orbit theorem of W. Schmid.

Finally, § 4 contains the prove of the results stated in § 1. We took the opportunity to reformulate and to simplify some of the basic constructions from [13], § 5, in the more general set up needed here. It will be obvious for the reader, that we took a lot of ideas from D. Mumford's geometric invariant theory [7].

I would like to express my thanks to H el ene Esnault for her helpful comments. The main new ingredient, the proof that 2.7 follows from 2.6, was found regarding similar constructions in our common paper [1].

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Conventions. We keep the conventions from part two [14]. Especially the reader should keep in mind, that all schemes are supposed to be separated and of finite type over \mathbf{C} , and that all points are \mathbf{C} -valued points.

If \mathcal{L} is an invertible sheaf on Z and Γ a divisor, we write $\mathcal{L}(\Gamma)$ instead of $\mathcal{L} \otimes \mathcal{O}_Z(\Gamma)$. For example $\mathcal{L}^a(\Gamma) = \mathcal{L}^a \otimes \mathcal{O}_Z(\Gamma)$.

We refer to [13], n.m, by I, n.m, and to [14] by II, n.m.

§ 1. Discussion of the main results

1.1. Let h be a polynomial of degree n and \mathcal{M}'_h the moduli functor of pairs of complex compact normal varieties with at most rational Gorenstein singularities, and ample invertible sheaves with Hilbert polynomial $h(v)$. Hence $\mathcal{M}'_h(\mathbf{C}) = \mathcal{M}'_h(\text{Spec}(\mathbf{C}))$ is the set of isomorphy classes of pairs (F, \mathcal{H}) where:

- a) F is a projective, non uniruled, normal Gorenstein variety with at most rational singularities.
- b) \mathcal{H} is an ample invertible sheaf on F , and $\chi(F, \mathcal{H}^v) = h(v)$.
- c) (F, \mathcal{H}) and (F', \mathcal{H}') are isomorphic in $\mathcal{M}'_h(\mathbf{C})$, if there are isomorphisms $\tau: F \rightarrow F'$ and $\mathcal{H} \rightarrow \tau^* \mathcal{H}'$.

Correspondingly, for any scheme S , defined over \mathbf{C} , we write $\mathcal{M}'_h(S)$ for the set of isomorphy classes of pairs $(f: X \rightarrow S, \mathcal{H})$, where f is flat, projective and surjective and \mathcal{H} an invertible sheaf on X such that $(F, \mathcal{H}|_F) \in \mathcal{M}'_h(\mathbf{C})$ for all fibres F of f . Isomorphisms are S -isomorphisms $\tau: X \rightarrow X'$ such that \mathcal{H} and $\tau^* \mathcal{H}'$ differ by the pullback of some invertible sheaf on S .

1.2. Let \mathcal{M}''_h be a subfunctor of \mathcal{M}'_h which is a bounded and separated moduli functor and given by a locally closed condition. For example, by [3], we may take

$$\text{for } n=2: \mathcal{M}''_h = \mathcal{M}'_h$$

$$\text{for } n=3: \mathcal{M}''_h(S) = \{(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}'_h(S); h^0(F, \omega_F) \neq 0 \\ \text{for all fibres } F \text{ of } f\}$$

$$\text{for } n \geq 3: \mathcal{M}''_h(S) = \{(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}'_h(S); f \text{ smooth}\}.$$

Since \mathcal{M}_h'' is supposed to be bounded, we can find some $\nu > 0$ such that for all $(F, \mathcal{H}) \in \mathcal{M}_h''(\mathbb{C})$ the sheaf \mathcal{H}^ν is very ample and $H^i(F, \mathcal{H}^\nu) = 0$ for $i > 0$. Replacing ν by $\nu(n+2)$ we can as well assume that $\mathcal{H}^\nu \otimes \mathcal{L} \otimes \omega_F$ is very ample for all numerically effective invertible sheaves \mathcal{L} on F . In fact, we have

Lemma 1.3. *Let F be a normal compact variety with rational Gorenstein singularities, \mathcal{H} a very ample invertible sheaf on F and \mathcal{L} a numerically effective invertible sheaf on F . Then*

- a) $\mathcal{H}^{\dim F + 1} \otimes \mathcal{L} \otimes \omega_F$ is generated by its global sections.
- b) $\mathcal{H}^{\dim F + 2} \otimes \mathcal{L} \otimes \omega_F$ is very ample.

Proof. By the Grauert-Riemenschneider vanishing theorem one has a surjection

$$H^0(F, \mathcal{H}^{\dim F + 1} \otimes \mathcal{L} \otimes \omega_F) \rightarrow H^0(H, \mathcal{H}^{\dim F} \otimes \mathcal{L} \otimes \omega_H)$$

where H is a smooth zero divisor of a section of \mathcal{H} . By induction on $\dim(F)$ one obtains *a. b* follows directly from *a.*

1.4. Let c be the highest coefficient of h and ν as above. Let us write $e = (n!) \cdot c \cdot \nu^n + 1$. In other terms, $e = c_1(\mathcal{H}^\nu)^n + 1$ for $(F, \mathcal{H}) \in \mathcal{M}_h''$. If ω_F happens to be numerically effective then by our choice of ν the sheaf $\mathcal{H}^\nu \otimes \omega_F^e$ is very ample and, by the Grauert-Riemenschneider vanishing theorem $H^i(F, \mathcal{H}^\nu \otimes \omega_F^e) = 0$ for $i > 0$. Since we do not want to consider the problem whether the condition that ω_F is numerically effective is open, we take instead \mathcal{M}_h to be the moduli functor given by

$$\mathcal{M}_h(\mathbb{C}) = \{(F, \mathcal{H}) \in \mathcal{M}_h''(\mathbb{C}); \mathcal{H}^\nu \otimes \omega_F^e\}$$

is very ample and $H^i(F, \mathcal{H}^\nu \otimes \omega_F^e) = 0$ for $i > 0$.

In general the polynomial $h'(\eta) = \chi(F, (\mathcal{H}^\nu \otimes \omega_F^e)^\eta)$ will not be the same for all $(F, \mathcal{H}) \in \mathcal{M}_h(\mathbb{C})$. However, since our moduli functor is bounded, there will be only finitely many h' occurring, and h' is constant for the fibres of $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ as soon as S is connected. Therefore, if $\mathcal{M}_{h,h'}$ is given by

$$\mathcal{M}_{h,h'}(\mathbb{C}) = \{(F, \mathcal{H}) \in \mathcal{M}_h(\mathbb{C}); h'(\eta) = \chi(F, \mathcal{H}^\nu \otimes \omega_F^e)^\eta\}$$

then $\mathcal{M}_h(S) = \bigcup_{h'} \mathcal{M}_{h,h'}(S)$ for connected S . Therefore, instead of constructing a coarse quasi-projective moduli scheme for \mathcal{M}_h it is enough to construct one for each $\mathcal{M}_{h,h'}$.

By abuse of notations we will assume in the sequel that $\mathcal{M}_h = \mathcal{M}_{h,h'}$ for some h' , i.e.: that $\chi(F, (\mathcal{H}^\nu \otimes \omega_F^e)^\eta)$ is independent of $(F, \mathcal{H}) \in \mathcal{M}_h(\mathbb{C})$. The same argument allows to assume that the polynomial

$$h''(\eta) = \chi(F, \mathcal{H}^{\nu \cdot \eta + 1} \otimes \omega_F^e)^\eta$$

is constant for $(F, \mathcal{H}) \in \mathcal{M}_h(\mathbb{C})$.

Claim 1.5. Let c be the highest coefficient of h and v, \mathcal{M}_h as above. Write $e = (n!) \cdot c \cdot v^n + 1$. Then one has for all $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$:

- a) $\mathcal{H}^v \otimes \omega_{X/S}^e$ is relatively very ample for f and $f_* \mathcal{H}^v \otimes \omega_{X/S}^e$ is locally free of rank r and commutes with arbitrary base change.
- b) $f_* \mathcal{H}^v$ is locally free and commutes with base change.
- c) For all fibres F of f one has $e = c_1(\mathcal{H}^v|_F)^n + 1$.
- d) There exists some separated scheme H of finite type over \mathbb{C} and $(g: \mathcal{X} \rightarrow H, \mathcal{H}) \in \mathcal{M}_h(H)$ together with an isomorphism

$$\varphi: \mathbb{P}(g_*(\omega_{\mathcal{X}/H}^e \otimes \mathcal{H}^v)) \rightarrow \mathbb{P}^{r-1} \times H$$

such that for all

$$(f: X \rightarrow Y, \mathcal{L}) \in \mathcal{M}_h(Y) \quad \text{and} \quad \rho: \mathbb{P}(f_*(\omega_{X/Y}^e \otimes \mathcal{L}^v)) \rightarrow \mathbb{P}^{r-1} \times Y$$

there exists a unique morphism $Y \rightarrow H$ such that (f, \mathcal{L}, ρ) is obtained from $(g, \mathcal{H}, \varphi)$ by pullback.

- e) The action of $G = \text{Sl}(r, \mathbb{C})$ on H obtained by “change of coordinates in \mathbb{P}^{r-1} ” is proper.

Proof. By assumption, \mathcal{H}^v is very ample for $(F, \mathcal{H}) \in \mathcal{M}_h''(\mathbb{C})$. Let $\mathcal{H} \text{ ilb}_h$ be the Hilbert scheme of subschemes F of \mathbb{P}^{r-1} with Hilbert polynomial $h(v \cdot t)$ carrying an invertible sheaf \mathcal{L} with $\chi(\mathcal{L} \otimes \mathcal{O}_F(t)) = h(t \cdot v + 1)$. Obviously the condition that $\mathcal{L}^v = \mathcal{O}_F(1)$ defines a closed subscheme. The points corresponding to $\mathcal{M}_h''(\mathbb{C})$ are a locally closed subset, as in [7] V, §2 or [9], 2.9. In fact, if $F \in \mathcal{M}_h''(\mathbb{C})$ is allowed to have rational Gorenstein singularities one has to use in addition, that a small deformation of such singularities is again a rational Gorenstein singularity. The additional condition posed for $\mathcal{M}_h(\mathbb{C})$ is open and hence we find some Hilbert scheme H' and a universal family $(g': \mathcal{X}' \rightarrow H', \mathcal{H}')$ parametrizing the $(F, \mathcal{H}) \in \mathcal{M}_h(\mathbb{C})$ with $F \rightarrow \mathbb{P}(H^0(F, \mathcal{H}^v)) \approx \mathbb{P}^{r-1}$.

Let us consider now the Hilbert scheme $\mathcal{H} \text{ ilb}_h$ parametrizing subschemes F of \mathbb{P}^{r-1} with Hilbert polynomial $h'(t)$ together with an invertible sheaf \mathcal{L} such that $\chi(\mathcal{L} \otimes \mathcal{O}_F(t)) = h'(t)$. Again the points corresponding to \mathcal{M}_h will form a locally closed subscheme H of $\mathcal{H} \text{ ilb}_h$. For example, one can consider the $\mathbb{P}\text{Gl}(r, \mathbb{C})$ bundle on H' given by

$$\mathbb{P}_0 = \mathbb{P}\text{Gl}(g'_*(\mathcal{H}'^v \otimes \omega_{\mathcal{X}'/H'}^e))$$

and the induced map $\mathbb{P}_0 \rightarrow \mathcal{H} \text{ ilb}_h$ and take H to be the image.

Since \mathcal{M}_h is separated the action of G on H will be proper.

1.6. Some G -linearized sheaves

Let $\sigma: G \times H \rightarrow H$ be the group action considered in 1.5, iii). Recall that in [7], Definition 1.6, a) G -linearization of a sheaf λ is defined to be an isomorphism $\sigma^* \lambda \approx \text{pr}_2^* \lambda$. Moreover, if λ is invertible, there is at most one G -linearization ([7], Prop. 1.4).

a) The G -action lifts to $\sigma': G \times \mathcal{X} \rightarrow \mathcal{X}$ and obviously $\omega_{\mathcal{X}/H}$ is G -linearized on \mathcal{X} . If \mathcal{H} is the polarization of $g: \mathcal{X} \rightarrow H$, then $\sigma'^* \mathcal{H}$ and $\text{pr}_2^* \mathcal{H}$ can differ by $(\text{id} \times g)^* \mathcal{N}$ for an invertible sheaf \mathcal{N} on $G \times H$. Hence

$$\sigma^* g_* \mathcal{H} = (\text{id} \times g)_* \sigma'^* \mathcal{H} = ((\text{id} \times g)_* \text{pr}_2^* \mathcal{H}) \otimes \mathcal{N} = (\text{pr}_2^* g_* \mathcal{H}) \otimes \mathcal{N}$$

and, for $r' = \text{rk}(g_* \mathcal{H}^{\nu})$, we have

$$\sigma^* \det(g_* \mathcal{H}^{\nu})^{-1} = \text{pr}_2^* \det(g_* \mathcal{H}^{\nu})^{-1} \otimes \mathcal{N}^{-r' \cdot \nu}.$$

Then $\mathcal{E} = (\bigotimes_{\nu}^{r'} g_*(\mathcal{H}^{\nu} \otimes \omega_{\mathcal{X}/H}^e)) \otimes \det(g_* \mathcal{H}^{\nu})^{-1}$ and $\lambda = \det(\mathcal{E})$ are both G -linearized.

b) for $\mu \geq 0$ the multiplication map

$$m: S^{\mu}(g_*(\mathcal{H}^{\nu} \otimes \omega_{\mathcal{X}/H}^e)) \rightarrow g_*(\mathcal{H}^{\mu \cdot \nu} \otimes \omega_{\mathcal{X}/H}^{e \cdot \mu})$$

will be surjective and $\ker(m)_x$ will generate the ideal of $g^{-1}(x)$ in \mathbb{P}^{r-1} . If

$$r(\mu) = \text{rk}(g_*(\mathcal{H}^{\mu \cdot \nu} \otimes \omega_{\mathcal{X}/H}^{e \cdot \mu}))$$

then

$$\mathcal{L}_0 = \det(g_*(\mathcal{H}^{\nu} \otimes \omega_{\mathcal{X}/H}^e))^{-\mu \cdot r(\mu)} \otimes \det(g_*(\mathcal{H}^{\mu \cdot \nu} \otimes \omega_{\mathcal{X}/H}^{e \cdot \mu}))^r$$

will again be G -linearized and ample. In fact, m induces a morphism from H to some Grassmannian, quasi-finite over its image, and \mathcal{L}_0 is nothing but the pullback of the tautological bundle.

c) Let us consider for $\eta > 0$ the sheaf

$$\mathcal{L}_{\eta} = \mathcal{L}_0 \otimes \lambda^{\eta} = \det(g_*(\mathcal{H}^{\mu \cdot \nu} \otimes \omega_{\mathcal{X}/H}^{e \cdot \mu}))^r \otimes \det(g_*(\mathcal{H}^{\nu} \otimes \omega_{\mathcal{X}/H}^e))^{a(\eta)} \otimes \det(g_* \mathcal{H}^{\nu})^{b(\eta)}$$

where $a(\eta) = r' \cdot r'^{-1} \cdot \eta - \mu \cdot r(\mu)$ and $b(\eta) = -r' \cdot \eta$. This sheaf is of course again G -linearized.

Theorem 1.7. *Under the assumptions made in 1.5 and using the notations introduced above one has for $\eta \geq 0$*

$$H = H(\mathcal{L}_{\eta})^s.$$

Recall that, as in I, $H(\mathcal{L}_{\eta})^s$ denotes the stable points of H under the G -action and with respect to \mathcal{L}_{η} , which have finite stabilizers. As in [7], Thm. 1.10, iii) and V, Prop. 5.4, or [9], §2, we obtain

Corollary 1.8. *The geometric quotient H/G exists and, for $p \geq 0$, $\eta \geq 0$, G -invariant sections of \mathcal{L}_{η}^p define an embedding of H/G into some projective space.*

Corollary 1.9. *Let \mathcal{M}'_h be as in 1.1 and \mathcal{M}_h the submoduli functor considered in 1.4, then there exists a coarse quasi-projective moduli scheme M_h for \mathcal{M}_h .*

Remarks 1.10. a) If $H_0 \subseteq H$ is a closed G -invariant subscheme, then the quotient H_0/G exists as well (I, 5.5, a), or [7], Prop. 1.18). Therefore, 1.7 implies the existence of coarse quasi-projective moduli schemes whenever for each $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ we can choose some natural polarization numerically

equivalent to the given one. This is the case for abelian varieties, as explained in [7], VI, §2. For canonically polarized varieties, we can, of course, assume that we take $\mathcal{H} = \omega_{X/S}$. Therefore we can state the known (see II).

Corollary. i) *There exist coarse quasi-projective moduli schemes for polarized abelian varieties.*

ii) *There exist coarse quasi-projective moduli schemes for canonically polarized manifolds.*

b) The moduli functor \mathcal{M}'_h considered in 1.1 is not the moduli functor \mathcal{P}'_h of polarized complex compact normal varieties with at most rational double points and with Hilbert polynomial $h(v)$. $\mathcal{P}'_h(\mathbb{C})$ is the set of isomorphy classes of pairs (F, \mathcal{H}) . But, if

$$NS(F) = \text{Div}(F)/(\text{numerical equivalence})$$

and \mathcal{H}_{in} a generator of the maximal rank one \mathbb{Z} -submodule $[\mathcal{H}]$ of $NS(F)$ containing \mathcal{H} , then $\chi(F, \mathcal{H}_{in}^v) = h(v)$. Moreover, one allows all isomorphisms $\tau: F \rightarrow F'$, with $[\mathcal{H}'] = [\tau^* \mathcal{H}]$.

For singular varieties this is however not the “right” definition. As J. Kollár pointed out, the sheaf \mathcal{H}_{in} can jump in families. Therefore one either takes \mathcal{P}'_h to be the subfunctor of smooth families or one considers the slightly different functor considered in [3]. Let us restrict ourselves to the smooth case:

Applying the same constructions as in 1.1–1.4 one ends up with the functor \mathcal{P}_h where

$$\mathcal{P}_h(\mathbb{C}) = \{(F, \mathcal{H}) \in \mathcal{P}'_h(\mathbb{C}); \mathcal{H}_{in}^v \otimes \omega_F^e \text{ is very ample and } H^i(F, \mathcal{H}_{in}^v \otimes \omega_F^e) = 0 \text{ for } i > 0\}.$$

By [7] or [9], 2.10, each element of $\mathcal{P}_h(S)$ is in the étale topology locally defined by pairs $(f: X \rightarrow S, \mathcal{H})$ such that $\mathcal{H}|_F = (\mathcal{H}|_F)_{in}^v$. As in 1.5 one can consider the Hilbert scheme H and by 1.7 the quotient scheme H/G . Let P_h be the analytic coarse moduli space for \mathcal{P}_h (see [7] p. 172 or [3]). Then the fibre of the natural map $H/G \rightarrow P_h$ over the point corresponding to F is nothing but $\text{Pic}^c(F)$. Unfortunately we were not able to divide out this part of the equivalence relation in the category of quasi-projective schemes.

c) If however the irregularity $h^0(F, \Omega_F^1)$ (or $h^1(F, \mathcal{O}_F)$ in the singular case) is zero for all $(F, \mathcal{H}) \in \mathcal{P}_h(\mathbb{C})$ then for v sufficiently big the map

$$\phi(\mathbb{C}): \mathcal{P}_h(\mathbb{C}) \rightarrow \mathcal{M}_{h(v-t)} \quad \text{with } \phi(\mathbb{C})(F, \mathcal{H}) = (F, \mathcal{H}_{in}^v)$$

will be injective. In fact, since \mathcal{P}_h is bounded, we can find some v such that $v \cdot \text{Pic}^c(F) = 0$ for all F . Again ϕ extends to a natural transformation, at least if we sheafify \mathcal{P}_h and $\mathcal{M}_{h(v-t)}$ with respect to the étale topology (see [7] or [9] p. 20). Then however, 1.7 implies the existence of a quasi-projective P_h :

Corollary. *Let \mathcal{P}_h'' be the moduli functor of polarized compact complex manifolds with Hilbert polynomial h and \mathcal{P}_h the submoduli functor considered in b). If for all $F \in \mathcal{P}_h(\mathbb{C})$ one has $h^0(F, \Omega_F^1) = 0$, then there exists a coarse quasi-projective moduli scheme P_h for \mathcal{P}_h .*

d) In general one is not interested in our moduli functor $\mathcal{M}_h(\mathbf{C})$ defined in 1.4, but rather in $\mathcal{M}_h^{\text{nef}}$ given by

$$\mathcal{M}_h^{\text{nef}}(\mathbf{C}) = \{(F, \mathcal{H}) \in \mathcal{M}_h'', \omega_F \text{ numerically effective}\}.$$

By our choice of ν and 1.3 we have $\mathcal{M}_h^{\text{nef}}(\mathbf{C}) \subseteq \mathcal{M}_h(\mathbf{C})$. Therefore 1.9 shows the existence of some quasi-projective scheme M_h , some of whose points parametrize $\mathcal{M}_h^{\text{nef}}(\mathbf{C})$. We do not know, however, whether $\mathcal{M}_h^{\text{nef}}(\mathbf{C})$ defines a subscheme of M_h . For $n=2$, this is obviously the case and we can state

Corollary. *There exist coarse quasi-projective moduli schemes for non singular complex compact, non uniruled and minimal*

- i) *surfaces of general type.*
- ii) *polarized surfaces with irregularity zero.*
- iii) *polarized abelian surfaces.*
- iv) *pairs of surfaces, consisting of a surface S and an ample invertible sheaf \mathcal{H} .*

Corollary 1.11. *Let \mathcal{M}_h be one of the moduli functors considered in 1.9 or 1.10 and M_h the corresponding coarse moduli scheme. Then there exists some $\mu, \eta, p \gg 0$ and an invertible ample sheaf \mathcal{L} on M_h with:*

For $(f: X \rightarrow S, \mathcal{H}) \in \mathcal{M}_h(S)$ let $\varphi: S \rightarrow M_h$ be the induced morphism. Then

$$\varphi^* \mathcal{L} = (\det(f_* (\mathcal{H}^{\mu \cdot \nu} \otimes \omega_{X/S}^e))^r \otimes \det(f_* (\mathcal{H}^\nu \otimes \omega_{X/S}^e))^{a(\eta)} \otimes \det(f_* \mathcal{H}^\nu)^{b(\eta)})^p$$

for

$$r = \text{rk}(f_* (\mathcal{H}^\nu \otimes \omega_{X/S}^e)), r' = \text{rk}(f_* \mathcal{H}^\nu), a(\eta) = r' \cdot r'^{-1} \cdot \eta - \mu \cdot r(\mu) \text{ and } b(\eta) = -r' \cdot \eta.$$

Needless to say that this ample sheaf is not the best possible one and that it would be nice to bound η and μ . In the case of canonically polarized varieties, the sheaf obtained in II is already nicer, not talking about the one obtained by D. Mumford or D. Gieseker in the curve or surface case (see II, 6.2 and 6.4).

§2. Weak positivity, continued

Let us recall the following definition from [1]:

Definition 2.1. Let Z be a normal variety with rational Gorenstein singularities, \mathcal{M} an invertible sheaf and Γ an effective Cartier divisor on Z . As usual, $\left[\frac{\Gamma}{N} \right]$ denotes the integral part of the \mathbf{Q} -divisor $\frac{1}{N} \cdot \Gamma$.

a) Let $\tau: Z' \rightarrow Z$ be a blowing up, such that $\Gamma' = \tau^* \Gamma$ is a normal crossing divisor. Then

$$e(\Gamma) = \text{Min} \left\{ N \in \mathbf{N} - \{0\}; \tau_* \omega_{Z'/Z} \left(- \left[\frac{\Gamma'}{N} \right] \right) = \mathcal{O}_Z \right\}$$

b)

$$e(\mathcal{M}) = \text{Max} \{ e(\Gamma); \Gamma \text{ zero divisor of } s \in H^0(Z, \mathcal{M}) \}.$$

In the situation of a)

$$\tau_* \omega_{Z'/Z} \left(- \left[\frac{\Gamma'}{N} \right] \right) = \tau_* \omega_{Z'/Z} = \mathcal{O}_Z,$$

for $N \gg 0$, since Z has rational Gorenstein singularities. Moreover, $e(\Gamma)$ does not depend on the blowing up $\tau: Z' \rightarrow Z$ chosen (see [12], 2.3).

Properties 2.2. a) If \mathcal{M} is very ample and Z compact, then

$$e(\mathcal{M}^v) \leq v \cdot c_1(\mathcal{M})^{\dim Z} + 1, \quad \text{for } v > 0.$$

Moreover, if $X = Z \times \dots \times Z$ (r -times) and $\mathcal{H} = \bigotimes_{i=1}^r \text{pr}_i^* \mathcal{M}$, then

$$e(\mathcal{H}^v) \leq v \cdot c_1(\mathcal{M})^{\dim Z} + 1$$

as well.

b) Let $f: X \rightarrow Y$ be a morphism of schemes, all of whose fibres are normal Gorenstein varieties with at most rational singularities and assume that Y is non singular. Let $Z = f^{-1}(y)$ be given and let Γ be a Cartier divisor on X , not containing Z . Then there is some open neighbourhood U of y with $e(\Gamma|_{f^{-1}(U)}) \leq e(\Gamma|_Z)$ and $e(\Gamma|_{f^{-1}(u)}) \leq e(\Gamma|_Z)$ for all $u \in U$.

Proof. The proof of a) is given in [1], 2.3. We assumed there that Z is non singular, but in fact we only used that $\tau_* \omega_{Z'} = \omega_Z$ and (see [12], 2.3) that $R^i \tau_* \omega_{Z'} \left(- \left[\frac{\Gamma'}{N} \right] \right) = 0$ for $N, i > 0$. Both remain true under the weaker assumptions made in 2.1.

We prove b) by induction on $\dim(Y)$. Let D be any reduced divisor on Y which contains y . Let $\pi: X' \rightarrow X$ be a desingularization such that $\Gamma' = \pi^* \Gamma$ has normal crossings. Again, by [12], 2.3,

$$\mathcal{E}(e) = \text{coker} \left(\pi^* \omega_{X'} \left(- \left[\frac{\Gamma'}{e} \right] \right) \rightarrow \omega_{X'} \right)$$

is independent of X' . Let $H = f^{-1}(D)$. If D is non singular $H \rightarrow D$ satisfies again the assumptions made in c. By induction we find some U such that $e(\Gamma|_{H \cap f^{-1}(U)}) \leq e = e(\Gamma|_Z)$. By [1], 2.2 (again this holds under the weaker assumptions made here) we find that

$$\text{Supp}(\mathcal{E}(e)) \cap H \cap f^{-1}(U) = \emptyset.$$

Since H contains Z we can make U a little bit smaller and we get $e(\Gamma|_{f^{-1}(U)}) \leq e$.

The inequality $e(\Gamma|_{f^{-1}(u)}) \leq e$ holds obviously for all u in some open dense subscheme U' of U . Let D a component of a divisor containing $Y - U'$. It can happen that D is singular. Let $\delta: D'' \rightarrow D$ be a desingularization and $H'' = D'' \times_D H$. By induction, again, we can find some neighbourhood V of $\delta^{-1}(y)$ such that $e(\text{pr}_2^* \Gamma|_{\text{pr}_1^{-1}(u)}) \leq e$ for all $u \in V$. Repeating this for a finite number of divisors we find a smaller neighbourhood of y such that b holds.

2.3. Let $Y_0 \xrightarrow{j} Y \xleftarrow{\delta} Y'$ be morphisms of quasi-projective reduced schemes, let \mathcal{F}_0 be a locally free sheaf on Y_0 of (constant) rank r and \mathcal{F}' a coherent torsion free sheaf on Y' of the same rank r . Assume that j is a compactification and δ a desingularization and that for all $\nu > 0$ one has inclusions

$$S^\nu(\mathcal{F}_0) \rightarrow j^* \delta_* S^\nu(\mathcal{F}').$$

Recall that, for $U \subset Y_0$ open, we called in II, 2.2, \mathcal{F}_0 *weakly positive over U with respect to (Y', \mathcal{F}')* if for all (or one) ample invertible sheaves \mathcal{H} on Y and for all $a > 0$ one finds some $b > 0$ such that $S^{a \cdot b}(\mathcal{F}_0) \otimes j^* \mathcal{H}^b$ is globally generated over U by

$$H^0(Y, \delta_* S^{a \cdot b}(\mathcal{F}') \otimes \mathcal{H}^b) \cap H^0(Y_0, S^{a \cdot b}(\mathcal{F}_0) \otimes j^* \mathcal{H}^b).$$

Let T be a finite dimensional representation. As we did in I, 2.1 and 1.8, ii), we consider the induced tensor sheaves $T(\mathcal{F}_0)$ and $T(\mathcal{F}')$. The second one is the reflexive hull of $T(\mathcal{F}'|_V)$, if V is the largest open subscheme of Y' where \mathcal{F}' is locally free.

Lemma 2.4. i) *The following two conditions are equivalent:*

a) \mathcal{F}_0 is weakly positive over U with respect to (Y', \mathcal{F}') .

b) For some $\eta > 0$, $\bigotimes^{\eta}(\mathcal{F}_0)$ is weakly positive over U with respect to $(Y', \bigotimes^{\eta} \mathcal{F}')$.

Moreover, if T is a positive representation (see I, 2.2), then a implies that:

c) $T(\mathcal{F}_0)$ is weakly positive over U with respect to $(Y', T(\mathcal{F}'))$.

ii) If \mathcal{F}_0 and \mathcal{G}_0 are weakly positive over U with respect to (Y', \mathcal{F}') and (Y', \mathcal{G}') respectively, then $\mathcal{F}_0 \oplus \mathcal{G}_0$ is weakly positive over U with respect to $(Y', \mathcal{F}' \oplus \mathcal{G}')$.

Remark. In [11], I and II, I used ii) several times assuming the proof to be obvious. Y. Kawamata pointed out, that the arguments needed are more complicated than I thought.

Proof. To see that b) implies a), one just has to use the natural maps

$$\bigotimes^{\eta}(\mathcal{F}_0) \rightarrow S^{\eta}(\mathcal{F}_0) \quad \text{and} \quad \bigotimes^{\eta}(\mathcal{F}') \rightarrow S^{\eta}(\mathcal{F}').$$

Therefore we just have to show that a) implies c). Using II, 2.4, b), this follows exactly by the same argument used in I, 3.6.

Before proving ii) we remark, that one can replace the words

“some $b > 0$ such that”

in 2.3 by

“some $b_0 > 0$ such that for all $b > b_0$ the sheaf”.

In fact, this is shown in [11], 3.2 i).

Let \mathcal{H} be any ample invertible sheaf on Y , $\mathcal{H}_0 = j^* \mathcal{H}$ and $\mathcal{H}' = \delta^* \mathcal{H}$. Using II, 2.4, we may replace \mathcal{F}_0 , \mathcal{G}_0 , \mathcal{F}' and \mathcal{G}' by $\mathcal{F}_0 \otimes \mathcal{H}_0$, $\mathcal{G}_0 \otimes \mathcal{H}_0$, $\mathcal{F}' \otimes \mathcal{H}'$ and $\mathcal{G}' \otimes \mathcal{H}'$. Hence we can assume that for some $b_0 > 0$ and all $b > b_0$ both sheaves,

$S^b(\mathcal{F}_0)$ and $S^b(\mathcal{G}_0)$ are globally generated over U by sections coming from global sections of $S^b(\mathcal{F}')$ and $S^b(\mathcal{G}')$ respectively. For $a, c > 0$ one has

$$S^{a \cdot c}(\mathcal{F}_0 \oplus \mathcal{G}_0) \otimes \mathcal{H}_0^c = \bigoplus_d S^d(\mathcal{F}_0) \otimes S^{a \cdot c - d}(\mathcal{G}_0) \otimes \mathcal{H}_0^c.$$

If c is large enough, we may assume

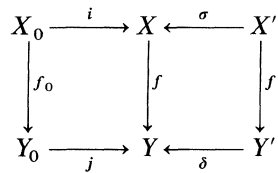
$$S^v(\mathcal{F}_0) \otimes \mathcal{H}_0^c \quad \text{and} \quad S^v(\mathcal{G}_0) \otimes \mathcal{H}_0^c$$

to be globally generated for $v = 0, 1, \dots, b_0$ by sections living on Y' . If $a \cdot c > 2 \cdot b_0$ all direct summands of

$$S^{a \cdot c}(\mathcal{F}_0 \oplus \mathcal{G}_0) \otimes \mathcal{H}_0^c$$

will be generated over U by sections, coming from global sections of the corresponding sheaves on Y' .

Assumptions 2.5. Let



be a commutative diagram of morphisms of reduced quasi-projective schemes such that

- a) i and j are compactifications and σ and δ desingularizations.
- b) f and f' are surjective and $X_0 = f^{-1}(Y_0)$.
- c) f_0 is flat and all fibres of f_0 are reduced normal varieties of dimension n with at most rational Gorenstein singularities.
- d) *Notations.* If (...) is a sheaf or a divisor on X then (...) ₀ will always denote the pullback under i and (...) _' the pullback under σ .

The main technical tool which will be proved in §3 using “our version of O. Gabber’s extension theorem” (II, 1.7) is:

Theorem 2.6. *Using notations and assumptions from 2.5, let \mathcal{L} be an invertible sheaf and Γ an effective Cartier divisor on X . Let $e \in \mathbb{N}$ be a number such that for all fibres F of f_0 one has $F \cap \Gamma_0 \neq F$ and $e(\Gamma_0|_F) \leq e$. Assume moreover, that $(\mathcal{L}_0^e(-\Gamma_0))^N$ is globally generated over X_0 by $H^0(X_0, (\mathcal{L}_0^e(-\Gamma_0))^N) \cap H^0(X', (\mathcal{L}'^e(-\Gamma'))^N)$, for some $N \geq 0$, and that $f_{0*}(\mathcal{L}_0 \otimes \omega_{X_0/Y_0})$ is locally free and compatible with arbitrary base change (see however II, 4.3). Then $f_{0*}(\mathcal{L}_0 \otimes \omega_{X_0/Y_0})$ is weakly positive over Y_0 with respect to $(Y', f'_*(\mathcal{L}' \otimes \omega_{X'/Y'}))$.*

Corollary 2.7. *Using the notations and assumptions stated in 2.5, let \mathcal{M} be an invertible sheaf on X . Assume that:*

- a) $\mathcal{M}_0|_F$ is very ample for each fibre F of f_0 .
- b) $r' = h^0(F, \mathcal{M}|_F)$ and $e = c_1(\mathcal{M}|_F)^n + 1$ are the same for all fibres F of f_0 .
- c) $f_0^* f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e) \rightarrow (\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)$ is surjective.
- d) $f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)$ and $f_{0*} \mathcal{M}_0$ are locally free and compatible with arbitrary base change.

Then

$$\left(\bigotimes^{r'} f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)\right) \otimes \det(f_{0*} \mathcal{M}_0)^{-1}$$

is weakly positive over Y_0 with respect to

$$(Y', \left(\bigotimes^{r'} f_{*}'(\mathcal{M}' \otimes \omega_{X'/Y'}^e)\right) \otimes \det(f_{*}' \mathcal{M}')^{-1}).$$

Example 2.8. Let us return for a moment to 1.5, and assume that

$$(f_0: X_0 \rightarrow Y_0, \mathcal{M}_0) \in \mathcal{M}_h(Y_0).$$

Assume moreover that \mathcal{M}_0 has the properties \mathcal{H}^ν should have in 1.5. Then the assumptions we made in 2.7 hold true.

Proof of 2.7, assuming 2.6. For $\eta > 0$ let us consider the η -fold products

$$X_0^\eta = X_0 \times_{Y_0} \dots \times_{Y_0} X_0, X^\eta = X \times_Y \dots \times_Y X \text{ and } X'^\eta = X' \times_{Y'} \dots \times_{Y'} X'.$$

Let $\gamma: X^{(\eta)} \rightarrow X'^\eta$ be a desingularization.

Claim 2.9. The induced diagram

$$\begin{array}{ccccc} X_0^\eta & \xrightarrow{i^\eta} & X^\eta & \xleftarrow{\sigma^{(\eta)}} & X^{(\eta)} \\ \downarrow f_0^\eta & & \downarrow f^\eta & & \downarrow f^{(\eta)} \\ Y_0 & \xrightarrow{j} & Y & \xleftarrow{\delta} & Y' \end{array}$$

satisfies again the assumptions made in 2.5.

Proof. a and b are trivial and since $\omega_{X_0^\eta/Y_0} = \bigotimes_{v=1}^\eta \text{pr}_v^* \omega_{X_0/Y_0}$ the morphism f_0^η is

again Gorenstein. To see that the fibres of f_0^η have rational singularities one just has to apply flat base change (as we did in [10], 3.6):

If $\tilde{}$ denotes a desingularization and if F and F' have rational singularities consider

$$\begin{array}{ccc} \tilde{F} \times \tilde{F}' & \xrightarrow{\varepsilon'} & F \times \tilde{F}' \\ \downarrow p_1 & & \downarrow \pi_1 \\ \tilde{F} & \xrightarrow{\varepsilon} & F \end{array} \quad \begin{array}{ccc} F \times \tilde{F}' & \xrightarrow{\gamma'} & F \times F' \\ \downarrow \pi_2 & & \downarrow p_2 \\ \tilde{F}' & \xrightarrow{\gamma} & F' \end{array}$$

Then

$$R^v \varepsilon'_* \mathcal{O}_{\tilde{F} \times \tilde{F}'} = \pi_1^* R^v \varepsilon_* \mathcal{O}_{\tilde{F}} = \begin{cases} 0 & \text{for } v > 0 \\ \mathcal{O}_{F \times \tilde{F}} & \text{for } v = 0 \end{cases}$$

and

$$R^v (\gamma' \circ \varepsilon')_* \mathcal{O}_{\tilde{F} \times \tilde{F}'} = R^v \gamma'_* \mathcal{O}_{F \times \tilde{F}'} = p_2^* R^v \gamma_* \mathcal{O}_{\tilde{F}'} = \begin{cases} 0 & \text{for } v > 0 \\ \mathcal{O}_{F \times F'} & \text{for } v = 0. \end{cases}$$

Now let \mathcal{N} be any invertible sheaf on X and write $\mathcal{N}_0^{(\eta)} = \bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}_0$ and $\mathcal{N}^{(\eta)} = \gamma^* \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' \right) = \sigma^{(\eta)*} \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N} \right)$. Similar to 3.6 in [10] we have:

Claim 2.10. a) There are natural maps

$$\left(\bigotimes_{v=1}^{\eta} f'_* \mathcal{N}' \right)^{**} \rightarrow \left(f_*^{(\eta)} \mathcal{N}^{(\eta)} \right)^{**}$$

and

$$\left(f_*^{(\eta)} \mathcal{N}^{(\eta)} \otimes \omega_{X^{(\eta)}/Y'} \right)^{**} \rightarrow \left(\bigotimes_{v=1}^{\eta} f'_* (\mathcal{N}' \otimes \omega_{X'/Y'}) \right)^{**}.$$

b)

$$f_{0*}^{\eta} (\mathcal{N}_0^{(\eta)} \otimes \omega_{X_0^{\eta}/Y_0}^e) = \bigotimes_{v=1}^{\eta} f_{0*} (\mathcal{N}_0 \otimes \omega_{X_0/Y_0}^e)$$

for all $e \geq 0$.

c) If f' is semi stable in codimension one (i.e.: if $f'^{-1}(y)$ is a reduced normal crossing divisor for all general points y of components of $Y' - \delta^{-1}(Y_0)$ which are of codimension one) then for all $e \geq 0$

$$\left(\bigotimes_{v=1}^{\eta} f'_* (\mathcal{N}' \otimes \omega_{X'/Y'}^e) \right)^{**} = \left(f_*^{(\eta)} (\mathcal{N}^{(\eta)} \otimes \omega_{X^{(\eta)}/Y'}) \right)^{**}.$$

Proof. Since we are taking reflexive hulls in *a* and *c*, we may as well replace Y' by the complement of a codimension two subscheme and assume thereby that f' is flat and Gorenstein. Then, by flat base change, the maps in *a* are just the natural maps

$$f_*^{\eta} \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' \right) \rightarrow f_*^{\eta} \gamma_* \gamma^* \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' \right)$$

and

$$f_*^{\eta} \gamma_* \left(\omega_{X^{(\eta)}/Y'} \otimes \gamma^* \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' \right) \right) \rightarrow f_*^{\eta} \left(\omega_{X'/Y'} \otimes \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' \right) \right).$$

b follows directly by flat base change and induction on η . Finally in c we may assume, making Y' smaller again, that $f'^{-1}(Y' - \delta^{-1}(Y_0))$ is a reduced relative normal crossing divisor and $Y' - \delta^{-1}(Y_0)$ smooth. Then, as in [10] p. 338, one finds that X'' has rational Gorenstein singularities. The equality in c is induced by

$$\gamma_* \left(\omega_{X(m)/Y'}^e \otimes \gamma^* \left(\bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' \right) \right) = \gamma_* \omega_{X(m)/Y'}^e \otimes \bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}' = \omega_{X'/\eta/Y'}^e \otimes \bigotimes_{v=1}^{\eta} \text{pr}_v^* \mathcal{N}'.$$

In order to prove 2.7 we are allowed to replace Y' by some finite cover. To be more precise, if we have a diagram

$$\begin{array}{ccccc} W_0 & \xrightarrow{\iota} & W & \xleftarrow{\gamma} & W' \\ \downarrow \tau_0 & & \downarrow \tau & & \downarrow \tau' \\ Y_0 & \longrightarrow & Y & \longleftarrow & Y' \end{array}$$

where τ_0, τ and τ' are finite, ι a compactification and γ a desingularization, and if the trace map from $\tau_{0*} \mathcal{O}_{W_0}$ to the integral closure of \mathcal{O}_{Y_0} has its image in \mathcal{O}_{Y_0} , then, using II, 2.4, a), and II, 2.5, b), it is enough to verify 2.7 for the pullback family.

Let us first choose W_0 such that $\tau_0^* \det(f_{0*} \mathcal{M}_0)$ becomes an r' -th power of some invertible sheaf. To this aim we can write $\det(f_{0*} \mathcal{M}_0)$ as the difference of two very ample sheaves \mathcal{H}_1 and \mathcal{H}_2 on Y . Then we take non singular coverings $Z_i \xrightarrow{\pi_i} \mathbb{P}(H^0(Y, \mathcal{H}_i))$ such that the pullback of the tautological bundle is an r' -th power. Then we can take for W and W_0 the pullback of Y and Y_0 .

In the same way, we may assume that $\det(f'_* \mathcal{M}')$ is an r' -th power. Here we have to use II, 1.10, to enforce the condition on the trace map. Moreover, as in [6], 4.6, or [10], 6.1, we can make semi stable reduction in codimension one.

Alltogether, we can assume thereby that $\det(f_{0*} \mathcal{M}_0) = \mathcal{O}_{Y_0}$ and $\det(f'_* \mathcal{M}') = \mathcal{O}_{Y'}$, and that the assumption of 2.10, c), holds true.

Let \mathcal{H} be an ample invertible sheaf on Y , $\mathcal{H}_0 = j^* \mathcal{H}$ and $\mathcal{H}' = \delta^* \mathcal{H}$. Let us consider

$$m = \text{Min} \{ \mu \in \mathbb{N}; \mathcal{H}_0^{\mu \cdot e - 1} \otimes f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e) \}$$

is weakly positive over Y_0 with respect to

$$(Y', \mathcal{H}'^{\mu \cdot e - 1} \otimes f'_*(\mathcal{M}' \otimes \omega_{X'/Y'}^e)).$$

Claim 2.11. $m < e + 1$.

The proof of 2.7 ends by the usual argument: $\mathcal{H}_0^{e^2 - 1} \otimes f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)$ is weakly positive over Y_0 with respect to $(Y', \mathcal{H}'^{e^2 - 1} \otimes f'_*(\mathcal{M}' \otimes \omega_{X'/Y'}^e))$. The exponent $e^2 - 1$ is independent of Y and \mathcal{H} and remains the same for all coverings. By II, 2.4, b), and II, 2.5, b), we obtain that $f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)$ is weakly positive over Y_0 with respect to $(Y', f'_*(\mathcal{M}' \otimes \omega_{X'/Y'}^e))$ and, using 2.4, we are done.

Proof of 2.11. For some $N > 0$ $S^N(\mathcal{H}_0^{m \cdot e - 1} \otimes f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)) \otimes \mathcal{H}_0^N$ will be globally generated over Y_0 by sections belonging to

$$H^0(Y', S^N(\mathcal{H}'^{m \cdot e - 1} \otimes f'_*(\mathcal{M}' \otimes \omega_{X'/Y'}^e)) \otimes \mathcal{H}'^N).$$

By assumption c) in 2.7 we can find some effective divisor E on X' , with $\text{codim}_{Y'}(f'(E)) \geq 2$, such that $(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e \otimes f_0^* \mathcal{H}_0^{m \cdot e})^N$ is globally generated by sections lying in

$$H^0(X', (\mathcal{M}' \otimes \omega_{X'/Y'}^e \otimes f'^* \mathcal{H}'^{m \cdot e})^N \otimes \mathcal{O}_{X'}(E)).$$

Replacing \mathcal{M}' by $\mathcal{M}'(E)$ we may assume that $E=0$.

Let us return, for $\eta=r'$, to the diagram considered in 2.9, and write $\mathcal{M}_0^{(r')}$ and $\mathcal{M}^{(r')}$ for the induced sheaves on $X_0^{r'}$ and $X^{(r')}$. Write

$$\mathcal{L}_0 = \mathcal{M}_0^{(r')} \otimes \omega_{X_0^{r'}/Y_0}^{e-1} \otimes f_0'^* \mathcal{H}_0^{m(e-1) \cdot r'}$$

and

$$\mathcal{L}' = \mathcal{M}^{(r')} \otimes \omega_{X^{(r')}/Y'}^{e-1} \otimes f'^* \mathcal{H}'^{m(e-1) \cdot r'}.$$

By 2.10, b) and a), we have

$$\varphi_0: \mathcal{O}_{Y_0} = \det(f_{0*} \mathcal{M}_0) \rightarrow \bigotimes_{v=1}^{r'} f_{0*} \mathcal{M}_0 = f_{0*} \mathcal{M}_0^{(r')}$$

and

$$\varphi': \mathcal{O}_{Y'} = \det(f'_* \mathcal{M}') \rightarrow \left(\bigotimes_{v=1}^{r'} f'_* \mathcal{M}' \right)^{**} \rightarrow (f'^* \mathcal{M}^{(r')})^{**}.$$

For some divisor E on $X^{(r')}$ with $\text{codim}_{Y'}(f'^{(r')}(E)) \geq 2$, the right hand side will be $f'^{(r')} \mathcal{M}^{(r')}(E)$.

Let Γ_0 and Γ' be the zero divisors induced by φ_0 and φ' on $X_0^{r'}$ and $X^{(r')}$. Then the N -th power of

$$\begin{aligned} \mathcal{L}_0^e(-\Gamma_0) &= (\mathcal{M}_0^{(r')} \otimes \omega_{X_0^{r'}/Y_0}^e \otimes f_0'^* \mathcal{H}_0^{m \cdot e \cdot r'})^{e-1} \\ &= \bigotimes_{v=1}^{r'} \text{pr}_v^* (\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e \otimes f_0^* \mathcal{H}_0^{m \cdot e})^{e-1} \end{aligned}$$

is globally generated over $X_0^{r'}$ by sections belonging to

$$H^0(X^{(r')}, (\mathcal{L}'^e(e \cdot E - \Gamma'))^N) = H^0(X^{(r')}, (\mathcal{M}^{(r')}(E) \otimes \omega_{X^{(r')}/Y'}^e \otimes f'^* \mathcal{H}'^{m \cdot e \cdot r'})^{(e-1) \cdot N}),$$

at last if we choose E large enough. In fact, this vector space contains, by 2.10, c,

$$\bigotimes_{v=1}^{r'} \text{pr}_v^* H^0(X', (\mathcal{M}' \otimes \omega_{X'/Y'}^e \otimes f'^* \mathcal{H}'^{m \cdot e})^{(e-1) \cdot N}).$$

By construction Γ_0 can not contain a whole fibre $F^{r'}$ of $f_0^{r'}$ and, by 2.2, c),

$$e(\Gamma_0|_{F^{r'}}) \leq e(\mathcal{M}_0^{(r')}|_{F^{r'}}) \leq c_1(\mathcal{M}_0|_F)^n + 1 = e.$$

By assumption d) of 2.7 and 2.10, b), the sheaf $f_{0*}(\mathcal{L}_0 \otimes \omega_{X'_0/Y_0})$ is locally free and compatible with base change. Hence we can apply 2.6 and find

$$f_{0*}^{r'}(\mathcal{L}_0 \otimes \omega_{X'_0/Y_0}) = \bigotimes^{r'} (f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e) \otimes \mathcal{H}_0^{m(e-1)})$$

to be weakly positive over Y_0 with respect to $(Y', f_*^{r'}(\mathcal{L}'(E) \otimes \omega_{X(r')/Y'}))$. Again, by 2.10, c), the reflexive hull of the last sheaf is

$$\left(\bigotimes^{r'} (f_*'(\mathcal{M}' \otimes \omega_{X'/Y'}^e) \otimes \mathcal{H}'^{m(e-1)}) \right)^{**}.$$

By our choice of m as a minimum, this is only possible if

$$(m-1) \cdot e - 1 < m(e-1) \quad \text{or} \quad m < e+1.$$

§3. The proof of 2.6

The strategy of the proof of 2.6 will be similar to the one used in II, §3 in order to prove II, 2.10. First we choose a section and consider the problem locally (using II, 2.11) and then we construct a category \mathcal{L} of compactifying triples which covers Y_0 and an extension \mathcal{G} of $\mathcal{F}_0 = f_{0*}(\mathcal{L}_0 \otimes \omega_{X_0/Y_0})$ to \mathcal{L} . The reader finds this terminology in II, 1.5. Let us start with some preliminary remarks.

3.1. We can replace in 2.6 e) by $N \cdot e$ and Γ by $N \cdot \Gamma$ in order to get $N = 1$.

Let $y \in Y_0$ be a point. By II, 2.11, we are allowed to replace Y_0 by some neighbourhood of y , whenever it is convenient to do so.

Let $F_y = f_0^{-1}(y)$ and A_y the image of

$$A = H^0(X_0, \mathcal{L}_0^e(-\Gamma_0)) \cap H^0(X', \mathcal{L}'^e(-\Gamma'))$$

in $H^0(F_y, \mathcal{L}_0^e(-\Gamma_0)|_{F_y})$. Let us choose a general section of A_y and lift it to $s \in A$. We denote by D_0 and D' the zero divisions of s on X_0 and X' respectively.

If $g: F' \rightarrow F_y$ is a desingularization such that $\gamma^*(\Gamma_0|_{F_y})$ has normal crossings then $\gamma^*(D_0|_{F_y})$ will be a smooth divisor which intersects $\gamma^*(\Gamma_0|_{F_y})$ properly. Therefore $e(\Gamma_0|_{F_y}) = e((\Gamma_0 + D_0)|_{F_y})$.

Claim 3.2. Replacing Y_0 by some neighbourhood of y we may assume

- D_0 does not contain any fibre of f_0 .
- $e((\Gamma_0 + D_0)|_F) \leq e$ for all fibres F of f_0 .
- For $Y'_0 = \delta^{-1}(Y_0)$ consider the fibre product

$$\begin{array}{ccc} X_0 & \xleftarrow{\sigma_0} & X''_0 \\ \downarrow f_0 & & \downarrow f'_0 \\ Y_0 & \xleftarrow{\delta_0} & Y'_0 \end{array}$$

and $\Gamma'_0 = \sigma_0^* \Gamma_0$ and $D'_0 = \sigma_0^* D_0$. Then $e(\Gamma'_0 + D'_0) \leq e$.

Proof. Evidently we may assume a) to hold true. Let us consider the morphisms and divisors given in c). By 2.2, b) we can find some neighbourhood of $\delta_0^{-1}(y)$ such that both, b) and c) hold true over this neighbourhood. Shrinking Y_0 we are done.

Claim 3.3. We may assume that $\mathcal{L}_0^e(-\Gamma_0) = \mathcal{O}_{X_0}$ and $\mathcal{L}'^e(-\Gamma') = \mathcal{O}_{X'}$.

Proof. By 3.2 $D_0 + \Gamma_0$ will again satisfy the assumptions made in 2.6. Therefore we can replace Γ_0 by $\Gamma_0 + D_0$ and Γ' by $\Gamma' + D'$.

As in II, 3.18, let us consider some kind of basic diagram. It will serve to define a Schmid extension of $f_{0*}(\omega_{X_0/Y_0} \otimes \mathcal{L}_0)$ and it will at the same time reappear when we formulate the main implication of II, 1.7, in our situation.

Statement 3.4. There exists a second diagram

$$\begin{array}{ccccc}
 V_0 & \xrightarrow{\alpha} & V & \xleftarrow{\tau} & V' \\
 \downarrow g_0 & & \downarrow g & & \downarrow g' \\
 W_0 & \xrightarrow{\beta} & W & \xleftarrow{\rho} & W'
 \end{array}$$

satisfying the assumptions made in 2.5 and morphisms

$\pi_0: W_0 \rightarrow Y_0, \pi: W \rightarrow Y, \pi': W' \rightarrow Y', \gamma_0: V_0 \rightarrow X_0, \gamma: V \rightarrow X, \gamma': V' \rightarrow X'$ such that

a) all possible diagrams commute and

$$\begin{array}{ccc}
 V_0 & \xrightarrow{\gamma_0} & X_0 \\
 \downarrow g_0 & & \downarrow f_0 \\
 W_0 & \xrightarrow{\pi_0} & Y_0
 \end{array}$$

is a fibre product.

b) π_0 is finite dominant and the trace map from $\pi_{0*} \mathcal{O}_{W_0}$ to the integral closure of \mathcal{O}_{Y_0} factors over \mathcal{O}_{Y_0} .

Let us denote $\gamma^* \mathcal{L}$ by \mathcal{M} and $\gamma^* \Gamma$ by Δ . We use the convention made in 2.5, d). Recall that $\mathcal{M}_0^e = \mathcal{O}_{V_0}(\Delta_0)$ and $\mathcal{M}'^e = \mathcal{O}_{V'}(\Delta')$. Let $\varphi: T \rightarrow V'$ be the normalization of the covering obtained by taking the e -th root out of Δ' and let $\varepsilon: T' \rightarrow T$ be a desingularization. Write $\varphi' = \varphi \circ \varepsilon$ and $h' = g' \circ \varphi': T' \rightarrow W'$. Then we want in addition:

c) Δ' is a normal crossing divisor.

d) There exists some open dense subscheme W'_s of $\rho^{-1}(W_0)$ such that $W' - W'_s$ is a normal crossing divisor and the induced morphism $h'_s: T_s = h'^{-1}(W'_s) \rightarrow W'_s$ is smooth.

3.5. We have

$$\phi'_* \omega_{T'/W'} = \bigoplus_{\nu=0}^{e-1} \left(\mathcal{M}'^\nu \left(- \left[\begin{array}{c} \nu \cdot \Delta' \\ e \end{array} \right] \right) \otimes \omega_{V'/W'} \right)$$

and for $\nu=1$ we obtain

$$\mathcal{F}' = g'_* \left(\mathcal{M}' \left(- \left[\begin{array}{c} \Delta' \\ e \end{array} \right] \right) \otimes \omega_{V'/W'} \right)$$

as a direct summand of $h'_* \omega_{T'/W'}$.

By 3.2 and 3.3 the divisor Γ_0 does not contain any fibre of f_0 and $e(\Gamma_0|_F) \leq e$ for all fibres F of f_0 . Therefore $e(\Delta'|_F) \leq e$ for all fibres F of g' lying over $\rho^{-1}(W_0)$. By 2.2, b) and since we assumed $f_{0*}(\mathcal{L}_0 \otimes \omega_{X_0/Y_0})$ to be compatible with arbitrary pullbacks, we find that \mathcal{F}' and $\rho^* \beta_* g_{0*}(\mathcal{M}_0 \otimes \omega_{V_0/W_0})$ coincide over $\rho^{-1}(W_0)$.

Definition 3.6. We say that $\mathcal{F}_0 = g_{0*}(\mathcal{M}_0 \otimes \omega_{V_0/W_0})$ has a Schmid extension, if the monodromies of $R^n h'_{s*} \mathbf{C}_{T'_s}$ around the components of $W' - W'_s$ are unipotent.

In this case we call $\mathcal{F}' = g'_* \left(\mathcal{M}' \left(- \left[\begin{array}{c} \Delta' \\ e \end{array} \right] \right) \otimes \omega_{V'/W'} \right)$ the Schmid extension of \mathcal{F}_0 to W' (or if we want to indicate the role of Δ ; with respect to Δ).

Lemma 3.7. Assume that we have a Schmid extension \mathcal{F}' of \mathcal{F}_0 to W' . Then \mathcal{F}' is locally free and weakly positive over W' . Moreover \mathcal{F}' is compatible with pullback under generically finite morphisms.

Proof. As in II, 3.1 and 3.3, $h'_* \omega_{T'/W'}$ is locally free and weakly positive over W' . Moreover, $h'_{s*} \omega_{T'_s/W'_s}$ is compatible with pullbacks and, since the canonical extension behaves nice under generically finite morphisms $h'_* \omega_{T'/W'}$ is compatible with pullbacks for those morphisms. Everything carries over to the direct summand \mathcal{F}' .

Lemma 3.8. Let Z be a smooth irreducible divisor in W' such that $Z_0 = W'_0 \cap Z$ is the complement of a normal crossing divisor and let η be the restriction of g' to the closure R' of $g'^{-1}(Z_0)$ in V' . Assume that $\eta_*(\omega_{R'/Z} \otimes \mathcal{M}')|_{Z_0}$ has a Schmid extension \mathcal{F}'_Z with respect to $\Delta'|_{R'}$. Then $\mathcal{F}'|_Z = \mathcal{F}'_Z$.

The proof of 3.8 is exactly the same as the proof of 3.22 in II: $\mathcal{F}'|_{Z_0} \approx \mathcal{F}'_Z|_{Z_0}$ and by II, 3.11, this isomorphism extends to a surjection $\mathcal{F}' \rightarrow \mathcal{F}'_Z$.

Lemma 3.9. Under the assumptions made in 2.6, 3.2 and 3.3 we can always find morphisms as in 3.4 such that $g_{0*}(\mathcal{M}_0 \otimes \omega_{V_0/W_0})$ has a Schmid extension \mathcal{F}' to W' with respect to Δ .

Proof. For $\pi_0 = \text{id}$ we can find those morphisms. Just, the local monodromies of $R^n h'_{s*} \mathbf{C}_{T'_s}$ will be only quasiunipotent. As in II, 3.23 we may use [6], 4.5 to find a finite non singular covering of W' where the monodromy condition holds for the pullback of h'_s . By II, 1.10 we get the corresponding covering of W_0 which satisfies 3.4, b). Then, we just have to desingularize to get the morphisms wanted.

Theorem 3.10. Under the assumptions made in 2.6, 3.2 and 3.3 we can find morphisms as in 3.4 and a locally free sheaf \mathcal{F} on W such that $\rho^* \mathcal{F} = \mathcal{F}'$ is the Schmid extension of $g_{0*}(\mathcal{M}_0 \otimes \omega_{V_0/W_0})$ with respect to Δ .

Proof. We construct a category of compactifying triples which covers Y_0 (see II, 1.5 for the definition): If S_0 is a closed subscheme of Y_0 we can use 3.9 to get morphisms as in 3.4 (for Y_0, X_0 replaced by $S_0, f_0^{-1}(S_0)$) such that the Schmid extension \mathcal{F}' exists. We may assume that in addition $W' - \rho^{-1}(W_0)$ is a normal crossing divisor. Then we take \mathcal{Z} to be the smallest complete category containing all those $\underline{W}' = (W', \rho^{-1}(W_0), \rho|_{\rho^{-1}(W_0)})$.

An extension \mathcal{G} of $f_{0*}(\omega_{X_0/Y_0} \otimes \mathcal{L}_0)$ to \mathcal{Z} is given by: $\mathcal{G}_{\underline{W}'}$ is the Schmid extension of $g_{0*}(\omega_{V_0/W_0} \otimes \mathcal{M}_0)$. By 3.7 we get $\mathcal{G}_{\underline{Z}}$ for all $\underline{Z} \in \mathcal{Z}$ which are generically finite over \underline{W}' . Then, by 3.8, those sheaves are compatible with inclusions $\tau: \underline{Z} \rightarrow \underline{Z}'$ and, by induction on the codimension and 3.7, with all morphisms in \mathcal{Z} . By II, 1.7 we are done.

Proof of 2.6. As mentioned already we can assume that 3.2 and 3.3 hold true. By II, 2.5, a), we can blow Y' up. Hence we may assume in 3.10 that the ramification locus of $\pi': W' \rightarrow Y'$ in Y' is a normal crossing divisor. Since, by [6], 4.7, every finite cover, whose ramification divisor has normal crossings, is dominated by a non singular cover, we may even assume that π' is finite.

Since $\delta^* \mathcal{F} = \mathcal{F}'$ is weakly positive over W' , \mathcal{F} is weakly positive over W with respect to $g'_*(\omega_{V'/W'} \otimes \mathcal{M}')$ by II, 2.4, c). Using II, 2.4, a), and II, 2.5, b), we find $f_{0*}(\omega_{X_0/Y_0} \otimes \mathcal{L}_0)$ to be weakly positive over Y_0 with respect to $(Y', f'_*(\omega_{X'/Y'} \otimes \mathcal{L}'))$.

§ 4. The proofs of 1.7 and 1.12

The proof of 1.7 is similar to I, 5.7–5.17. However, there we only considered canonically polarized varieties and therefore we should indicate the necessary modifications of the proof. A good opportunity to try to present I, 5.17, in a more readable way.

Let H be the Hilbert scheme considered in 1.5 d) and $G = \text{Sl}(r, \mathbb{C})$. As in I, § 5 we can use [7], Proposition 1.16, and replace H by H_{red} .

Claim 4.1. The stabilizer $S(x)$ of $x \in H$ is finite.

Proof. $S(x)$ is a subgroup of $\text{Sl}(r, \mathbb{C})$ and of $\mathcal{A} \text{ut}(g^{-1}(x))$. Since $F \in \mathcal{M}_h(\mathbb{C})$ was supposed to be non uniruled, $\mathcal{A} \text{ut}(g^{-1}(x))$ can not contain a linear subgroup and $S(x)$ must be finite (see [5]).

Claim 4.2. Let $\mathcal{L}_\eta = \mathcal{L}_0 \otimes \lambda^\eta$ be the sheaf considered in 1.6, c). Then for all $\eta \geq 0$ \mathcal{L}_η is ample.

Proof. We just have to recall that

$$\lambda = \det(\mathcal{E}) \quad \text{for } \mathcal{E} = \left(\bigotimes^{r'} g_* (\mathcal{H}^\vee \otimes \omega_{\mathcal{X}/H}^e) \right) \otimes \det(g_* \mathcal{H}^\vee)^{-1}$$

where $(g: \mathcal{X} \rightarrow H, \mathcal{H}) \in \mathcal{M}_h(H)$ is the universal family considered in 1.5, d). Especially (see 2.8 and 2.7) \mathcal{E} is weakly positive over Y_0 and, by 2.4, c), λ as well. Hence 4.2 follows from I, 3.2.

The main technical tool is (as in I, 5.9).

Proposition 4.3. For $x \in H$ let H_x be the G -orbit. Then we can find some projective compactification $\iota: H \rightarrow H'$ and some $\eta_0 > 0$, such that we have for the closure H'_x of H_x in H' :

- a) There is some effective Cartier divisor Γ_x with on H'_x with $(\Gamma_x)_{\text{red}} = H'_x - H_x$.
- b) For all multiples η of η_0 we can find some $\alpha > 0$ and a coherent subsheaf $\mathcal{L}_\eta^{(\alpha)}$ of $\iota_*(\mathcal{L}_\eta)^\alpha$ such that:
 - i) $\mathcal{L}_\eta^{(\alpha)}$ is generated by its global sections.
 - ii) There is an inclusion

$$\mathcal{O}_{H'_x}(\Gamma_x) \rightarrow \mathcal{L}_\eta^{(\alpha)}|_{H'_x},$$

isomorphic over H_x .

4.4. Now the proof of 1.7 is easy: First, replacing α by some multiple, we can assume that for some $V \subset H^0(H', \mathcal{L}_\eta^{(\alpha)})$ the morphism $H \rightarrow \mathbb{P}(V)$ is an embedding. Moreover we can assume that V generates $\mathcal{L}_\eta^{(\alpha)}$. By [7], Chap. I, §1, we may assume as well that V is a G -invariant subspace of $H^0(H, (\mathcal{L}_\eta)^\alpha)$. Finally, replacing H' by the closure of H in $\mathbb{P}(V)$, G will act on H' and $\mathcal{L}_\eta^{(\alpha)} = \mathcal{O}_{H'}(1)$ is G -linearized. We still have a section of $\mathcal{O}_{H'_x}(1)$, whose zero divisor is exactly supported in $H'_x - H_x$. Hence by I, 5.4 we have $H_x \subset H'_x(\mathcal{O}_{H'_x}(1))^s$ and by I, 5.5

$$H'_x(\mathcal{O}_{H'_x}(1))^s = H'_x \cap H'(\mathcal{O}_{H'}(1))^s$$

and

$$H \cap H'(\mathcal{O}_{H'}(1))^s \subset H(\mathcal{O}_H(1))^s = H(\mathcal{L}_\eta^\alpha)^s.$$

Since “stable” is an open property we can choose η big enough such that $H = H(\mathcal{L}_\eta^\alpha)^s$.

Proof of 4.3. The group $\mathbb{P}\text{Gl}(r, \mathbb{C})$ is an open subvariety of $\mathbb{P} = \mathbb{P}(\bigoplus^r \mathbb{C})$. Let us choose a projective compactification H' such that $H' - H$ is exactly the support of some effective Cartier divisor Γ and such that H'_x is non singular.

Choose Y to be a compactification of $G \times H$ such that the three morphisms

$$\sigma: G \times H \rightarrow H \quad \text{pr}_2: G \times H \rightarrow H \quad \text{pr}_1: G \times H \rightarrow \mathbb{P}\text{Gl}(r, \mathbb{C})$$

extend to morphisms

$$\varphi: Y \rightarrow H' \quad p_2: Y \rightarrow H' \quad p_1: Y \rightarrow \mathbb{P}.$$

Let us write $U = \varphi^{-1}(H)$ and $V = p_2^{-1}(H)$. Then, since $(\sigma, \text{pr}_2): G \times H \rightarrow H \times H$ is proper, we have $U \cap V = G \times H$. We can consider the pullback of

$$(g: \mathcal{X} \rightarrow H, \mathcal{H}) \in \mathcal{M}_h(H)$$

under $\varphi|_U$ and $p_2|_V$. Over $U \cap V$ both are isomorphic. In fact, for $\gamma \in G$, h and $\gamma \circ h$ parametrise the same subvariety of \mathbb{P}^{r-1} , up to change of coordinates. Therefore we can glue both pullbacks together and obtain for $Y_0 = U \cup V$

some $(f_0: X_0 \rightarrow Y_0, \mathcal{M}_0) \in \mathcal{M}_h(Y_0)$. Let us write $U'_y = \varphi^{-1}(y)$ for $y \in H$ and $U_y = U'_y \cap V = \sigma^{-1}(y)$. For our given point x we can assume that U'_x is non singular, as well as H'_x .

4.5. Compactifying and desingularizing we find a diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & X & \longleftarrow & X' \\ \downarrow f_0 & & \downarrow f & & \downarrow f' \\ Y_0 & \xrightarrow{j} & Y & \xleftarrow{\delta} & Y' \end{array}$$

as in 2.5, and we may assume that \mathcal{M}_0 extends to some invertible sheaf \mathcal{M} on X . As in II, 2.3, we can assume that we have chosen Y such that $\mathcal{O}_Y = \delta_* \mathcal{O}_{Y'} \cap j_* \mathcal{O}_{Y_0}$.

Let us write

$$\mathcal{F}' = (\bigotimes^{r'} f'_*(\mathcal{M}' \otimes \omega_{X'/Y'}^e)) \otimes \det(f'_* \mathcal{M}')^{-1}$$

and

$$\mathcal{F}_0 = (\bigotimes^{r'} f_{0*}(\mathcal{M}_0 \otimes \omega_{X_0/Y_0}^e)) \otimes \det(f_{0*} \mathcal{M}_0)^{-1}$$

where e is as in 1.5, and $r' = \text{rk}(f_{0*} \mathcal{M}_0)$. By 2.8 and 2.7 again, we know that \mathcal{F}_0 is weakly positive over Y_0 with respect to (Y', \mathcal{F}') . By II, 2.3, the sheaf $\mathcal{F} = j_* \mathcal{F}_0 \cap \delta_* \mathcal{F}'$ will be weakly positive over Y . Blowing up again, we can assume that \mathcal{F} is locally free. Therefore we extended $p_2|_V^* \mathcal{E}$ to some locally free sheaf \mathcal{F} on Y which is weakly positive over Y_0 .

Of course by 2.4, c) we know that $\det(\mathcal{F})$ is weakly positive over Y_0 as well. May be, the main idea behind 1.7 is, that the group action enforces some stronger statement:

Claim 4.6. Blowing up Y , if necessary, we can find some effective Cartier divisor Δ on Y with

- a) $\Delta_{\text{red}} = p_2^*(\Gamma)_{\text{red}}$.
- b) $\mathcal{O}_Y(\Delta) \otimes \det(\mathcal{F})^{r' \cdot r - 1}$ is weakly positive over Y_0 .
- c) For $y \in H$ let $\pi_y: U'_y \rightarrow \mathbb{P}$ be the restriction of $p_1: Y \rightarrow \mathbb{P}$, and $\Delta_y = \Delta|_{U'_y}$. Then $(\Delta_y)_{\text{red}} = U'_y - U_y$ and $\pi_y^* \mathcal{O}_{\mathbb{P}}(r \cdot r' - 2) = \mathcal{O}_{U'_y}(\Delta_y)$.

Proof. \mathcal{E} is the direct sum of several copies of some invertible sheaf. Therefore

we can write $\mathcal{F}|_V = \bigoplus^{r' \cdot r} \mathcal{N}_0$. In fact, if $g_*(\mathcal{H} \otimes \omega_{X/H}^e) = \bigoplus^r \mathcal{N}_1$ we obtain $\mathcal{N}_0 = p_2|_V^* \mathcal{N}_1' \otimes \det(g_* \mathcal{H})^{-1}$. Let \mathcal{N} be some extension of \mathcal{N}_0 to Y , such that

the induced inclusion $\mathcal{N}_0 \rightarrow \bigoplus^{r' \cdot r} \mathcal{F}|_V$ extends to $\varepsilon: \mathcal{N} \rightarrow \bigoplus^{r' \cdot r} \mathcal{F}$. Blowing up Y , we may assume that \mathcal{N} is invertible and splits locally. ε induces an inclusion

$$\underline{s}: \bigoplus^{r' \cdot r} \mathcal{N} \rightarrow \mathcal{F}.$$

Let us choose Δ to be the zero divisor of $\det(\underline{s}) \in H^0(Y, \det(\mathcal{F}) \otimes \mathcal{N}^{-r' \cdot r})$. Let us prove c) first:

The morphism $\pi_y: U'_y \rightarrow \mathbb{P}$ maps $(\gamma, \gamma^{-1} \cdot y) \in U_y$ to γ . Hence $\pi_y|_{U_y}$ is finite over $\mathbb{P}\mathrm{Gl}(r, \mathbb{C})$ and $U_y \approx G$. p_2 induces the inverse action of G on H_y . The trivialization $\bigoplus^r \mathcal{N}_1 \approx g_* (\mathcal{H} \otimes \omega_{\mathbb{A}^1/H}^2)$ gives under pullback to $U_y \approx G$, the “universal basis” $\bigoplus^r \mathcal{O}_G \approx \mathbb{C}^r \otimes \mathcal{O}_G$. This morphism is induced by

$$\underline{s}'': \bigoplus^r \mathcal{O}_{\mathbb{P}}(-1) \rightarrow \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}}$$

and corresponds to the tautological map

$$\mathcal{O}_{\mathbb{P}}(-1) \rightarrow \bigoplus^r \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}}.$$

Taking the r' -th tensor product we find

$$\underline{s}': \bigoplus^{r \cdot r'} \mathcal{O}_{\mathbb{P}}(-r') \rightarrow \bigotimes^{r'} \bigoplus^r \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}}.$$

Since our morphism f is trivial over U'_y , the pullback of the right hand side is $\mathcal{F}|_{U_y}$ and that of the left hand side is $\bigoplus^{r \cdot r'} \mathcal{N}|_{U_y}$ and $\pi_y^*(\underline{s}') = \underline{s}|_{U_y}$. Therefore

$$\mathcal{O}_{U_y}(\Delta_y) = \mathcal{N}^{r \cdot r'}|_{U_y} = \pi_y^* \mathcal{O}_{\mathbb{P}}(r \cdot r'^2).$$

Moreover, Δ_y is the pullback of the locus of degeneration of \underline{s}' and $(\Delta_y)_{\mathrm{red}}$ must be the pullback of the degeneration locus of \underline{s}'' which is just $\mathbb{P} - \mathbb{P}\mathrm{Gl}(r, \mathbb{C})$.

Since we can add to Δ divisor supported in $Y - Y_0$, and since $Y_0 = V \cup (\bigcup_{y \in H} U_y)$, c) implies a).

To prove b) we consider the dual of ε and its symmetric product

$$S^{r \cdot r'} \left(\bigoplus^{r \cdot r'} \mathcal{F}^\vee \right) \rightarrow \mathcal{N}^{-r \cdot r'}$$

and

$$S^{r \cdot r'} \left(\bigoplus^{r \cdot r'} (\mathcal{F}^\vee \otimes \det(\mathcal{F})) \right) \rightarrow \mathcal{N}^{-r \cdot r'} \otimes \det(\mathcal{F})^{r \cdot r'} = \mathcal{O}_Y(\Delta) \otimes \det(\mathcal{F})^{r \cdot r' - 1}.$$

The sheaf on the left hand side is nothing but $S^{r \cdot r'} \left(\bigoplus^{r \cdot r'} A^{r \cdot r' - 1} \mathcal{F} \right)$ and b) follows from 2.4, c).

4.7. To prove 4.3 we just have to descend the “weak positivity” in 4.6, b) to H' . By abuse of notations we replace G by the $\mathbb{P}\mathrm{Gl}(r, \mathbb{C})$. Replacing Γ by some multiple we can assume that $\Delta \leq p_2^*(\Gamma)$ and, if $\rho: U'_x \rightarrow H'_x$ is the restriction of p_2 , that $\rho^*(\Gamma_x) \geq \Delta_x$. Therefore we have a trace map $\rho^* \mathcal{O}_{U'_x}(\Delta_x) \rightarrow \mathcal{O}_{H'_x}(\Gamma_x)$.

For some $\beta > 0$ the ample sheaf \mathcal{L}_0 will extend to some ample invertible sheaf $\mathcal{L}_0^{(\beta)}$ on H' , may be after further blowing ups. Then, for some $\mu > 0$, divisible by $r \cdot r'^2$, the sheaf

$$\mathcal{L} = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}}(\mu) \otimes \mathrm{pr}_2^* \mathcal{L}_0^{(\beta)}$$

will be ample on $\mathbb{P} \times H'$. The natural map $(p_1, p_2): Y \rightarrow \mathbb{P} \times H'$ is isomorphic over $G \times H$ and, blowing up Y , we find some exceptional divisor F such that $\mathcal{O}_Y(-F)$ is (p_1, p_2) -ample and $(p_1, p_2)_* \mathcal{O}_Y(-F)$ contained in $\mathcal{O}_{\mathbb{P} \times H'}$. Choosing β and μ large enough,

$$\mathcal{L}' = \mathcal{O}_Y(-F) \otimes (p_1, p_2)^* \mathcal{L}$$

will be ample on Y .

By 4.5 and I, 3.2, the sheaf

$$\mathcal{P}_\eta = \mathcal{L}' \otimes \det(\mathcal{F})^{\eta \cdot (r' \cdot r - 1)} \otimes \mathcal{O}_Y(\eta \cdot \Delta)$$

is ample with respect to Y_0 for all $\eta \geq 0$ (see I, 1.16 for this notation).

Claim 4.8. There is some $\eta'_0 > 0$ such that for all $\eta \geq \eta'_0$ one can find an inclusion $\mathcal{O}_{U'_x} \left(\frac{2 \cdot \mu}{r \cdot r'^2} \rho^* \Gamma_x \right) \rightarrow \mathcal{P}_\eta|_{U'_x}$ which is surjective over U_x .

Proof. As we have seen in 4.6, c), $p_1^* \mathcal{O}_{\mathbb{P}}(\mu) = \mathcal{O}_{U'_x} \left(\frac{-\mu}{r \cdot r'^2} \Delta_x \right)$. Moreover F meets U'_x at most in $U'_x - U_x$ and $\mathcal{L}_\eta^{(\beta)}|_{U'_x} = \mathcal{O}_{U'_x}$. Hence for some divisor A with support in $U'_x - U_x$ we will have $\mathcal{L}'|_{U'_x} = \mathcal{O}_{U'_x}(A)$. Since $\det(\mathcal{F})|_{U'_x} = \mathcal{O}_{U'_x}$ we can find η_0 .

4.9. Let us choose some multiple η of $\beta \cdot \eta'_0$. Since U'_x is compact and contained in Y_0 we can find some $N > 0$ such that $H^0(Y, \mathcal{P}_\eta^N)$ generates $\mathcal{P}_\eta^N|_{U'_x}$. Then the subsheaf of $\mathcal{O}_{U'_x}(*\Delta_x)$ which is generated by $H^0(Y, \mathcal{P}_\eta^N)$ will contain $\mathcal{O}_{U'_x} \left(\frac{2 \cdot \mu \cdot N}{r \cdot r'^2} \rho^* \Gamma_x \right)$ and this inclusion is an isomorphism over U_x .

By our choice of F we have

$$(p_1, p_2)_* \mathcal{P}_\eta^N \rightarrow \mathcal{P}_\eta^{*N} = \text{pr}_1^* \mathcal{O}_{\mathbb{P}}(N \cdot \mu) \otimes \text{pr}_2^* (t_* \mathcal{L}_0^{\beta \cdot N} \otimes \lambda^{\eta \cdot (r' \cdot r - 1) \cdot N}).$$

Hence we obtain a morphism from

$$H^0(\mathbb{P} \times H', \mathcal{P}_\eta^{*N}) \otimes \mathcal{O}_{\mathbb{P} \times H'} \quad \text{to} \quad \mathcal{O}_{U'_x}(*\Delta_x)$$

whose image still contains $\mathcal{O}_{U'_x} \left(\frac{2 \cdot \mu \cdot N}{r \cdot r'^2} \rho^* \Gamma_x \right)$. Applying pr_{2*} and the trace map we find

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu \cdot N)) \otimes H^0(H, \mathcal{L}_0^{\beta \cdot N} \otimes \lambda^{\eta \cdot (r' \cdot r - 1) \cdot N}) \otimes \mathcal{O}_{H'} \rightarrow \rho_* \mathcal{O}_{U'_x}(*\Delta_x) \rightarrow \mathcal{O}_{H'_x}(*\Gamma_x)$$

and the image of the composed map will contain $\mathcal{O}_{H'_x} \left(\frac{2 \cdot \mu \cdot N}{r \cdot r'^2} \Gamma_x \right)$. As we have seen in 4.6, c), $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu \cdot N))$ generates $\mathcal{O}_{U'_x} \left(\frac{\mu \cdot N}{r \cdot r'^2} \Delta_x \right)$ and, under the trace map, they end up in $\mathcal{O}_{H'_x} \left(\frac{\mu \cdot N}{r \cdot r'^2} \Gamma_x \right)$. Therefore, the subsheaf of $\mathcal{O}_{H'_x}(*\Gamma_x)$ which

is generated by $H^0(H, \mathcal{L}_0^{\beta \cdot N} \otimes \lambda^{\eta(r' \cdot r - 1) \cdot N})$ contains $\mathcal{O}_{H_x} \left(\frac{\mu \cdot N}{r \cdot r'^2} \Gamma_x \right)$, and of course also $\mathcal{O}_{H_x}(\Gamma_x)$, isomorphic over H_x . Hence, in 4.3, we just have to choose $\eta_0 = \eta'_0 \cdot (r' \cdot r - 1)$ and $\alpha = \beta \cdot N$.

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Notes added in proof

(a) A. Fujiki and G. Schumacher study in their article “The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics” natural Kähler metrics on analytic moduli spaces (to appear in *Publ. RIMS, Kyoto Univ.*). Especially they prove, that compact analytic subspaces of the moduli space of canonically polarized manifolds are always projective. A better understanding of the relations between their methods and ours should clarify the properties of line bundles on moduli spaces.

(b) It seems that in remark 1.10 I overlooked some quite simple argument, which allows to descend “quasi projectivity” from M_h to P_h . A short proof of the second corollary in 1.10 without the assumption on the irregularity is in preparation.

